

Hilbert's 1st problem: Cantor's Continuum Hypothesis

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Abstract: There is no set X with $\aleph_0 < \text{card}X < 2^{\aleph_0}$.

Introduction The continuum hypothesis reflects Cantor's inability to construct a set with cardinality between that of the natural numbers and that of the real numbers. His approach was constructive, But if he was right, such set cannot be constructed, and he needed a proof by contradiction.

That contradiction remained out of reach at Cantor's time. Even when Hilbert presented Cantor's Continuum Hypothesis as his first problem, the tools for the solution did not exist. Tarski obtained the essential lemma only in 1948. But it was not utilized and the problem remained open.

In 1963, Cohen proved that if the commonly accepted postulates of set theory are consistent, then adding the negation of the hypothesis does not result in inconsistency. [1, p.97]. That left the impression that Hilbert's first problem was either solved, or is unsolvable. But it became commonly accepted that the problem was closed.

Not that the question was settled. According to [2,p.189], "Mathematicians do not tend to assume the Continuum Hypothesis as an additional axiom of set theory mostly since they cannot convince themselves that this statement is "true" as many of them have done for the axioms of ZFC including the axiom of choice. However, a mathematician trying to prove a theorem will usually regard a proof of the theorem from the generalized continuum hypothesis as a partial success".

Cohen result was interpreted to mean that there is another set theory that utilizes the negation of the continuum hypothesis. However, the alternative set theory was never developed, and we shall show here that the Continuum Hypothesis can be proved under the assumptions of Cantor's set theory.

Proof: To understand the gap between cardinal numbers, it is natural to examine the convergence $\aleph_0^n \rightarrow \aleph_0^{\aleph_0}$, and try to pinpoint where the jump from $\aleph_0 (= \aleph_0^1)$ to $2^{\aleph_0} (= \aleph_0^{\aleph_0})$ occurs. This idea¹ motivates our proof.

By [3, p.173], the sequence

$$\alpha_n \equiv \aleph_0 + \aleph_0^2 + \dots + \aleph_0^n,$$

converges to the series

$$\aleph_0 + \aleph_0^2 + \dots + \aleph_0^n + \dots = \sum_{n=1}^{\infty} \aleph_0^n.$$

The series has a well-defined sum α , which is a cardinal number that does not depend on the order of the summation.

By [3, p.174],

$$\alpha \text{ is } \geq \text{any component of the series.}$$

In particular,

$$\alpha \geq \aleph_0 \aleph_0 \dots = \prod_{n=1}^{\infty} \aleph_0 = \prod_{n \in \mathbb{N}} \aleph_0.$$

By [2, p.106], (or [3, p.183])

$$\prod_{n \in \mathbb{N}} \aleph_0 = \aleph_0^{\text{card} \mathbb{N}}.$$

Therefore,

$$\alpha \geq \aleph_0^{\text{card} \mathbb{N}} = 2^{\aleph_0}.$$

That is,

¹We actually use sequences of partial sums of series of cardinal numbers

$$\alpha \geq 2^{\aleph_0}.$$

Now, suppose that there is a set X with

$$\aleph_0 < \text{card}X < 2^{\aleph_0}.$$

Then, for any finite n ,

$$\alpha_n = \aleph_0 + \aleph_0^2 + \aleph_0^3 + \dots + \aleph_0^n \leq \text{card}X.$$

Tarski ([4], or [3, p.174]) proved that for any cardinal numbers, the inequalities

$$m_1 + m_2 + \dots + m_n \leq m, \text{ for } n = 1, 2, \dots, n \dots$$

imply the inequality

$$m_1 + m_2 + \dots + m_n + \dots \leq m.$$

Since for any n

$$\alpha_n \leq \text{card}X,$$

by Tarski result

$$\alpha \leq \text{card}X.$$

Combining this with

$$\text{card}X < 2^{\aleph_0},$$

yields by transitivity of cardinal inequalities [3, p. 147],

$$\alpha < 2^{\aleph_0},$$

which contradicts $\alpha \geq 2^{\aleph_0}$.

Discussion: Why did Cantor fail to prove his hypothesis? Tarski result was not available to Cantor. Furthermore, Cantor was familiar with the partial sums

$$\aleph_0 + \aleph_0^2 + \dots + \aleph_0^n,$$

that are the cardinality of all the roots of all the polynomials in integer coefficients of degree $\leq n$. In that context, n must be always finite, and the partial sum never exceeds \aleph_0 . Only when we take infinite n , we obtain terms such as $\aleph_0^{\aleph_0} = 2^{\aleph_0} > \aleph_0$. The infinite summation over all n , is the key to our proof. In spite of the degenerate character of \aleph_0^n , there is a jump to 2^{\aleph_0} that forbids a cardinal number between $CardN$, and $CardR$.

The gap between cardinal numbers may be traced to the jump between finite cardinal numbers n , and the first infinite cardinal \aleph_0 . That incomprehensible jump may be the reason for the jump from the cardinal number

$$\aleph_0 + \aleph_0^2 + \aleph_0^3 + \dots \aleph_0^n = \aleph_0^n = \aleph_0$$

to the cardinal

$$\sum_{n \in N} \aleph_0^n = \aleph_0^{\aleph_0} = 2^{\aleph_0}.$$

Perhaps, there is no cardinality between the integers and the real numbers, because there is no infinite cardinal number between n , and \aleph_0 . But an infinite cardinality between n , and \aleph_0 , will contradict the definition of \aleph_0 , as the first infinite cardinal.

Was Cohen right? In earlier stages of this work, we wondered whether our proof for the Continuum Hypothesis proves Cohen wrong. But our further studies affirm Cohen's result. Under the assumptions of Cantor's set theory, we can prove the Continuum Hypothesis. But with one assumption changed, we can construct a Non-Cantorian set theory, where there is a set X with $CardN < CardX < CardR$. Cohen's result predicts the vulnerability of Cantor's set theory, and makes it easier for us to present the NonCantorian set theory [5].

References

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- [4] Tarski, A. “*Axiomatic and algebraic aspects on two theorems on sums of cardinals.*” *Fund. Math.* 35 (1948), p. 79-104.
- [5] Dannon, H. Vic. “*The Continuum Hypothesis is equivalent to $\aleph_0^2 = \aleph_0$.*” www.gauge-institute.org