Cantor's Set, the Cardinality of the Reals, and the Continuum hypothesis

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Abstract: The Cantor set is obtained from the closed unit interval [0,1] by a sequence of deletions of middle third open intervals.

Apparently, Cantor constructed this set while attempting to find a cardinality between $CardN$, and 2^{CardN}. The length-less, nowhere dense Cantor set which is almost a void in (0,1), has cardinality equal to $CardR = 2^{CardN}$, and perhaps led Cantor to his Continuum Hypothesis that there is no set X with $CardN < CardX < 2^{CardN}$.

The puzzling Cantor's set establishes that cardinality is unrelated to measure, and that $CardR$ equals the power of a nowhere-dense set.

How can such a meager set have a cardinality greater than $CardN$?

Is it possible that $Card(0,1)$ may not be any greater than $CardN$?

In the following, we attempt to answer these questions.

1. Observations about infinite Cardinalities

Cantor's diagonal proof that $CardN < CardR$ may not apply to infinite sets

In his proof, Cantor listed the reals in a countable column, and exhibited the diagonal element as the one not counted for, in the list.

By [1, p. 57], every real number between 0, and 1, has a unique infinite decimal representation

$$c_1^{(n)}c_2^{(n)}c_3^{(n)}....$$

Missing from that listing is the real number

$$c_1c_2c_3....$$

where

$$c_n = 0, \text{ if } c_n^{(n)} \neq 0, \quad \text{and} \quad c_n = 1, \text{ if } c_n^{(n)} = 0.$$

It is unclear why the one missing diagonal element cannot be added to the listing. In fact, the listing will remain countable after countably many such elements will be added to it.

Two infinite sets have the same power, even if one set is “half” of the other. For instance, the natural numbers, and the odd natural numbers have the same power. For infinite sets, we can tolerate the missing of countably many elements before we conclude a contradiction. In other words, the diagonal proof does not apply credibly to infinite sets.

Card(P(A)) may not be strictly greater than Card(A).

The inequality $CardN < 2^{CardN}$, may seem plausible by $n < 2^n$, but as $n \to \infty$, we have $\lim n \leq \lim 2^n$. Is the inequality in $CardN < 2^{CardN}$ indeed strict?

By [1, p. 87], Cantor proved that

The set of all subsets of any given set A is of greater power than the set A

The proof aims to show the strict inequality

$$Card(A) < Card(P(A)),$$

to conclude that for every cardinal number there is a greater cardinal number. This will certainly establish $Card(N) < Card(P(N)) = Card(R)$.

But Cantor's proof uses the concept of set of sets which for infinite sets is not well-understood, since it may lead to the Russell paradoxial set. Russell (1903) defined his set y by

$$x \in y \leftrightarrow x \notin x.$$

Then, in particular, $y \in y \leftrightarrow y \notin y$ which is a contradiction.

In fact, as pointed out in [2, p.87], if we apply Cantor's theorem to the universal class of all objects V, every subset of V is also a member of V, and we have

$$Card(V) = Card(P(V)).$$

Avoiding this fact by claiming that V is not a set, while leaving the definition of what is a set vague enough to suit other results, does not establish the credibility of Cantor's claim. One may conclude that for infinite sets strict inequality may not exist, and we have only

$$Card(A) \le Card(P(A)).$$

Card(R) is the power of the nowhere-dense Cantor set.

The association of *CardR* with the "*Power of the continuum*", gives the impression that cardinality is related to measure, and that *CardR* must be greater than *CardN*.

But Cantor's set, which cardinality equals to *CardR*, is nowhere dense in the interval $(0,1)$. It has no length, and it is almost a void in $(0,1)$.

Cantor's set establishes that cardinality is unrelated to measure, and that *CardR* equals the power of a nowhere-dense set.

The Cantor set demonstrates that cardinality fails to distinguish between discrete sets and intervals. The measure of the interval $(0,1)$ is 1, but the cardinality of $(0,1)$ equals that of the nowhere dense Cantor set. Thus, it is conceivable that $Card(0,1)$ may not be any greater than *CardN*.

We present five arguments in favor of the equality $CardN = CardR$.

1st Argument: Injection from the real numbers into the rationals

The dictionary listing of the Real numbers

We list the real numbers in $[0,1]$, in the lexicographic or dictionary order, using their binary representation.

The first row has the sequence that starts with 0, and the sequence that starts with 1, each followed by infinitely many zeros.

$$(0,0,0,...,0,0,0...) \equiv x_{1,1}, \qquad (1,0,0,...,0,0,0...) \equiv x_{1,2}$$

The second has the 2^2 sequences

$(0,0,0,...,0,0,0...) \equiv x_{2,1}$, $(0,1,0,...,0,0,0...) \equiv x_{2,2}$, $(1,0,0,...,0,0,0...) \equiv x_{2,3}$

$(1,1,0,...,0,0,0...) \equiv x_{2,4}$

The third row has the 2^3 sequences

$(0,0,0,0,...) \equiv x_{3,1}$, $(0,0,1,0,...) \equiv x_{3,2}$, $(0,1,0,0,...) \equiv x_{3,3}$, $(0,1,1,0,...) \equiv x_{3,4}$,

$(1,0,0,0,...) \equiv x_{3,5}$, $(1,0,1,0,...) \equiv x_{3,6}$, $(1,1,0,0,...) \equiv x_{3,7}$, $(1,1,1,0,...) \equiv x_{3,8}$.

The nth row lists the 2^n sequences that start with $(0,0,0,0,...) \equiv x_{n,1}$,

and end with $(1,1,1,1,...1,0,...) \equiv x_{n,2^n}$.

The sequences in each row are rational numbers, but the infinitely many rows contain all the real numbers.

Denote

$$X_n \equiv \{x_{n,1}, x_{n,2}, x_{n,3}, ...x_{n,2^n}\}$$

Then,

$$X_1 \subset X_2 \subset ... \subset X_n \subset X_{n+1} \subset ...$$

We order the sequences=numbers as follows

$$\begin{array}{c}
0 \;\to\; 1 \\
00 \;\to\; 01 \;\to\; 10 \;\to\; 11 \\
000 \;\to\; 001 \;\to\; 010 \;\to\; 011 \;\to\; 100 \;\to\; 101 \;\to\; 110 \;\to\; 111 \\
...\;\;...\;\;...\;\;...\;\;...\;\;...\;\;...\;\;...\;\;...\;\;...\;\;...\;\;...\;\;...\;\;...\;\;... \\
\to\;\;...\;\;...\;\;...\;\;...\;\;\to\;\;...\;\;...\;\;...\;\;...\;\;...\;\;...\;\;...\;\;...\;\;\to\;\;...\;\;\to \\
...\;\;...\;\;...\;\;...\;\;...\;\;...\;\;...\;\;...\;\;...\;\;...\;\;...\;\;...\;\;...\;\;...\;\;...\;\;...\;\;...
\end{array}$$

We are guaranteed that this listing will enumerate all the real numbers between 0, and 1.

$$[0,1] \subseteq \bigcup_{n=1}^{\infty} X_n$$

We locate the rational truncation of $\sqrt{2}/2$, up to specified number of digits, in one of the rows. But the irrational number $\sqrt{2}/2$ is only in the last row.

The last row starts with the infinitely many sequences

$$(0,0,0,0,0,0,...,0,0,0,...)$$

$$(0,1,0,0,0,0,...,0,0,0,...)$$

$$(0,0,1,0,0,0,...,0,0,0,...)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$(0,0,0,0,0,0,...,0,0,0,...1)$$

Then , the infinitely many sequences that have two 1’s and infinitely many zeros, Then, the infinitely many sequences that have three 1’s, etc.

Thus, the last row has infinitely many infinite sequences, and it is not clear that, according to [1, p. 38], such sequencing makes the reals in (0,1) *effectively countable***,** as it makes the rational numbers sequenced by the Cantor Zig-Zag.

Therefore, we proceed to exhibit an injection from the real numbers into the rationals

Injection from the real numbers into the rationals

We want to map each real number one-one to a distinct rational number. All the rationals in the range of the map, have to be different from each other. Such rationals are the rational endpoints produced in the generation of the Cantor set [3].

The Cantor set is obtained from the closed unit interval [0,1] by a sequence of deletions of middle third open intervals.

First, we delete the open interval (1/3,2/3). The numbers left in the intervals [0,1/3], and [2/3,1] have either 0, or 2 of the fraction 1/3 in their expansion in base 3. The two rational endpoints

$$\frac{1}{3} = y_{1,1} \ , \ \text{ and } \ \frac{2}{3} = y_{1,2}$$

remain in the Cantor set after indefinitely many deletions. We denote

$$Y_1 \equiv \{y_{1,1}, y_{1,2}\}.$$

Second, we remove the open intervals (1/9,2/9) and (7/9,8/9). The numbers left in the intervals [0,1/9], [2/9,1/3], [2/3,7/9], and [8/9,1], have either 0, or 2 of the fraction 1/9 in their expansion in base 3. The four rational endpoints

$$\frac{1}{9} = y_{2,1}, \quad \frac{2}{9} = y_{2,2}, \quad \frac{7}{9} = y_{2,3}, \quad \frac{8}{9} = y_{2,4},$$

remain in the Cantor set after indefinitely many deletions.

$$Y_2 \equiv \{y_{2,1}, y_{2,2}, y_{2,3}, y_{2,4}\}$$

Third, we remove the open intervals (1/27,2/27), (7/27,8/27), (19/27,20/27), (25/27,26/27). The numbers left in the 8 closed intervals have either 0, or 2 of the fraction 1/27 in their expansion in base 3. The eight rational endpoints

$$\frac{1}{27} = y_{3,1}, \ \frac{2}{27} = y_{3,2}, \ \frac{7}{27} = y_{3,3}, \ \frac{8}{27} = y_{3,4}, \ \frac{19}{27} = y_{3,5}, \ \frac{20}{27} = y_{3,6}, \ \frac{25}{27} = y_{3,7}, \ \frac{26}{27} = y_{3,8},$$

remain in the Cantor set after indefinitely many deletions.

$$Y_3 \equiv \{y_{3,1}, y_{3,2}, y_{3,3}, y_{3,4}, y_{3,5}, y_{3,6}, y_{3,7}, y_{3,8}\}$$

In the n-th step, we remove 2^{n-1} open intervals leaving 2^n closed intervals, which numbers have either 0, or 2 of the fraction $1/3^n$ in their expansion in base 3. The 2^n rational endpoints of the removed open intervals, remain in the Cantor set after indefinitely many deletions.

$\bigcup_{n=1}^{\infty} Y_n$ is a subset of the rationals in [0,1] because the midpoints of the removed intervals, ½, 1/6, 5/6,.. are not in any Y_n.

We define a one-one function between the real numbers listed in dictionary order, and the rational endpoints of the Cantor set.

The two reals in X_1 are assigned by their order to the two rational endpoints produced in the first deletion 1/3, and 2/3, by the injection

$$f_1 : X_1 \to Y_1$$

$$(0, 0, 0,..., 0, 0,0, ...) \to 1/3 ,$$

$$(1, 0, 0,0....0, 0, 0....) \to 2/3 .$$

The four reals in X_2 are assigned by their order to each of the four endpoints of the two deleted open intervals,

1/9, 2/9, 7/9, 8/9.

by the injection

$$f_2 : X_2 \to Y_2 .$$

The eight reals in the X_3 are assigned by their order to each of the eight endpoints of the four deleted open intervals,

1/27, 2/27, 7/27, 8/27, 19,27, 20/27, 25/27, 26/27.

by the injection

$$f_3 : X_3 \to Y_3$$

The 2^n reals in X_n are assigned by their order to each of the 2^n endpoints of the 2^{n-1} deleted open intervals, by the injection

$$f_n : X_n \to Y_n .$$

Now define

$$f : \bigcup_{n=1}^{\infty} X_n \to \bigcup_{n=1}^{\infty} Y_n \quad \text{so that} \quad f\big|_{X_n} = f_n$$

The mapping f is one-one from the real numbers in $\bigcup_{n=1}^{\infty} X_n$ onto the rational numbers in $\bigcup_{n=1}^{\infty} Y_n$. Consequently,

$$CardR = Card[0,1] \le Card \bigcup_{n=1}^{\infty} X_n = Card \bigcup_{n=1}^{\infty} Y_n \le CardQ \le CardR .$$

Hence $Card[0,1] = CardQ$. This completes our first argument of reals countability.

Second Argument: Tarski result for Cardinals.

The 1st row in the dictionary listing has 2^1 reals. The 2nd row has 2^2 reals. The 3rd row has 2^3 reals. The n-th row has 2^n reals.

Summing the number of the reals along the listing, for $n = 1,2,3....$,

$$2+2^2+2^3+\ldots+2^n<\aleph_0. \quad (1)$$

Thus,

$$2^{n+1}-2<\aleph_0.$$

That is,

$$2^n<\aleph_0. \quad (2)$$

Tarski ([4], or [1, p.174]) proved that for any sequence of cardinal numbers, $m_1, m_2, m_3,\ldots$, and a cardinal m, the partial sums inequalities

$$m_1+m_2+\ldots+m_n\le m,$$

for $n=1,2,3\ldots.$ imply the series inequality

$$m_1+m_2+\ldots+m_n+\ldots.\le m.$$

Applying Tarski result to (1), we obtain

$$2+2^2+2^3+\ldots.+2^n+\ldots<\aleph_0.$$

Regarding (2), this says

$$2^{\aleph_0}\le\aleph_0.$$

Since $2^{\aleph_0}=CardR\ge CardQ=\aleph_0$,

$$2^{\aleph_0}=\aleph_0.$$

Third Argument: Cardinals and Ordinals

By [1, p.277], every ordinal number α has a next ordinal number

$$\alpha + 1 > \alpha ,$$

and no intermediate ordinal number ξ with $\alpha + 1 > \xi > \alpha$.

The ordinal number $\omega + 7$ is preceded by $\omega + 6$, and is classified as 1st kind. The smallest ordinal number that is not preceded by any ordinal, denoted by ω, is of 2nd kind. By [1, p.288] any 2nd kind ordinal number is the limit of an increasing transfinite sequence of ordinal numbers. In particular,

$$\omega = \lim_{n<\omega} 2^n .$$

By [1, p. 318, Theorem 1], the function $f(n) = 2^n$ is continuous in n, and

$$\omega = 2^{\lim_{n<\omega} n} = 2^{\omega} .$$

Now, by [2, p. 88, Corollary 2.19]

ω is a cardinal number.

By [2, p.90, Corollary 2.33],

$$\aleph_0 = \omega .$$

Thus, $\omega = 2^{\omega}$ says

$$\aleph_0 = 2^{\aleph_0} .$$

Fourth Argument: Cardinality of Ordinals

By a Theorem of Schonflies (1913) [2, p. 126, Theorem 2.11], for ordinal numbers α, and β

$$Card(\alpha^{\beta}) = \max(Card(\alpha), Card(\beta)).$$

Therefore,

$$Card(2^{\omega}) = \max(Card(2), Card(\omega)) = \aleph_0 .$$

On the other hand, by [2, p.126, (2.9)], exponentiation is a repeated multiplication , and for all α , and β

$$\alpha^{\beta} = \prod_{\gamma<\beta} \alpha .$$

Hence,

$$2^{\omega} = \prod_{n<\omega} 2 .$$

Therefore,

$$Card(2^{\omega}) = Card \prod_{n<\omega} 2 .$$

Now, by [2, p. 106, proposition 4.15], if a is a well ordered cardinal,

$$\prod_{x\in u} a = a^{Card(u)} .$$

Therefore,

$$\prod_{n<\omega} 2 = 2^{Card(\omega)} = 2^{\aleph_0} .$$

In conclusion,

$$\aleph_0 = 2^{\aleph_0}.$$

Fifth Argument: Effective Countability of Ordinals

By the axiom of Choice, the set of ordinals up to the ordinal 2^{ω} is well ordered, by magnitude, and is sequenced. Therefore, it is well known to be effectively countable [5], and its power is $\aleph_0$.

Cardinalities of Transcendental, and Irrational numbers

The rationals can be sequenced by Cantor's Zig-Zag, and are effectively countable. That is, $\aleph_0^2 = \aleph_0$. For the algebraic numbers

$$\aleph_0 = CardN \le Card(Algebraic\ Numbers) \le CardR = \aleph_0.$$

Hence,

$$Card(Algebraic\ Numbers) = \aleph_0.$$

By [6], *if a is non-zero, real algebraic number, then e^a is a transcendental number.* The function

$$a \to e^a$$

is an injection from the algebraic numbers into the transcendental numbers.

Therefore,

$$\aleph_0 = CardR \ge Card(Transcendental\ Numbers) \ge Card(Algebraic\ Numbers) = \aleph_0.$$

Thus,

$$Card(Transcendental\ \ Numbers) = \aleph_0 .$$

Finally,

$$\aleph_0 = CardR \geq Card(Irrational\ \ Numbers) \geq Card(Transcendental\ \ Numbers) = \aleph_0 .$$

Hence,

$$Card(Irrational\ \ Numbers) = \aleph_0 .$$

The Continuum Hypothesis.

The Continuum Hypothesis says that there is no set X with $\aleph_0 < CardX < 2^{\aleph_0}$.

In 1963, Cohen proved that if the commonly accepted postulates of set theory are consistent, then the addition of the negation of the hypothesis does not result in inconsistency [7].

Cohen's result was interpreted to mean that there is a set theory where the negation of the Continuum Hypothesis holds. However, Non-Cantorian cardinal numbers were not found, and the Non-Cantorian set theory was never developed.

Recently, we proved the Continuum Hypothesis in Cantor's set theory [8]. But according to Cohen, the Continuum Hypothesis is an assumption. How can an assumption be proved? Close scrutiny of our proof reveals that Rationals countability is equivalent to the Continuum Hypothesis [9].

Since rationals countability is a fact, so is the Continuum hypothesis.

While the continuum hypothesis assumes that $CardN < CardR$, it is self-evidently true if $CardN = CardR$,

If our arguments that $CardN = CardR$ are valid, the Continuum Hypothesis holds with no alternative, and there is no alternative set theory based on the negation of the Continuum Hypothesis. Perhaps, no such theory was ever constructed because it does not exist.

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