Power Means Calculus Product Calculus, Harmonic Mean Calculus, and Quadratic Mean Calculus

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Abstract

Each Power Mean of order $r \neq 0$, $\left(\frac{a_1^r + a_2^r + ...a_n^r}{n}\right)^{\frac{1}{r}}$

is associated with a *Power Mean Derivative of order* r, $D^{(r)}$.

We describe the

Arithmetic Mean Calculus obtained if $r = 1$,

Geometric Mean Calculus obtained if $r \to 0$,

Harmonic Mean Calculus obtained if $r = -1$,

Quadratic Mean Calculus obtained if $r = 2$

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Contents

Introduction

We describe a generalized calculus that was suggested by Michael Spivey's [Spiv] observation of the relation between the Geometric Mean of a function over an interval, and its product integral.

We will see that each Power Mean of order $r \neq 0$,

$$\left(\frac{a_1^r + a_2^r + \dots a_n^r}{n} \right)^{\frac{1}{r}}$$

is associated with a *Power Mean Derivative of order* r*,*

$$D^{(r)}.$$

The Fermat/Newton/Leibnitz Derivative

$$D^{(1)} = D \equiv \frac{d}{dx}$$

is associated with the Arithmetic Mean

$$\frac{a_1 + a_2 + \dots a_n}{n},$$

which is Power Mean of order $r = 1$.

The Geometric Mean Derivative

$$D^{(0)}$$

is associated with the Geometric Mean

$$\left(a_1 a_2 \dots a_n \right)^{1/n}$$

which is Power Mean of order $r \to 0$ [Kaza].

Product Integration is an operation inverse to the Geometric Mean Derivative. Both are multiplicative operations, that apply naturally to products, and in particular to $\Gamma(z)$, the analytic extension of the factorial function

The Harmonic Mean Derivative

$$D^{(-1)}$$

is associated with the Harmonic Mean

$$\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+...\frac{1}{a_n}}$$

which is Power Mean of order $r = -1$.

The Quadratic Mean Derivative

$$D^{(2)}$$

is associated with the Power Mean of order $r = 2$,

$$\left(\frac{a_1^2+a_2^2+...a_n^2}{n}\right)^{\frac{1}{2}}.$$

The inverse operation, the Quadratic Mean Integration transforms a function to its L^2 norm squared.

We proceed with the definition of the Arithmetic Mean Derivative.

1

Arithmetic Mean Calculus

1.1 The Arithmetic Mean of $f(x)$ over $[a,b]$

Given a function $f(x)$ that is Riemann integrable over the interval $[a,b]$, partition the interval into n sub-intervals, of equal length

$$\Delta x = \frac{b-a}{n},$$

choose in each subinterval a point

$$c_i,$$

and consider the Arithmetic Mean of $f(x)$,

$$\frac{f(c_1)+f(c_2)+...f(c_n)}{n} = \frac{1}{b-a}\big(f(c_1)+f(c_2)+...f(c_n)\big)\Delta x$$

As $n \to \infty$, the sequence of the Arithmetic Means converges to

$$\tfrac{1}{b-a}\int_{x=a}^{x=b} f(x)dx = \tfrac{F(b)-F(a)}{b-a},$$

where

$$F(x) = \int_{t=0}^{t=x} f(t)dt.$$

Therefore, the *Arithmetic Mean of* $f(x)$ *over* $[a,b]$ is defined by

$$\boxed{\tfrac{1}{b-a}\int_{x=a}^{x=b} f(x)dx}$$

1.2 Mean Value Theorem for the Arithmetic Mean

There is a point $a < c < b$*, so that* $\frac{1}{b-a}\int_{x=a}^{x=b} f(x)dx = f(c)$.

Proof: Since $F(x) = \int_{t=0}^{t=x} f(t)dt$ is continuous on $[a,b]$, and differentiable in (a,b), by Lagrange Intermediate Value Theorem there is a point

$$a < c < b\,,$$

so that

$$\frac{F(b)-F(a)}{b-a} = f(c).$$

That is,

$$\frac{1}{b-a}\int_{x=a}^{x=b} f(x)dx = f(c).\square$$

1.3 The Arithmetic Mean of $f(x)$ over $[x, x+dx]$

The Arithmetic Mean of $f(x)$ *over* $[x, x+dx]$ *is the Inverse operation to Integration*

Proof: By 1.2, there is

$$x < c < x + \Delta x\,,$$

so that

$$\frac{1}{\Delta x}\int_{t=x}^{t=x+\Delta x} f(t)dt = f(c)$$

Letting $\Delta x \to 0$, the Arithmetic Mean of $f(x)$ at x, equals $f(x)$.

$$\lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{t=x}^{t=x+\Delta x} f(t)dt = f(x).$$

Thus, the operation of finding the Arithmetic Mean of $f(x)$ x, is inverse to integration.$\square$

This leads to the definition of the Arithmetic Mean Derivative.

1.4 Arithmetic Mean Derivative of $F(x) = \int_{t=0}^{t=x} f(t)dt$ at x

The Arithmetic Mean Derivative of

$$F(x) = \int_{t=0}^{t=x} f(t)dt$$

at x *is defined as the Arithmetic Mean of* $f(x)$ *over* $[x, x+dx]$

$$\boxed{D^{(1)}F(x) \equiv \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{t=x}^{t=x+\Delta x} f(t)dt}$$

1.5 The Arithmetic Mean Derivative is the Fermat-Newton-Leibnitz derivative

$$\boxed{D^{(1)}F(x) = \frac{dF(x)}{dx}}$$

Proof: $$D^{(1)}F(x) = \text{Standard Part of } \frac{1}{dx} \int_{t=x}^{t=x+dx} f(t)dt$$

$$= \text{Standard Part of } \frac{F(x+dx) - F(x)}{dx}$$

$$= \frac{dF(x)}{dx} = DF(x).\square$$

1.6 The Arithmetic Mean Derivative is an Additive Operator

$$D\Big(F_1(x) + F_2(x)\Big) = DF_1(x) + DF_2(x)$$

Thus, the Arithmetic Mean Derivative applies effectively to infinite series.

1.7 The Arithmetic Mean Derivative is not a multiplicative operator

$$D\Big(F_1(x)F_2(x)\Big) = \Big(DF_1(x)\Big)F_2(x) + F_1(x)\Big(DF_2(x)\Big)$$

Thus, Arithmetic Mean Derivative does not apply easily to infinite products.

2

The Product Integral

2.1 Growth Problems

The Arithmetic Mean Derivative is unsuitable when we are interested in the quotient

$$\frac{\text{Present Value}}{\text{Invested Value}}$$

Similarly, attenuation or amplification is measured by

$$\frac{\text{Out-Put Signal}}{\text{In-Put Signal}}.$$

The need for a multiplicative derivative operator motivated the creation of the product integration.

2.2 The Product Integral of $e^{r(t)}$ over the interval $[a,b]$

An amount A compounded continuously at rate $r(t)$ over time dt becomes

$$Ae^{r(t)dt}.$$

Over n equal sub-intervals of the time interval $[a,b]$,

$$\Delta t = \tfrac{b-a}{n},$$

we obtain the sequence of finite products

$$Ae^{r(t_1)\Delta t}e^{r(t_2)\Delta t}...e^{r(t_n)\Delta t} = Ae^{[r(t_1)+r(t_2)+...+r(t_n)]\Delta t}.$$

As $n \to \infty$, the sequence converges to

$$A e^{\int_{t=a}^{t=b} r(t)dt}$$

The amplification factor

$$e^{\int_{t=a}^{t=b} r(t)dt}$$

is called

the product integral of $e^{r(t)}$ *over the interval* $[a,b]$

and is denoted

$$\prod_{t=a}^{t=b} e^{r(t)dt}.$$

Thus, the Product Integral of $e^{r(t)}$ over the interval $[a,b]$ is

$$\boxed{\prod_{t=a}^{t=b} e^{r(t)dt} \equiv e^{\int_{t=a}^{t=b} r(t)dt}}$$

2.3 The Product Integral of $f(x)$ over the interval $[a,b]$

Given a Riemann integrable, positive $f(x)$ on $[a,b]$, partition the interval into n sub-intervals, of equal length

$$\Delta x = \frac{b-a}{n},$$

choose in each subinterval a point

$$c_i,$$

and consider the finite products,

$$f(c_1)^{\Delta x} f(c_2)^{\Delta x} ... f(c_n)^{\Delta x} = e^{\ln[f(c_1)^{\Delta x} f(c_2)^{\Delta x} ... f(c_n)^{\Delta x}]}$$

$$= e^{[\ln f(c_1) + \ln f(c_2) + ... + \ln f(c_n)]\Delta x}.$$

As $n \to \infty$, the sequence of products converges to

$$e^{\int_{x=a}^{x=b} (\ln f(x))dx} > 0.$$

We call this limit

the product integral of $f(x)$ *over the interval* $[a,b]$,

and denote it by

$$\prod_{x=a}^{x=b} f(x)^{dx}.$$

Thus, the Product Integral of $f(x)$ over the interval $[a,b]$ is

$$\boxed{\prod_{x=a}^{x=b} f(x)^{dx} \equiv e^{\int_{x=a}^{x=b} (\ln f(x))dx}}$$

2.4 Intermediate Value Theorem for the Product Integral

There is a point $a < c < b$, *so that* $\prod_{x=a}^{x=b} f(x)^{dx} = f(c)^{(b-a)}$

Proof: Since $\varphi(x) = \int_{t=0}^{t=x} (\ln f(t))dt$ is continuous on $[a,b]$, and differentiable in (a,b), by Lagrange Intermediate Value Theorem there is a point

$$a < c < b\,,$$

so that

$$\int_{x=a}^{x=b} \big(\ln f(x)\big)dx = \varphi(b) - \varphi(a) = \big(\ln f(c)\big)(b-a).$$

Hence,

$$e^{\int_{x=a}^{x=b} \left(\ln f(x)\right)dx} = e^{\left(\ln f(c)\right)(b-a)} = f(c)^{(b-a)}.\square$$

2.5 The Product Integral is a multiplicative operator

If $a < c < b$, $\prod_{x=a}^{x=b} f(x)^{dx} = \left(\prod_{x=a}^{x=c} f(x)^{dx}\right)\left(\prod_{x=c}^{x=b} f(x)^{dx}\right)$

Proof: If $a < c < b$,

$$\prod_{x=a}^{x=b} f(x)^{dx} = e^{\int_{x=a}^{x=c} \left(\ln f(x)\right)dx + \int_{x=c}^{x=b} \left(\ln f(x)\right)dx} = \left(\prod_{x=a}^{x=c} f(x)^{dx}\right)\left(\prod_{x=c}^{x=b} f(x)^{dx}\right).\square$$

The inverse operation to product integration is the Geometric Mean Derivative.

3

Geometric Mean and Geometric Mean Derivative

3.1 The Power Mean with $r \to 0$ is the Geometric Mean

$$\left(\frac{a_1^r + a_2^r + \ldots a_n^r}{n}\right)^{\frac{1}{r}} \underset{r \to 0}{\longrightarrow} \left(a_1 a_2 \ldots a_3\right)^{\frac{1}{n}}$$

Proof: Let $r \to 0$ in

$$\left(\frac{a_1^r + a_2^r + \ldots a_n^r}{n}\right)^{\frac{1}{r}} = e^{\frac{1}{r}\left[\log\left(a_1^r + a_2^r + \ldots a_n^r\right) - \log n\right]}.$$

Then, the exponent $\frac{1}{r}\left[\log(a_1^r + a_2^r + \ldots a_n^r) - \log n\right]$ is of the form $\frac{0}{0}$,

and by L'Hospital, its limit is

$$\lim_{r \to 0} \frac{D_r \left\{\log(a_1^r + a_2^r + \ldots a_n^r) - \log n\right\}}{D_r r}$$

$$= \lim_{r \to 0} \frac{1}{a_1^r + a_2^r + \ldots a_n^r}\left(a_1^r \ln a_1 + a_2^r \ln a_2 + \ldots a_n^r \ln a_n\right)$$

$$= \frac{1}{n}\left(\ln a_1 + \ln a_2 + \ldots + \ln a_n\right)$$

$$= \ln\left(a_1 a_2 \ldots a_n\right)^{\frac{1}{n}}.$$

Therefore,

$$\left(\frac{a_1^r + a_2^r + \ldots a_n^r}{n}\right)^{\frac{1}{r}} \xrightarrow[r \to 0]{} e^{\ln\left(a_1 a_2 \ldots a_n\right)^{\frac{1}{n}}} = \left(a_1 a_2 \ldots a_n\right)^{\frac{1}{n}}. \square$$

3.2 Geometric Mean of $f(x)$ over $[a,b]$

Given an integrable function $f(x)$ that is positive over $[a,b]$, partition the interval, into n sub-intervals, of equal length

$$\Delta x = \frac{b-a}{n},$$

choose in each subinterval a point

$$c_i,$$

and consider the Geometric Mean of $f(x)$

$$\left(f(c_1)f(c_2)...f(c_n)\right)^{1/n} = e^{\frac{1}{b-a}\left(\ln f(c_1)+\ln f(c_2)+...\ln f(c_n)\right)\Delta x}.$$

As $n \to \infty$, the sequence of Geometric Means converges to

$$e^{\frac{1}{b-a}\int_{x=a}^{x=b}\left(\ln f(x)\right)dx} = \frac{G(b)}{G(a)},$$

where

$$G(x) = e^{\int_{t=a}^{t=x}\left(\ln f(t)\right)dt}$$

Therefore,

$$\boxed{e^{\frac{1}{b-a}\int_{x=a}^{x=b}\left(\ln f(x)\right)dx}}$$

is defined as the *Geometric Mean of* $f(x)$ *over* $[a,b]$.□

3.3 Mean Value Theorem for the Geometric Mean

There is a point $a < c < b$, *so that* $e^{\frac{1}{b-a}\int_{x=a}^{x=b}\left(\ln f(x)\right)dx} = f(c)$

Proof: By 2.3, and 2.4.$\square$

3.4 The Geometric Mean of $f(x)$ over $[x, x+dx]$

The Geometric Mean of $f(x)$ *over* $[x, x+dx]$*, is the Inverse Operation to Product Integration*

Proof: By 3.3, there is

$$x < c < x + \Delta x\,,$$

so that

$$e^{\frac{1}{\Delta x}\int\limits_{t=x}^{t=x+\Delta x} \ln f(t)dt} = f(c)\,.$$

Letting $\Delta x \to 0$, the Geometric Mean of $f(x)$ at x equals $f(x)$.

$$\lim_{\Delta x \to 0} e^{\frac{1}{\Delta x}\int\limits_{t=x}^{t=x+\Delta x} \ln f(t)dt} = f(x)$$

Thus, the operation of finding the Geometric Mean of $f(x)$ over $[x, x+dx]$, is inverse to product integration over $[x, x+dx]$.$\square$

This leads to the definition of the Geometric Mean Derivative

3.5 Geometric Mean Derivative

The Geometric Mean Derivative of

$$G(x) = e^{\int\limits_{t=a}^{t=x} (\ln f(t))dt}$$

at x *is defined as the Geometric Mean of* $f(x)$ *over* $[x, x+dx]$

$$\boxed{D^{(0)}G(x) \equiv \lim_{\Delta x \to 0} e^{\frac{1}{\Delta x}\int\limits_{t=x}^{t=x+\Delta x}\left(\ln f(t)\right)dt}}$$

3.6 $$\boxed{D^{(0)}G(x) = e^{D\log G(x)}}$$

Proof:
$$D^{(0)}G(x) = \lim_{\Delta x \to 0} e^{\frac{1}{\Delta x}\int\limits_{t=x}^{t=x+\Delta x}\left(\ln f(t)\right)dt}$$
$$= e^{\lim\limits_{\Delta x \to 0}\frac{1}{\Delta x}\int\limits_{t=x}^{t=x+\Delta x}\left(\ln f(t)\right)dt}$$
$$= \text{Standard part of } e^{\frac{1}{dx}\int\limits_{t=x}^{t=x+dx}\left(\ln f(t)\right)dt}$$
$$= \text{Standard Part of } \left(\frac{G(x+dx)}{G(x)}\right)^{\frac{1}{dx}}$$
$$= \text{Standard Part of } e^{\frac{1}{dx}\left[\log G(x+dx) - \log G(x)\right]}$$
$$= e^{D\log G(x)}. \square$$

3.7 $$\boxed{D^{(0)}G(x) = e^{\frac{DG(x)}{G(x)}}}$$

3.8 ***The Geometric Mean Derivative is non- additive operator***

Proof: $D^{(0)}(G_1 + G_2)(x) = e^{\frac{D\left(G_1(x)+G_2(x)\right)}{G_1(x)+G_2(x)}}. \square$

3.9 *The Geometric Mean Derivative is a multiplicative operator*

Proof: $D^{(0)}\left[G_1(x)G_2(x)\right] = e^{D\log[G_1(x)G_2(x)]}$

$$= e^{\left(\frac{DG_1(x)}{G_1(x)} + \frac{DG_2(x)}{G_2(x)}\right)}$$

$$= e^{\frac{DG_1(x)}{G_1(x)}} e^{\frac{DG_2(x)}{G_2(x)}}$$

$$= \left(D^{(0)}G_1(x)\right)\left(D^{(0)}G_2(x)\right). \square$$

3.10 Geometric Mean Derivative Rules

$$\boxed{\left(D^{(0)}\right)^2 G(x) \equiv D^{(0)}e^{D\ln G(x)} = e^{D^2\ln G(x)}}$$

$$\boxed{\left(D^{(0)}\right)^n G(x) = e^{D^n \ln G(x)}}$$

$$\boxed{D^{(0)}\left(f(x)^{g(x)}\right) = f(x)^{Dg(x)} e^{g(x)D\ln f(x)}}$$

$$\boxed{D^{(0)}f(g(x)) = e^{\frac{\frac{df}{dg}\frac{dg}{dx}}{f(g(x))}}}$$

4

Geometric Mean Calculus

4.1 The Fundamental Theorem of the Product Calculus

$$\boxed{D^{(0)}\prod_{t=a}^{t=x} f(t)^{dt} = f(x)}$$

Proof: $D_x^{(0)}\prod_{t=a}^{t=x} f(t)^{dt} = D_x^{(0)} e^{\int_{t=a}^{t=x}\left(\ln f(t)\right)dt}$

$$= e^{\frac{1}{e^{\int_{t=a}^{t=x}\left(\ln f(t)\right)dt}} D_x e^{\int_{t=a}^{t=x}\left(\ln f(t)\right)dt}}$$

$$= e^{\frac{e^{\int_{t=a}^{t=x}\left(\ln f(t)\right)dt} D_x \int_{t=a}^{t=x}\left(\ln f(t)\right)dt}{e^{\int_{t=a}^{t=x}\left(\ln f(t)\right)dt}}}$$

$$= e^{D_x \int_{t=a}^{t=x}\left(\ln f(t)\right)dt}$$

$$= f(x).\square$$

4.2 Table of Geometric Mean Derivatives and integrals

We list some geometric Mean Derivatives, and Product Integrals. Some of these are given in [Spiv].

$f(x)$	$D^{(0)}f(x)$	$I^{(0)}f(x) = \prod_x f(x)$
1	1	1
2	1	2^x
a	1	a^x
e	1	e^x
x	$e^{1/x}$	$e^{x(\ln x - 1)} = e^{-x}x^x$
x^2	$e^{2/x}$	$e^{2x(\ln x - 1)} = e^{-2x}x^{2x}$
x^a	$e^{a/x}$	$e^{ax(\ln x - 1)} = e^{-ax}x^{ax}$
x^{-1}	$e^{-1/x}$	$e^{-x(\ln x - 1)} = e^x x^x$
$\ln x$	$e^{1/x \log x}$	$e^{\int \ln(\ln x)dx}$
e^{2x}	e^2	e^{x^2}
e^{ax}	e^a	$e^{ax^2/2}$
x^x	ex	$e^{x^2(\ln x - 1/2)/2} = e^{-\frac{x^2}{4}} x^{\frac{x^2}{2}}$
e^{x^n}	$e^{nx^{n-1}}$	$e^{x^{n+1}/(n+1)}$
$\sin x$	$e^{\cot x}$	$e^{\int \ln(\sin x)dx}$
$\cos x$	$e^{-\tan x}$	$e^{\int \ln(\cos x)dx}$
$\tan x$	$e^{2/\sin 2x}$	$e^{\int \ln(\tan x)dx}$
e^{e^x}	e^{e^x}	e^{e^x}
$e^{\sin x}$	$e^{\cos x}$	$e^{-\cos x}$
$e^{\cos x}$	$e^{-\sin x}$	$e^{\sin x}$

5

Product Differential Equations

5.1 Product Differential Equations

A Product Differential Equation involves powers of the Geometric Mean Derivative operator

$$D^{(0)},$$

and no sums, only products.

An ordinary differential equations involves sums of powers of the Arithmetic Mean Derivative operator D. Such equation is not suitable to the application of $D^{(0)}$, and does not convert easily into a product differential equation.

[Doll] attempts to write the solutions to ordinary differential equations in terms of product integral, but being unaware of the Geometric Mean Derivative, it fails to produce one product differential equation.

[Doll] demonstrates that products integrals are not natural solutions for ordinary differential equations.

The attempt made in [Doll] to interpret Summation Calculus in terms of the Product Integral alone, being oblivious to the product Calculus derivative, does not lead to better understanding of differential equations, or to any new results.

Only the basic equation

$$\frac{dy}{dx} = P(x)y$$

may be converted to a product differential equation, and be solved as such.

5.2 Product Calculus Solution of $\frac{dy}{dx} = P(x)y$.

Dividing both sides by $y(x)$,

$$\frac{y'}{y} = P(x).$$

$$e^{\frac{y'}{y}} = e^{P(x)}$$

$$D^{(0)}y = e^{P(x)}$$

$$y(x) = \prod_{t=0}^{t=x} e^{P(t)dt} = e^{\int_{t=0}^{t=x} P(t)dt}.\square$$

5.3 $\frac{dy}{dx} = P(x)y + Q(x)$ *may not be solved by Product Calculus*

Proof: We don't know of a Product Calculus method to solve the equation

$$\frac{dy}{dx} = P(x)y + Q(x).$$

In Arithmetic Mean Calculus, we multiply both sides by $e^{\int_{t=0}^{t=x} P(t)dt}$.

Then,

$$y' e^{\int_{t=0}^{t=x} P(t)dt} + yP(x)e^{\int_{t=0}^{t=x} P(t)dt} = Q(x)e^{\int_{t=0}^{t=x} P(t)dt}$$

$$\frac{d}{dx}(ye^{\int_{t=0}^{t=x} P(t)dt}) = Q(x)e^{\int_{t=0}^{t=x} P(t)dt}$$

$$ye^{\int_{t=0}^{t=x} P(t)dt} = \int_{u=0}^{u=x} Q(u)e^{\int_{t=0}^{t=u} P(t)dt}$$

$$y = \frac{\int_{u=0}^{u=x} Q(u)e^{\int_{t=0}^{t=u} P(t)dt}}{e^{\int_{t=0}^{t=x} P(t)dt}}$$

Writing this as

$$y = \frac{\int_{u=0}^{u=x} Q(u)\prod_{t=0}^{t=x} e^{P(t)dt}}{\prod_{t=0}^{t=x} e^{P(t)dt}}$$

demonstrates why the equation cannot be converted into a product differential equation, and cannot be solved as such: In product Calculus we need to have pure products. No summations.□

5.4 $y'' = P(x)y' + Q(x)y$ ***may not be solved by Product Calculus***

Proof: We may write

$$y'' = P(x)y' + Q(x)y$$

as a first order system

$$y' = z$$

$$z' = P(x)z + Q(x)y$$

Thus, in matrix form,

$$\frac{d}{dx}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ Q(x) & P(x) \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix}.$$

But we need the methods of summation Calculus, to obtain two independent solutions $y_1(x)$, and $y_2(x)$ that span the solution space for the equation.□

6

Product Calculus of $\dfrac{\sin x}{x}$

6.1 Euler's Product Representation for $\dfrac{\sin z}{z}$

For any complex number z, $\boxed{\dfrac{\sin z}{z} = \cos\frac{z}{2}\cos\frac{z}{4}\cos\frac{z}{8}\ldots}$

Proof: $\sin z = 2\cos\frac{z}{2}\sin\frac{z}{2}$

$$= 2\cos\tfrac{z}{2}2\cos\tfrac{z}{4}\sin\tfrac{z}{4}$$

$$= 2\cos\tfrac{z}{2}2\cos\tfrac{z}{4}2\cos\tfrac{z}{8}\sin\tfrac{z}{8}$$

$$= 2\cos\tfrac{z}{2}2\cos\tfrac{z}{4}2\cos\tfrac{z}{8}\ldots 2\cos\tfrac{z}{2^n}\sin\tfrac{z}{2^n}$$

$$= \left(z\tfrac{2^n}{z}\sin\tfrac{z}{2^n}\right)\cos\tfrac{z}{2}\cos\tfrac{z}{4}\cos\tfrac{z}{8}\ldots\cos\tfrac{z}{2^n}.$$

Therefore, for any complex number $z \neq 0$,

$$\left(\frac{\sin z}{z}\right)\frac{\frac{z}{2^n}}{\sin\frac{z}{2^n}} = \cos\tfrac{z}{2}\cos\tfrac{z}{4}\cos\tfrac{z}{8}\ldots\cos\tfrac{z}{2^n}$$

Letting $n \to \infty$,

$$\frac{\sin z}{z} = \cos\tfrac{z}{2}\cos\tfrac{z}{4}\cos\tfrac{z}{8}\ldots$$

This holds also for $z \to 0$. Hence, it holds for any complex number z.$\square$

6.2 Conversion to Trigonometric Series

Products of Cosines can be converted into summations, and the infinite product may be converted into a Trigonometric Series.

For instance,

$$\cos\alpha\cos\beta\cos\gamma = \left(\tfrac{1}{2}\cos(\alpha+\beta) + \tfrac{1}{2}\cos(\alpha-\beta)\right)\cos\gamma$$

$$= \tfrac{1}{2}\cos(\alpha+\beta)\cos\gamma + \tfrac{1}{2}\cos(\alpha-\beta)\cos\gamma$$

$$= \tfrac{1}{4}\cos(\alpha+\beta+\gamma) + \tfrac{1}{4}\cos(\alpha+\beta-\gamma)$$

$$+\tfrac{1}{4}\cos(\alpha-\beta+\gamma) + \tfrac{1}{4}\cos(\alpha-\beta-\gamma)$$

6.3 Geometric Mean Derivative of $\dfrac{\sin x}{x}$

$$\boxed{\frac{\cos x}{\sin x} - \frac{1}{x} = -\tfrac{1}{2}\tan\tfrac{x}{2} - \tfrac{1}{2^2}\tan\tfrac{x}{2^2} - \tfrac{1}{2^3}\tan\tfrac{x}{2^3} - \ldots}$$

Proof: Geometric Mean Differentiating both sides of 6.1,

$$D^{(0)}\frac{\sin x}{x} = \left(D^{(0)}\cos\tfrac{x}{2}\right)\left(D^{(0)}\cos\tfrac{x}{4}\right)\left(D^{(0)}\cos\tfrac{x}{8}\right)\ldots$$

$$e^{\left(\frac{D\frac{\sin x}{x}}{\frac{\sin x}{x}}\right)} = e^{\left(\frac{D\cos\frac{x}{2}}{\cos\frac{x}{2}}\right)} e^{\left(\frac{D\cos\frac{x}{2^2}}{\cos\frac{x}{2^2}}\right)} e^{\left(\frac{D\cos\frac{x}{2^3}}{\cos\frac{x}{2^3}}\right)}\ldots$$

$$e^{\frac{\cos x}{\sin x}-\frac{1}{x}} = e^{-\frac{1}{2}\tan\frac{x}{2}} e^{-\frac{1}{2^2}\tan\frac{x}{2^2}} e^{-\frac{1}{2^3}\tan\frac{x}{2^3}}$$

That is,

$$\frac{\cos x}{\sin x} - \frac{1}{x} = -\tfrac{1}{2}\tan\tfrac{x}{2} - \tfrac{1}{2^2}\tan\tfrac{x}{2^2} - \tfrac{1}{2^3}\tan\tfrac{x}{2^3} - \ldots \square$$

6.4 Second Geometric Mean Derivative of $\dfrac{\sin x}{x}$

$$\boxed{-\frac{1}{\sin^2 x}+\frac{1}{x^2}=-\frac{1}{2^2\cos^2\frac{x}{2}}-\frac{1}{2^4\cos^2\frac{x}{2^2}}-\frac{1}{2^6\cos^2\frac{x}{2^3}}....}$$

Proof: Second Geometric Mean Differentiation of 6.1 gives

$$D^{(0)}e^{\left(\frac{\cos x}{\sin x}-\frac{1}{x}\right)}=D^{(0)}e^{\left(-\frac{1}{2}\tan\frac{x}{2}\right)}\times D^{(0)}e^{\left(-\frac{1}{2^2}\tan\frac{x}{2^2}\right)}\times ....$$

$$e^{\left(-\frac{1}{\sin^2 x}+\frac{1}{x^2}\right)}=e^{\left(-\frac{1}{2^2\cos^2\frac{x}{2}}\right)}e^{\left(-\frac{1}{2^4\cos^2\frac{x}{2^2}}\right)}e^{\left(-\frac{1}{2^6\cos^2\frac{x}{2^3}}\right)}....$$

That is,

$$-\frac{1}{\sin^2 x}+\frac{1}{x^2}=-\frac{1}{2^2\cos^2\frac{x}{2}}-\frac{1}{2^4\cos^2\frac{x}{2^2}}-\frac{1}{2^6\cos^2\frac{x}{2^3}}....\square$$

The last series can be obtained by term by term series-differentiation of 6.3.

6.5 Product Integration of $\dfrac{\sin x}{x}$

$$\boxed{\int\log\frac{\sin x}{x}dx=-\frac{B_2}{4\cdot 3!}(2x)^3+\frac{B_4}{8\cdot 5!}(2x)^5-\frac{B_6}{12\cdot 7!}(2x)^7+\frac{B_8}{16\cdot 9!}(2x)^9-....}$$

Where the $B_2, B_4, B_{6,}...$ *are the Bernoulli Numbers.*

Proof: Product integrating 6.1,

$$e^{\int \log \frac{\sin x}{x} dx} = e^{\int \log \cos \frac{x}{2} dx} e^{\int \log \cos \frac{x}{4} dx} e^{\int \log \cos \frac{x}{8} dx} \ldots$$

By [Grob, p.113, 8b], for $|x| < \pi$, up to a constant,

$$\int \log \frac{\sin x}{x} dx = \int \log \sin x dx - \int \log x dx$$

$$= -\frac{B_2}{4 \cdot 3!}(2x)^3 + \frac{B_4}{8 \cdot 5!}(2x)^5 - \frac{B_6}{12 \cdot 7!}(2x)^7 + \frac{B_8}{16 \cdot 9!}(2x)^9 - \ldots,$$

where the $B_2, B_4, B_{6,} \ldots$ are the Bernoulli Numbers.

By [Grob, p.113, 9b], for $|x| < \pi$, up to a constant,

$$\int \log \cos \tfrac{x}{2} dx = 2\int \log \cos \tfrac{x}{2} d \tfrac{x}{2}$$

$$= 2\left(-\frac{(2^2 - 1)B_2}{4 \cdot 3!}(x)^3 + \frac{(2^4 - 1)B_4}{8 \cdot 5!}(x)^5 - \frac{(2^6 - 1)B_6}{12 \cdot 7!}(x)^7 + \ldots \right)$$

$$\int \log \cos \tfrac{x}{4} dx = 4\int \log \cos \tfrac{x}{4} d \tfrac{x}{4}$$

$$= 4\left(-\frac{(2^2 - 1)B_2}{4 \cdot 3!}(\frac{x}{2})^3 + \frac{(2^4 - 1)B_4}{8 \cdot 5!}(\frac{x}{2})^5 - \frac{(2^6 - 1)B_6}{12 \cdot 7!}(\frac{x}{2})^7 + \ldots \right)$$

…………………………………………………………………………..

Comparing the coefficients of $x^3, x^5, x^7, \ldots$ on both sides, does not yield any new result.

6.6 Euler's 2nd Product for $\frac{\sin z}{z}$

For any complex number z

$$\boxed{\frac{\sin z}{z} = \frac{4\cos^2\frac{z}{3} - 1}{3} \times \frac{4\cos^2\frac{z}{3^2} - 1}{3} \times \frac{4\cos^2\frac{z}{3^3} - 1}{3} \times ...}$$

Proof: Using the triple angle formula,

$$\sin 3z = 3\sin z - 4\sin^3 z,$$

[Zeid, p.57], we write

$$\sin z = 3\sin\tfrac{z}{3} - 4\sin^3\tfrac{z}{3}$$

$$= \sin\tfrac{z}{3}\Big(3 - 4[1 - \cos^2\tfrac{z}{3}]\Big)$$

$$= \Big(4\cos^2\tfrac{z}{3} - 1\Big)\sin\tfrac{z}{3}$$

$$= \Big(4\cos^2\tfrac{z}{3} - 1\Big)\Big(4\cos^2\tfrac{z}{3^2} - 1\Big)\sin\tfrac{z}{3^2}$$

$$= \Big(4\cos^2\tfrac{z}{3} - 1\Big) \times ... \times \Big(4\cos^2\tfrac{z}{3^n} - 1\Big)\sin\tfrac{z}{3^n}$$

$$= x\frac{3^n}{x}\sin\tfrac{x}{3^n}\frac{4\cos^2\frac{z}{3} - 1}{3} \times ... \times \frac{4\cos^2\frac{z}{3^n} - 1}{3}$$

Therefore, for any complex number $z \neq 0$,

$$\frac{\sin z}{z}\frac{\frac{z}{3^n}}{\sin\frac{z}{3^n}} = \frac{4\cos^2\frac{z}{3} - 1}{3} \times ... \times \frac{4\cos^2\frac{z}{3^n} - 1}{3}$$

Letting $n \to \infty$,

$$\frac{\sin z}{z} = \frac{4\cos^2\frac{z}{3} - 1}{3} \times \frac{4\cos^2\frac{z}{3^2} - 1}{3} \times ...$$

Since this holds also for $z \to 0$, it holds for any complex number z including $z = 0$. □

6.7 Geometric Mean Derivative of Euler's 2nd Product Presentation for $\frac{\sin x}{x}$

$$\boxed{\frac{\cos x}{\sin x} - \frac{1}{x} = -\frac{4\sin\frac{2x}{3}}{3(4\cos^2\frac{x}{3} - 1)} - \frac{4\sin\frac{2x}{3^2}}{3^2(4\cos^2\frac{x}{3^2} - 1)} - \frac{4\sin\frac{2x}{3^3}}{3^3(4\cos^2\frac{x}{3^3} - 1)}\ldots}$$

Proof: Geometric Mean Differentiating 6.6,

$$D^{(0)}\frac{\sin x}{x} = D^{(0)}\frac{4\cos^2\frac{x}{3} - 1}{3} \times D^{(0)}\frac{4\cos^2\frac{x}{3^2} - 1}{3} \times \ldots$$

$$e^{\left(\frac{D\frac{\sin x}{x}}{\frac{\sin x}{x}}\right)} = e^{\left(\frac{4D\cos^2\frac{x}{3}}{4\cos^2\frac{x}{3}-1}\right)} e^{\left(\frac{4D\cos^2\frac{x}{3^2}}{4\cos^2\frac{x}{3^2}-1}\right)} e^{\left(\frac{4D\cos^2\frac{x}{3^3}}{4\cos^2\frac{x}{3^3}-1}\right)}\ldots$$

$$e^{\left(\frac{\cos x}{\sin x}-\frac{1}{x}\right)} = e^{\left(-\frac{4\sin\frac{2x}{3}}{3(4\cos^2\frac{x}{3}-1)}\right)} e^{\left(-\frac{4\sin\frac{2x}{3^2}}{3^2(4\cos^2\frac{x}{3^2}-1)}\right)} e^{\left(-\frac{4\sin\frac{2x}{3^3}}{3^3(4\cos^2\frac{x}{3^3}-1)}\right)}\ldots$$

$$\frac{\cos x}{\sin x} - \frac{1}{x} = -\frac{4\sin\frac{2x}{3}}{3(4\cos^2\frac{x}{3} - 1)} - \frac{4\sin\frac{2x}{3^2}}{3^2(4\cos^2\frac{x}{3^2} - 1)} - \frac{4\sin\frac{2x}{3^3}}{3^3(4\cos^2\frac{x}{3^3} - 1)}\ldots \square$$

7

Product Calculus of $\sin x$

7.1 Euler's Product Representation for $\sin z$

For any complex number z,

$$\boxed{\sin z = z\left(1-\frac{z^2}{\pi^2}\right)\left(1-\frac{z^2}{(2\pi)^2}\right)\left(1-\frac{z^2}{(3\pi)^2}\right)....}$$

The product converges absolutely in any disk $\left|z\right| < R$.

Proof:

$$\sin z = 2\sin\tfrac{z}{2}\cos\tfrac{z}{2}$$

$$= 2\cdot 2\sin\tfrac{z}{4}\cos\tfrac{z}{4}\sin\left(\tfrac{z}{2}+\tfrac{\pi}{2}\right)$$

$$= 2^2\sin\tfrac{z}{4}\sin\left(\tfrac{z}{4}+\tfrac{\pi}{2}\right)\sin\left(\tfrac{z}{2}+\tfrac{\pi}{2}\right)$$

$$= 2^2\sin\tfrac{z}{4}\sin\left(\tfrac{z}{4}+\tfrac{\pi}{2}\right)2\sin\left(\tfrac{z}{4}+\tfrac{\pi}{4}\right)\cos\left(\tfrac{z}{4}+\tfrac{\pi}{4}\right)$$

$$= 2^3\sin\tfrac{z}{4}\sin\left(\tfrac{z+2\pi}{4}\right)\sin\left(\tfrac{z+\pi}{4}\right)\cos\left(-\tfrac{z}{4}-\tfrac{\pi}{4}\right)$$

$$= 2^3\sin\tfrac{z}{4}\sin\left(\tfrac{z+2\pi}{4}\right)\sin\left(\tfrac{z+\pi}{4}\right)\sin\left(-\tfrac{z}{4}-\tfrac{\pi}{4}+\tfrac{\pi}{2}\right)$$

$$= 2^3\sin\tfrac{z}{4}\sin\left(\tfrac{z+2\pi}{4}\right)\sin\left(\tfrac{z+\pi}{4}\right)\sin\left(\tfrac{\pi-z}{4}\right)$$

$$= 2^7\sin\tfrac{z}{8}\sin\left(\tfrac{z+4\pi}{8}\right)\sin\left(\tfrac{z+\pi}{4}\right)\sin\left(\tfrac{\pi-z}{4}\right)\times$$

$$\times\sin\left(\tfrac{z+2\pi}{8}\right)\sin\left(\tfrac{2\pi-z}{8}\right)\sin\left(\tfrac{z+\pi 3}{8}\right)\sin\left(\tfrac{3\pi-z}{8}\right)$$

....

$$= 2^{2^n-1}\sin\tfrac{z}{2^n}\sin\tfrac{z+2^{n-1}\pi}{2^n}\sin\tfrac{z+\pi}{2^n}\sin\tfrac{\pi-z}{2^n}\times$$

$$\times\sin\left(\tfrac{z+2\pi}{2^n}\right)\sin\left(\tfrac{2\pi-z}{2^n}\right)\sin\left(\tfrac{z+3\pi}{2^n}\right)\sin\left(\tfrac{3\pi-z}{2^n}\right)\times\ldots$$

$$\ldots\times\sin\tfrac{z+(2^{n-1}-1)\pi}{2^n}\sin\tfrac{(2^{n-1}-1)\pi-z}{2^n}$$

Now,

$$\sin\tfrac{z+\pi}{2^n}\sin\tfrac{\pi-z}{2^n} = \left(2\sin\tfrac{z+\pi}{2^{n+1}}\cos\tfrac{z+\pi}{2^{n+1}}\right)\left(2\sin\tfrac{\pi-z}{2^{n+1}}\cos\tfrac{\pi-z}{2^{n+1}}\right)$$

$$= \left(2\sin\tfrac{z+\pi}{2^{n+1}}\cos\tfrac{\pi-z}{2^{n+1}}\right)\left(2\sin\tfrac{\pi-z}{2^{n+1}}\cos\tfrac{z+\pi}{2^{n+1}}\right)$$

$$= \left(\sin\tfrac{z}{2^n}+\sin\tfrac{\pi}{2^n}\right)\left(\sin\tfrac{\pi}{2^n}-\sin\tfrac{z}{2^n}\right)$$

$$= \sin^2\tfrac{\pi}{2^n}-\sin^2\tfrac{z}{2^n}$$

And,

$$\sin\tfrac{z+2\pi}{2^n}\sin\tfrac{2\pi-z}{2^n} = \sin^2\tfrac{2\pi}{2^n}-\sin^2\tfrac{z}{2^n}$$

Therefore,

- $$\sin z = 2^{2^n-1}\sin\tfrac{z}{2^n}\cos\tfrac{z}{2^n}\left(\sin^2\tfrac{\pi}{2^n}-\sin^2\tfrac{z}{2^n}\right)\times$$

$$\times\left(\sin^2\tfrac{2\pi}{2^n}-\sin^2\tfrac{z}{2^n}\right)\times\ldots\times\left(\sin^2\tfrac{(2^{n-1}-1)\pi}{2^n}-\sin^2\tfrac{z}{2^n}\right)$$

That is,

$$2^n\frac{\frac{z}{2^n}}{\sin\frac{z}{2^n}}\frac{\sin z}{z} = 2^{2^n-1}\cos\tfrac{z}{2^n}\left(\sin^2\tfrac{\pi}{2^n}-\sin^2\tfrac{z}{2^n}\right)\times$$

$$\times\left(\sin^2\tfrac{2\pi}{2^n}-\sin^2\tfrac{z}{2^n}\right)\times\ldots\times\left(\sin^2\tfrac{(2^{n-1}-1)\pi}{2^n}-\sin^2\tfrac{z}{2^n}\right)$$

Letting $z\to 0$,

$$2^n = 2^{2^n - 1}\sin^2\tfrac{\pi}{2^n}\sin^2\tfrac{2\pi}{2^n}\times ... \times \sin^2\tfrac{(2^{n-1}-1)\pi}{2^n}$$

Dividing ⬣ by this last equation,

$$\sin z = 2^n \sin\tfrac{z}{2^n}\cos\tfrac{z}{2^n}\left(1-\frac{\sin^2\tfrac{z}{2^n}}{\sin^2\tfrac{\pi}{2^n}}\right)\left(1-\frac{\sin^2\tfrac{z}{2^n}}{\sin^2\tfrac{2\pi}{2^n}}\right)\times .. \times\left(1-\frac{\sin^2\tfrac{z}{2^n}}{\sin^2\tfrac{(2^{n-1}-1)\pi}{2^n}}\right)$$

$$= z\left(\frac{\sin\tfrac{z}{2^n}}{\tfrac{z}{2^n}}\right)\cos\tfrac{z}{2^n}\left(1-\frac{\sin^2\tfrac{z}{2^n}}{\sin^2\tfrac{\pi}{2^n}}\right)\left(1-\frac{\sin^2\tfrac{z}{2^n}}{\sin^2\tfrac{2\pi}{2^n}}\right)\times .. \times\left(1-\frac{\sin^2\tfrac{z}{2^n}}{\sin^2\tfrac{(2^{n-1}-1)\pi}{2^n}}\right)$$

Letting $n \to \infty$, for any fixed natural number m we have

$$1-\frac{\sin^2\tfrac{z}{2^n}}{\sin^2\tfrac{m\pi}{2^n}} = 1-\left(\frac{\sin\tfrac{z}{2^n}}{\tfrac{z}{2^n}}\right)^2\frac{z^2}{(m\pi)^2}\left(\frac{\tfrac{m\pi}{2^n}}{\sin\tfrac{m\pi}{2^n}}\right)^2 \to 1-\frac{z^2}{(m\pi)^2}$$

Consequently, the infinite product

$$z\left(1-\frac{z^2}{\pi^2}\right)\left(1-\frac{z^2}{(2\pi)^2}\right)\left(1-\frac{z^2}{(3\pi)^2}\right)....$$

Converges to $\sin z$.

The convergence is absolute in any disk $|z| < R$, because the infinite series

$$\frac{z^2}{\pi^2}+\frac{z^2}{(2\pi)^2}+\frac{z^2}{(3\pi)^2}+...$$

converges absolutely in any disk $|z| < R$.

Indeed, in $\left|z\right| < R$,

$$\frac{\left|z\right|^2}{\pi^2} + \frac{\left|z\right|^2}{(2\pi)^2} + \frac{\left|z\right|^2}{(3\pi)^2} + ... = \left|z\right|^2 \frac{1}{\pi^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + ...\right)$$

$$= \left|z\right|^2 \frac{1}{\pi^2}\frac{\pi^2}{6}$$

$$< \frac{1}{6}R^2 . \square$$

7.2 Geometric Mean Derivative of $\sin x$

$$\boxed{\cot x = \frac{1}{x} - \frac{2x}{\pi^2 - x^2} - \frac{2x}{(2\pi)^2 - x^2} - \frac{2x}{(3\pi)^2 - x^2} - ...}$$

Proof: $D^{(0)} \sin x \;=\; D^{(0)} x D^{(0)}\left(1 - \frac{x^2}{\pi^2}\right) D^{(0)}\left(1 - \frac{x^2}{(2\pi)^2}\right) D^{(0)}\left(1 - \frac{x^2}{(3\pi)^2}\right) ....$

$$e^{\frac{\cos x}{\sin x}} = e^{\frac{1}{x}} e^{-\frac{2x}{\pi^2 - x^2}} e^{-\frac{2x}{(2\pi)^2 - x^2}} e^{-\frac{2x}{(3\pi)^2 - x^2}} ...$$

Thus,

$$\cot x = \frac{1}{x} - \frac{2x}{\pi^2 - x^2} - \frac{2x}{(2\pi)^2 - x^2} - \frac{2x}{(3\pi)^2 - x^2} - ... \square$$

7.3 The Wallis Product for π

$$\boxed{\frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7} ....}$$

Proof: Wallis product for π follows from the product formula for $\sin\frac{\pi}{2}$, [Bert, p. 424].

$$1 = \sin\tfrac{\pi}{2}$$

$$= \frac{\pi}{2}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right)....$$

$$= \frac{\pi}{2}\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{6}\right)\left(1+\frac{1}{6}\right)...$$

$$= \frac{\pi}{2}\frac{1}{2}\frac{3}{2}\frac{3}{4}\frac{5}{4}\frac{5}{6}\frac{7}{6}.... \square$$

8

Product Calculus of $\cos x$

8.1 Euler's Product Representation for $\cos z$,

For any complex number z,

$$\boxed{\cos z = \left(1-\left(\frac{2z}{\pi}\right)^2\right)\left(1-\left(\frac{2z}{3\pi}\right)^2\right)\left(1-\left(\frac{2z}{5\pi}\right)^2\right)...}$$

The convergence is absolute in any disk $|z| < R$.

Proof:

$$\cos z = \frac{\sin 2z}{2\sin z}$$

$$= \frac{1}{2}\frac{2z\left(1-\left(\frac{2z}{\pi}\right)^2\right)\left(1-\left(\frac{2z}{2\pi}\right)^2\right)\left(1-\left(\frac{2z}{3\pi}\right)^2\right)\left(1-\left(\frac{2z}{4\pi}\right)^2\right)\left(1-\left(\frac{2z}{5\pi}\right)^2\right)...}{z\left(1-\left(\frac{z}{\pi}\right)^2\right)\left(1-\left(\frac{z}{2\pi}\right)^2\right)\left(1-\left(\frac{z}{3\pi}\right)^2\right)\left(1-\left(\frac{z}{4\pi}\right)^2\right)\left(1-\left(\frac{z}{5\pi}\right)^2\right)...}$$

$$= \left(1-\left(\frac{2z}{\pi}\right)^2\right)\left(1-\left(\frac{2z}{3\pi}\right)^2\right)\left(1-\left(\frac{2z}{5\pi}\right)^2\right)... \square$$

8.2 Geometric Mean Derivative of $\cos x$

$$\boxed{\tan x = \frac{8x}{\pi^2-(2x)^2} + \frac{8x}{(3\pi)^2-(2x)^2} + \frac{8x}{(5\pi)^2-(2x)^2} + ...}$$

Proof: $D^{(0)}\cos x = D^{(0)}\left(1-\left(\frac{2x}{\pi}\right)^2\right)D^{(0)}\left(1-\left(\frac{2x}{3\pi}\right)^2\right)D^{(0)}\left(1-\left(\frac{2x}{5\pi}\right)^2\right)...$

$$e^{-\frac{\sin x}{\cos x}} = e^{-\frac{8x}{\pi^2-(2x)^2}}e^{-\frac{8x}{(3\pi)^2-(2x)^2}}e^{-\frac{8x}{(5\pi)^2-(2x)^2}}...$$

Thus,

$$\tan x = \frac{8x}{\pi^2-(2x)^2}+\frac{8x}{(3\pi)^2-(2x)^2}+\frac{8x}{(5\pi)^2-(2x)^2}+... \square$$

9

Product Calculus of $\tan x$

9.1 Product Representation for $\tan z$

For any complex number z,

$$\boxed{\tan z = \frac{z\left(1-\frac{z^2}{\pi^2}\right)\left(1-\frac{z^2}{(2\pi)^2}\right)\left(1-\frac{z^2}{(3\pi)^2}\right)...}{\left(1-\left(\frac{2z}{\pi}\right)^2\right)\left(1-\left(\frac{2z}{3\pi}\right)^2\right)\left(1-\left(\frac{2z}{5\pi}\right)^2\right)...}}$$

The convergence is absolute in any disk $|z| < R$.

Proof: 7.1 and 8.1□

9.2 Geometric Mean Derivative of $\tan x$

$$\frac{2}{\sin 2x} = \frac{1}{x} - \frac{2x}{\pi^2 - x^2} + \frac{8x}{\pi^2 - (2x)^2}$$

$$-\frac{2x}{(2\pi)^2 - x^2} + \frac{8x}{(3\pi)^2 - (2x)^2}$$

$$-\frac{2x}{(3\pi)^2 - x^2} + \frac{8x}{(5\pi)^2 - (2x)^2} - ...$$

Proof:

$$D^{(0)}\tan x = \frac{D^{(0)}xD^{(0)}\left(1-\left(\frac{x}{\pi}\right)^2\right)D^{(0)}\left(1-\left(\frac{x}{2\pi}\right)^2\right)D^{(0)}\left(1-\left(\frac{x}{3\pi}\right)^2\right)...}{D^{(0)}\left(1-\left(\frac{2x}{\pi}\right)^2\right)D^{(0)}\left(1-\left(\frac{2x}{3\pi}\right)^2\right)D^{(0)}\left(1-\left(\frac{2x}{5\pi}\right)^2\right)...}$$

$$e^{\frac{2}{\sin 2x}} = \frac{e^{\frac{1}{x}}e^{-\frac{2x}{\pi^2-x^2}}e^{-\frac{2x}{(2\pi)^2-x^2}}e^{-\frac{2x}{(3\pi)^2-x^2}}....}{e^{-\frac{8x}{\pi^2-(2x)^2}}e^{-\frac{8x}{(3\pi)^2-(2x)^2}}e^{-\frac{8x}{(5\pi)^2-(2x)^2}}}$$

Thus,

$$\frac{2}{\sin 2x} = \frac{1}{x} - \frac{2x}{\pi^2 - x^2} + \frac{8x}{\pi^2 - (2x)^2}$$

$$-\frac{2x}{(2\pi)^2 - x^2} + \frac{8x}{(3\pi)^2 - (2x)^2}$$

$$-\frac{2x}{(3\pi)^2 - x^2} + \frac{8x}{(5\pi)^2 - (2x)^2} - ... \square$$

10

Product Calculus of $\sinh x$

10.1 Product Representation of $\sinh z$

$$\boxed{\sinh z = z\left(1+\frac{z^2}{\pi^2}\right)\left(1+\frac{z^2}{(2\pi)^2}\right)\left(1+\frac{z^2}{(3\pi)^2}\right)....}$$

The convergence is absolute in any disk $|z| < R$.

Proof: $\sinh z = -i\sin iz$, and use 7.1□

10.2 Geometric Mean Derivative of $\sinh x$

$$\boxed{\coth x = \frac{1}{x} + \frac{2x}{\pi^2 + x^2} + \frac{2x}{(2\pi)^2 + x^2} + \frac{2x}{(3\pi)^2 + x^2} + ...}$$

Proof: $D^{(0)}\sinh x = D^{(0)}xD^{(0)}\left(1+\frac{x^2}{\pi^2}\right)D^{(0)}\left(1+\frac{x^2}{(2\pi)^2}\right)D^{(0)}\left(1+\frac{x^2}{(3\pi)^2}\right)....$

$$e^{\frac{D\sinh x}{\sinh x}} = e^{\frac{1}{x}}e^{\frac{2x}{\pi^2+x^2}}e^{\frac{2x}{(2\pi)^2+x^2}}e^{\frac{2x}{(3\pi)^2+x^2}}...$$

Thus,

$$\coth x = \frac{1}{x} + \frac{2x}{\pi^2 + x^2} + \frac{2x}{(2\pi)^2 + x^2} + \frac{2x}{(3\pi)^2 + x^2} + ...\square$$

11

Product Calculus of $\cosh x$

11.1 Product Representation of $\cosh z$

$$\boxed{\cosh z = \left(1+\left(\frac{2z}{\pi}\right)^2\right)\left(1+\left(\frac{2z}{3\pi}\right)^2\right)\left(1+\left(\frac{2z}{5\pi}\right)^2\right)....}$$

The convergence is absolute in any disk $|z| < R$.

Proof: $\cosh z = \cos iz$, and apply 8.1□

11.2 Geometric Mean Derivative of $\cosh x$

$$\boxed{\tanh x = \frac{8x}{\pi^2 + (2x)^2} + \frac{8x}{(3\pi)^2 + (2x)^2} + \frac{8x}{(5\pi)^2 + (2x)^2} + ...}$$

Proof: $$D^{(0)} \cosh x = D^{(0)}\left(1+\left(\frac{2x}{\pi}\right)^2\right) D^{(0)}\left(1+\left(\frac{2x}{3\pi}\right)^2\right) D^{(0)}\left(1+\left(\frac{2x}{5\pi}\right)^2\right)....$$

$$e^{\frac{D\cosh x}{\cosh x}} = e^{\frac{8x}{\pi^2+(2x)^2}} e^{\frac{8x}{(3\pi)^2+(2x)^2}} e^{\frac{8x}{(5\pi)^2+(2x)^2}}....$$

Thus,

$$\tanh x = \frac{8x}{\pi^2 + (2x)^2} + \frac{8x}{(3\pi)^2 + (2x)^2} + \frac{8x}{(5\pi)^2 + (2x)^2} + ...\square$$

12

Product Calculus of $\tanh x$

12.1 Product Representation for $\tanh z$

For any complex number z,

$$\boxed{\tanh z = \frac{z\left(1+\frac{z^2}{\pi^2}\right)\left(1+\frac{z^2}{(2\pi)^2}\right)\left(1+\frac{z^2}{(3\pi)^2}\right)...}{\left(1+\left(\frac{2z}{\pi}\right)^2\right)\left(1+\left(\frac{2z}{3\pi}\right)^2\right)\left(1+\left(\frac{2z}{5\pi}\right)^2\right)...}}$$

The convergence is absolute in any disk $|z| < R$.

Proof: 10.1 and 11.1□

12.2 Geometric Mean Derivative of $\tanh x$

$$\frac{2}{\sinh 2x} = \frac{1}{x} + \frac{2x}{\pi^2 + x^2} - \frac{8x}{\pi^2 + (2x)^2}$$

$$+\frac{2x}{(2\pi)^2 + x^2} - \frac{8x}{(3\pi)^2 + (2x)^2}$$

$$+\frac{2x}{(3\pi)^2 + x^2} - \frac{8x}{(5\pi)^2 + (2x)^2} + ...$$

Proof:

$$D^{(0)}\tanh x = \frac{D^{(0)}xD^{(0)}\left(1+\left(\frac{x}{\pi}\right)^2\right)D^{(0)}\left(1+\left(\frac{x}{2\pi}\right)^2\right)D^{(0)}\left(1+\left(\frac{x}{3\pi}\right)^2\right)\dots}{D^{(0)}\left(1+\left(\frac{2x}{\pi}\right)^2\right)D^{(0)}\left(1+\left(\frac{2x}{3\pi}\right)^2\right)D^{(0)}\left(1+\left(\frac{2x}{5\pi}\right)^2\right)\dots}$$

Now,

$$\frac{D\tanh x}{\tanh x} = \frac{\frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}}{\frac{\sinh x}{\cosh x}} = \frac{1}{\sinh x\cosh x} = \frac{2}{\sinh 2x}.$$

Therefore,

$$e^{\frac{2}{\sinh 2x}} = \frac{e^{\frac{1}{x}}e^{\frac{2x}{\pi^2+x^2}}e^{\frac{2x}{(2\pi)^2+x^2}}e^{\frac{2x}{(3\pi)^2+x^2}}\dots}{e^{\frac{8x}{\pi^2+(2x)^2}}e^{\frac{8x}{(3\pi)^2+(2x)^2}}e^{\frac{8x}{(5\pi)^2+(2x)^2}}\dots}$$

Thus,

$$\frac{2}{\sinh 2x} = \frac{1}{x} + \frac{2x}{\pi^2 + x^2} - \frac{8x}{\pi^2 + (2x)^2}$$

$$+\frac{2x}{(2\pi)^2 + x^2} - \frac{8x}{(3\pi)^2 + (2x)^2}$$

$$+\frac{2x}{(3\pi)^2 + x^2} - \frac{8x}{(5\pi)^2 + (2x)^2} + \dots \square$$

13

Product Calculus of e^x

13.1 Product representation of e^x

$$\boxed{\left(1+\frac{z}{n}\right)^n \underset{n\to\infty}{\to} e^z}$$

13.2 Geometric Mean Derivative of e^x

$$\boxed{D^{(0)}e^x = e}$$

Proof: $D^{(0)}\left(1+\frac{x}{n}\right)^n = \left[D^{(0)}\left(1+\frac{x}{n}\right)\right]^n$

$$= \left[e^{\frac{D(1+\frac{x}{n})}{1+\frac{x}{n}}}\right]^n$$

$$= \left[e^{\frac{\frac{1}{n}}{1+\frac{x}{n}}}\right]^n$$

$$= e^{\frac{1}{1+\frac{x}{n}}} \xrightarrow[n\to\infty]{} e$$

Thus, $D^{(0)}e^x = e$.$\square$

13.3 Geometric Mean Derivative of e^{e^x}

$$\boxed{D^{(0)}e^{e^x} = e^{e^x}}$$

Proof: $D^{(0)}\left(1+\frac{e^x}{n}\right)^n = \left[D^{(0)}\left(1+\frac{e^x}{n}\right)\right]^n$

$$= \left[e^{\frac{D(1+\frac{e^x}{n})}{1+\frac{e^x}{n}}}\right]^n$$

$$= \left[e^{\frac{\frac{e^x}{n}}{1+\frac{e^x}{n}}}\right]^n$$

$$= e^{\frac{e^x}{1+\frac{e^x}{n}}}$$

$$\underset{n\to\infty}{\to} e^{e^x}$$

Thus, $D^{(0)}e^{e^x} = e^{e^x}$. $\square$

13.4 Geometric Mean Derivative of e^{x^k}

$$\boxed{D^{(0)}e^{x^k} = e^{kx^{k-1}}}$$

Proof: $D^{(0)}\left(1+\frac{x^k}{n}\right)^n = \left[D^{(0)}\left(1+\frac{x^k}{n}\right)\right]^n$

$$= \left[e^{\frac{D(1+\frac{x^k}{n})}{1+\frac{x^k}{n}}} \right]^n$$

$$= \left[e^{\frac{k\frac{x^{k-1}}{n}}{1+\frac{x^k}{n}}} \right]^n$$

$$= e^{\frac{kx^{k-1}}{1+\frac{x^k}{n}}}$$

$$\underset{n\to\infty}{\longrightarrow} e^{kx^{k-1}}$$

Thus, $D^{(0)}e^{x^k} = e^{kx^{k-1}}$.$\square$

14

Geometric Mean Derivative by Exponentiation

Geometric Mean Derivative can be obtained by using the Product Calculus of the exponential function.

We demonstrate this method by examples.

14.1 $\boxed{D^{(0)}\sin x = e^{\cot x}}$

Proof: Since

$$\sin x = e^{\log\sin x} = \lim_{n\to\infty}(1+\tfrac{\log\sin x}{n})^n,$$

we apply the Geometric Mean Derivative to $(1+\tfrac{\log\sin x}{n})^n$.

$$D^{(0)}\underbrace{\left\{\left(1+\frac{\log\sin x}{n}\right)\times\ldots\times\left(1+\frac{\log\sin x}{n}\right)\right\}}_{n\text{ factors}} =$$

$$= \underbrace{D^{(0)}\left(1+\frac{\log\sin x}{n}\right)\times\ldots\times D^{(0)}\left(1+\frac{\log\sin x}{n}\right)}_{n\text{ factors}}$$

$$= \left[e^{\frac{D(1+\frac{\log\sin x}{n})}{1+\frac{\log\sin x}{n}}} \right]^n$$

$$= \left[e^{\frac{\frac{\cot x}{n}}{1+\frac{\log\sin x}{n}}} \right]^n$$

$$= e^{\frac{\cot x}{1+\log\sin x/n}}$$

$$\underset{n\to\infty}{\longrightarrow} e^{\cot x}.$$

Thus, $D^{(0)} \sin x = e^{\cot x}$. $\square$

14.2 $\boxed{D^{(0)} x^x = ex}$

Proof: Since

$$x^x = e^{x\log x} = \lim_{n\to\infty} (1 + \tfrac{x\log x}{n})^n,$$

we apply $D^{(0)}$ to $(1 + \frac{x\log x}{n})^n$.

$$D^{(0)}\left(1 + \frac{x\log x}{n}\right)^n = \left[D^{(0)}\left(1 + \frac{x\log x}{n}\right)\right]^n$$

$$= \left[e^{\frac{D(1+\frac{x\log x}{n})}{1+\frac{x\log x}{n}}} \right]^n$$

$$= \left[e^{\frac{\frac{1+\log x}{n}}{1+\frac{x\log x}{n}}} \right]^n$$

$$= e^{\frac{1+\log x}{1+\frac{x\log x}{n}}}$$

$$\underset{n\to\infty}{\longrightarrow} e^{1+\log x} = ex$$

Thus, $D^{(0)}x^x = ex$.$\square$

15

Product Calculus of $\Gamma(x)$

On the half line $x > 0$, Euler defined the real valued Gamma function by

$$\Gamma(x) = \int_{t=0}^{t=\infty} e^{-t} t^{x-1} dt .$$

In the half plane $\mathrm{Re}\, z > 0$, the complex valued integral

$$\int_{t=0}^{t=\infty} e^{-t} t^{z-1} dt$$

converges, and is differentiable with

$$D_z \int_{t=0}^{t=\infty} e^{-t} t^{z-1} dt = \int_{t=0}^{t=\infty} e^{-t} t^{z-1} \ln t dt .$$

Thus, the complex valued integral extends the Euler integral into an analytic function in the half plane $\mathrm{Re}\, z > 0$. It is denoted by

$$\Gamma(z) .$$

This function can be further extended to a product representation that is analytic for any z, except for simple poles that it has at $z = 0, -1, -2, -3, \ldots$

The function $1 / \Gamma(z)$, that is given by the inverse product, is analytic for any z, with simple zeros at $z = 0, -1, -2, -3, \ldots$

Therefore, the natural calculus for $\Gamma(z)$ and for $1 / \Gamma(z)$ in the complex plane is the product Calculus.

15.1 Euler's Product Representation for $\Gamma(z)$

$$\boxed{\Gamma(z) = \frac{\left(1+\frac{1}{1}\right)^z\left(1+\frac{1}{2}\right)^z\left(1+\frac{1}{3}\right)^z....}{z\left(1+\frac{z}{1}\right)\left(1+\frac{z}{2}\right)\left(1+\frac{z}{3}\right)....}}$$

Proof:

$$\Gamma(z) = \int_{t=0}^{t=\infty} e^{-t}t^{z-1}dt$$

$$= \int_{t=0}^{t=\infty} \lim_{n\to\infty}\left(1-\frac{t}{n}\right)^n t^{z-1}dt$$

Uniform convergence allows order change of limit, and integration

$$= \lim_{n\to\infty}\int_{t=0}^{t=\infty}\left(1-\frac{t}{n}\right)^n t^{z-1}dt$$

The change of variable, $u = t / n$, $du = dt / n$, gives

$$= \lim_{n\to\infty} n^z \int_{u=0}^{u=1}\left(1-u\right)^n u^{z-1}du$$

n^z can be written as a product

$$n^z = (n+1)^z \frac{n^z}{(n+1)^z}$$

$$= \left(1+\tfrac{1}{1}\right)^z\left(1+\tfrac{1}{2}\right)^z\left(1+\tfrac{1}{3}\right)^z...\left(1+\tfrac{1}{n}\right)^z \tfrac{n^z}{(n+1)^z}$$

Integrating by parts with respect to u, keeping z, and n fixed

$$\int_{u=0}^{u=1} (1-u)^n u^{z-1} du = \int_{u=0}^{u=1} (1-u)^n d\left(\frac{u^z}{z}\right)$$

$$= \left[\frac{u^z}{z}(1-u)^n\right]_{u=0}^{u=1} - \int_{u=0}^{u=1} \frac{u^z}{z} d(1-u)^n$$

$$= \frac{1}{z}\int_{u=0}^{u=1} u^z n(1-u)^{n-1} du$$

$$= \frac{n}{z}\int_{u=0}^{u=1} (1-u)^{n-1} d\left(\frac{u^{z+1}}{z+1}\right)$$

$$= \frac{n}{z}\frac{n-1}{z+1}\int_{u=0}^{u=1} (1-u)^{n-2} d\left(\frac{u^{z+2}}{z+2}\right)$$

$$= \frac{n}{z}\frac{n-1}{z+1}\frac{n-2}{z+2}\int_{u=0}^{u=1} (1-u)^{n-3} d\left(\frac{u^{z+3}}{z+3}\right)$$

....

$$= \frac{n}{z}\frac{n-1}{z+1}\frac{n-2}{z+2}\cdots\frac{n-(n-1)}{z+n-1}\int_{u=0}^{u=1} (1-u)^{n-n} d\left(\frac{u^{z+n}}{z+n}\right)$$

$$= \frac{n}{z}\frac{n-1}{z+1}\frac{n-2}{z+2}\cdots\frac{n-(n-1)}{z+n-1}\left(\frac{u^{z+n}}{z+n}\right)_{u=0}^{u=1}$$

$$= \frac{n}{z}\frac{n-1}{z+1}\frac{n-2}{z+2}\cdots\frac{n-(n-1)}{z+n-1}\frac{1}{z+n}$$

$$= \frac{n}{z}\frac{n-1}{z+1}\frac{n-2}{z+2}\cdots\frac{n-(n-1)}{z+n-1}\frac{1}{z+n}$$

$$= \frac{1}{z}\frac{1}{\left(\frac{z}{1}+1\right)}\frac{1}{\left(\frac{z}{2}+1\right)}\cdots\frac{1}{\left(\frac{z}{n-1}+1\right)}\frac{1}{\left(\frac{z}{n}+1\right)}$$

Therefore,

$$\lim_{n\to\infty} n^z \int_{u=0}^{u=1} \left(1-u\right)^n u^{z-1}du =$$

$$= \lim_{n\to\infty}\left\{\left(1+\tfrac{1}{1}\right)^z\left(1+\tfrac{1}{2}\right)^z\left(1+\tfrac{1}{3}\right)^z\cdots\left(1+\tfrac{1}{n}\right)^z\tfrac{n^z}{(n+1)^z}\times\right.$$

$$\left.\times\frac{1}{z}\frac{1}{\left(\frac{z}{1}+1\right)}\frac{1}{\left(\frac{z}{2}+1\right)}\cdots\frac{1}{\left(\frac{z}{n-1}+1\right)}\frac{1}{\left(\frac{z}{n}+1\right)}\right\}$$

$$= \lim_{n\to\infty}\frac{\left(1+\frac{1}{1}\right)^z\left(1+\frac{1}{2}\right)^z\left(1+\frac{1}{3}\right)^z\cdots\left(1+\frac{1}{n}\right)^z}{z\left(\frac{z}{1}+1\right)\left(\frac{z}{2}+1\right)\cdots\left(\frac{z}{n-1}+1\right)\left(\frac{z}{n}+1\right)}$$

$$= \frac{\left(1+\frac{1}{1}\right)^z\left(1+\frac{1}{2}\right)^z\left(1+\frac{1}{3}\right)^z\cdots\cdot}{z\left(1+\frac{z}{1}\right)\left(1+\frac{z}{2}\right)\left(1+\frac{z}{3}\right)\cdots\cdot}.\square$$

15.2 Geometric Mean Derivative of $\Gamma(x)$

$$\boxed{\frac{\Gamma'(x)}{\Gamma(x)} = \lim_{n\to\infty}\left(\log n - \frac{1}{x} - \frac{1}{x+1} - \frac{1}{x+2} - \ldots - \frac{1}{x+n}\right)}$$

Proof:

$$D^{(0)}\Gamma(x) = \frac{D^{(0)}\left(1+\frac{1}{1}\right)^x D^{(0)}\left(1+\frac{1}{2}\right)^x D^{(0)}\left(1+\frac{1}{3}\right)^x \ldots\ldots}{D^{(0)}x D^{(0)}\left(1+\frac{x}{1}\right) D^{(0)}\left(1+\frac{x}{2}\right) D^{(0)}\left(1+\frac{x}{3}\right)\ldots\ldots}$$

$$= \frac{e^{\frac{D2^x}{2^x}} e^{\frac{D(3/2)^x}{(3/2)^x}} e^{\frac{D(4/3)^x}{(4/3)^x}} \ldots\ldots}{e^{\frac{1}{x}} e^{\frac{1}{x+1}} e^{\frac{1}{x+2}} e^{\frac{1}{x+3}} \ldots\ldots}$$

$$= \frac{\frac{2}{1}\frac{3}{2}\frac{4}{3}\ldots\ldots}{e^{\frac{1}{x}+\frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x+3}\ldots\ldots}}$$

$$= \left(\frac{2}{1}\frac{3}{2}\frac{4}{3}\ldots\ldots\right) e^{-\frac{1}{x}-\frac{1}{x+1}-\frac{1}{x+2}-\frac{1}{x+3}\ldots\ldots}$$

$$= \lim_{n\to\infty} (n+1) e^{-\frac{1}{x}-\frac{1}{x+1}-\frac{1}{x+2}\ldots\ldots-\frac{1}{x+n}}$$

$$= \lim_{n\to\infty} n e^{-\frac{1}{x}-\frac{1}{x+1}-\frac{1}{x+2}\ldots\ldots-\frac{1}{x+n}}$$

$$= \lim_{n\to\infty} e^{\log n-\frac{1}{x}-\frac{1}{x+1}-\frac{1}{x+2}-\ldots\ldots-\frac{1}{x+n}}$$

$$= e^{\lim_{n\to\infty}\left(\log n-\frac{1}{x}-\frac{1}{x+1}-\frac{1}{x+2}-\ldots\ldots-\frac{1}{x+n}\right)}$$

Since $D^{(0)}\Gamma(x) = e^{\frac{\Gamma'(x)}{\Gamma(x)}}$,

$$\frac{\Gamma'(x)}{\Gamma(x)} = \lim_{n\to\infty}\left(\log n - \frac{1}{x} - \frac{1}{x+1} - \frac{1}{x+2} - \dots - \frac{1}{x+n}\right)\square$$

15.3 $\boxed{\Gamma(1)=1}$

Proof:

$$\Gamma(1) = \frac{\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\dots.}{1\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\dots.} = 1\,\square$$

15.4 $\boxed{\Gamma(z+1) = z\Gamma(z)}$

Proof:

$$\Gamma(z+1) = \frac{\left(1+\frac{1}{1}\right)^{z+1}\left(1+\frac{1}{2}\right)^{z+1}\left(1+\frac{1}{3}\right)^{z+1}\dots.}{z\left(1+\frac{z+1}{1}\right)\left(1+\frac{z+1}{2}\right)\left(1+\frac{z+1}{3}\right)\dots.}$$

$$= \frac{\left(1+\frac{1}{1}\right)^{z}2\left(1+\frac{1}{2}\right)^{z}\frac{3}{2}\left(1+\frac{1}{3}\right)^{z}\frac{4}{3}\dots.}{\left(z+1\right)\left(1+\frac{z+1}{1}\right)\left(1+\frac{z+1}{2}\right)\left(1+\frac{z+1}{3}\right)\dots.}$$

$$= \frac{\left(1+\frac{1}{1}\right)^{z}\left(1+\frac{1}{2}\right)^{z}\left(1+\frac{1}{3}\right)^{z}\dots}{\left(1+z\right)\frac{1}{2}\left(2+z\right)\frac{2}{3}\left(\frac{3}{2}+\frac{z}{2}\right)\frac{3}{4}\left(\frac{4}{3}+\frac{z}{3}\right)\dots}$$

$$= \frac{\left(1+\frac{1}{1}\right)^{z}\left(1+\frac{1}{2}\right)^{z}\left(1+\frac{1}{3}\right)^{z}\dots}{\left(1+z\right)\left(1+\frac{z}{2}\right)\left(1+\frac{z}{3}\right)\left(1+\frac{z}{4}\right)\dots}$$

$$= z\Gamma(z).\square$$

15.5 Product Reflection Formula for $\Gamma(z)$

$$\boxed{\Gamma(z)\Gamma(1-z) = \frac{1}{z\left(1-z^2\right)\left(1-\frac{z^2}{2^2}\right)\left(1-\frac{z^2}{3^2}\right)...}}$$

Proof:

$$\Gamma(z)\Gamma(1-z) = \frac{\left(1+\frac{1}{1}\right)^z\left(1+\frac{1}{2}\right)^z\left(1+\frac{1}{3}\right)^z...}{z\left(1+\frac{z}{1}\right)\left(1+\frac{z}{2}\right)\left(1+\frac{z}{3}\right)...}\frac{\left(1+\frac{1}{1}\right)^{1-z}\left(1+\frac{1}{2}\right)^{1-z}\left(1+\frac{1}{3}\right)^{1-z}...}{\left(1-z\right)\left(1+\frac{1-z}{1}\right)\left(1+\frac{1-z}{2}\right)\left(1+\frac{1-z}{3}\right)..}$$

$$= \frac{\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)...}{z\left(1+z\right)\left(1+\frac{z}{2}\right)\left(1+\frac{z}{3}\right)..\left(1-z\right)2\left(1-\frac{z}{2}\right)\frac{3}{2}\left(1-\frac{z}{3}\right)\frac{4}{3}\left(1-\frac{z}{4}\right)..}$$

$$= \frac{1}{z\left(1+\frac{z}{1}\right)\left(1+\frac{z}{2}\right)\left(1+\frac{z}{3}\right)..\left(1-z\right)\left(1-\frac{z}{2}\right)\left(1-\frac{z}{3}\right)\left(1-\frac{z}{4}\right)..}$$

$$= \frac{1}{z\left(1+z\right)\left(1-z\right)\left(1+\frac{z}{2}\right)\left(1-\frac{z}{2}\right)\left(1+\frac{z}{3}\right)\left(1-\frac{z}{3}\right)..}$$

$$= \frac{1}{z\left(1-z^2\right)\left(1-\frac{z^2}{2^2}\right)\left(1-\frac{z^2}{3^2}\right)...}.\square$$

15.6 $\boxed{\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}}$

Proof: By 15.5,

$$\Gamma(z)\Gamma(1-z) = \frac{1}{z\left(1-z^2\right)\left(1-\frac{z^2}{2^2}\right)\left(1-\frac{z^2}{3^2}\right)...}$$

$$= \frac{\pi}{\pi z\left(1-\frac{(\pi z)^2}{\pi^2}\right)\left(1-\frac{(\pi z)^2}{(2\pi)^2}\right)\left(1-\frac{(\pi z)^2}{(3\pi)^2}\right)..}$$

$$= \frac{\pi}{\sin \pi z}.\square$$

15.7 $$\boxed{\left(\Gamma(\tfrac{1}{2})\right)^2 = \pi}$$

Proof: Substituting $z = \frac{1}{2}$ in $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$, we obtain

$$\Gamma\left(\tfrac{1}{2}\right)\Gamma\left(1-\tfrac{1}{2}\right) = \frac{\pi}{\sin\frac{\pi}{2}}.$$

That is,

$$\left[\Gamma(\tfrac{1}{2})\right]^2 = \pi\,.\square$$

This can be obtained directly through the Wallis Product for π.

15.8 $$\boxed{\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = 2\cdot\frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\frac{8}{7}\frac{8}{9}\frac{10}{9}\frac{10}{11}\frac{12}{11}\frac{12}{13}...}$$

Proof: By 15.5,

$$\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = \frac{1}{\frac{1}{2}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right)\left(1-\frac{1}{8^2}\right)\left(1-\frac{1}{10^2}\right)...}$$

$$= \frac{1}{\frac{1}{2}\left(\frac{1\cdot 3}{2^2}\right)\left(\frac{3\cdot 5}{4^2}\right)\left(\frac{5\cdot 7}{6^2}\right)\left(\frac{7\cdot 9}{8^2}\right)\left(\frac{9\cdot 11}{10^2}\right)...}$$

$$= 2 \cdot \left(\frac{2}{1} \cdot \frac{2}{3}\right)\left(\frac{4}{3} \cdot \frac{4}{5}\right)\left(\frac{6}{5} \cdot \frac{6}{7}\right)\left(\frac{8}{7} \cdot \frac{8}{9}\right)\left(\frac{10}{9} \cdot \frac{10}{11}\right) \ldots . \square$$

Wallis Formula of 7.3, follows from 15.7, and 15.8.

16

Products of $\Gamma(z)$

16.1 $\dfrac{\Gamma(1+z_1)}{\Gamma(1+w_1)\Gamma(1+w_2)}$, **where** $z_1 = w_1 + w_2$

$$\frac{\Gamma(1+z_1)}{\Gamma(1+w_1)\Gamma(1+w_2)} = \frac{\left(1+w_1\right)\left(1+w_2\right)}{\left(1+z_1\right)} \times \frac{\left(1+\frac{w_1}{2}\right)\left(1+\frac{w_2}{2}\right)}{\left(1+\frac{z_1}{2}\right)} \times \frac{\left(1+\frac{w_1}{3}\right)\left(1+\frac{w_2}{3}\right)}{\left(1+\frac{z_1}{3}\right)} \times \dots$$

Proof:

$$\frac{\Gamma(1+z_1)}{\Gamma(1+w_1)\Gamma(1+w_2)} =$$

$$= \frac{\left(1+\frac{1}{1}\right)^{1+z_1}\left(1+\frac{1}{2}\right)^{1+z_1}\left(1+\frac{1}{3}\right)^{1+z_1}\dots.}{\left(1+z_1\right)\left(1+\frac{1+z_1}{1}\right)\left(1+\frac{1+z_1}{2}\right)\left(1+\frac{1+z_1}{3}\right)\dots.} \times$$

$$\times \frac{\left(1+w_1\right)\left(1+\frac{1+w_1}{1}\right)\left(1+\frac{1+w_1}{2}\right)\left(1+\frac{1+w_1}{3}\right)\dots}{\left(1+\frac{1}{1}\right)^{1+w_1}\left(1+\frac{1}{2}\right)^{1+w_1}\left(1+\frac{1}{3}\right)^{1+w_1}\dots} \times$$

$$\times \frac{\left(1+w_2\right)\left(1+\frac{1+w_2}{1}\right)\left(1+\frac{1+w_2}{2}\right)\left(1+\frac{1+w_2}{3}\right)\dots}{\left(1+\frac{1}{1}\right)^{1+w_2}\left(1+\frac{1}{2}\right)^{1+w_2}\left(1+\frac{1}{3}\right)^{1+w_2}\dots}.$$

$$= \frac{\left(1+\frac{1}{1}\right)^{z_1}\left(1+\frac{1}{2}\right)^{z_1}\left(1+\frac{1}{3}\right)^{z_1}\dots.}{\left(1+z_1\right)\left(1+\frac{z_1}{2}\right)\left(1+\frac{z_1}{3}\right)\left(1+\frac{z_1}{4}\right)\dots.} \times$$

$$\times\frac{\left(1+w_1\right)\left(1+\frac{w_1}{2}\right)\left(1+\frac{w_1}{3}\right)\left(1+\frac{w_1}{4}\right)\ldots}{\left(1+\frac{1}{1}\right)^{w_1}\left(1+\frac{1}{2}\right)^{w_1}\left(1+\frac{1}{3}\right)^{w_1}\ldots}\times$$

$$\times\frac{\left(1+w_2\right)\left(1+\frac{w_2}{2}\right)\left(1+\frac{w_2}{3}\right)\left(1+\frac{w_2}{4}\right)\ldots}{\left(1+\frac{1}{1}\right)^{w_2}\left(1+\frac{1}{2}\right)^{w_2}\left(1+\frac{1}{3}\right)^{w_2}\ldots}.$$

$$=\frac{\left(1+w_1\right)\left(1+\frac{w_1}{2}\right)\left(1+\frac{w_1}{3}\right)\ldots\left(1+w_2\right)\left(1+\frac{w_2}{2}\right)\left(1+\frac{w_2}{3}\right)\ldots}{\left(1+z_1\right)\left(1+\frac{z_1}{2}\right)\left(1+\frac{z_1}{3}\right)\left(1+\frac{z_1}{4}\right)\ldots}$$

$$=\frac{\left(1+w_1\right)\left(1+w_2\right)}{\left(1+z_1\right)}\times\frac{\left(1+\frac{w_1}{2}\right)\left(1+\frac{w_2}{2}\right)}{\left(1+\frac{z_1}{2}\right)}\times\frac{\left(1+\frac{w_1}{3}\right)\left(1+\frac{w_2}{3}\right)}{\left(1+\frac{z_1}{3}\right)}\times\ldots$$

16.2 $$\frac{\Gamma(1)}{\Gamma(1+ix)\Gamma(1-ix)}=\sinh x$$

Proof:

$$\frac{\Gamma(1)}{\Gamma(1+ix)\Gamma(1-ix)}=\left(1+ix\right)\left(1-ix\right)\left(1+\frac{ix}{2}\right)\left(1-\frac{ix}{2}\right)\left(1+\frac{ix}{3}\right)\left(1-\frac{ix}{3}\right)\ldots$$

$$=\left(1+x^2\right)\left(1+\frac{x^2}{2^2}\right)\left(1+\frac{x^2}{3^2}\right)\ldots$$

$$=\sinh x\,.\square$$

Similarly, we obtain

16.3 $\dfrac{\Gamma(1+z_1)\Gamma(1+z_2)}{\Gamma(1+w_1)\Gamma(1+w_2)\Gamma(1+w_3)}$ **, where** $z_1 + z_2 = w_1 + w_2 + w_3$.

$$\frac{\Gamma(1+z_1)\Gamma(1+z_2)}{\Gamma(1+w_1)\Gamma(1+w_2)\Gamma(1+w_3)} =$$

$$= \frac{\left(1+w_1\right)\left(1+w_2\right)\left(1+w_3\right)}{\left(1+z_1\right)\left(1+z_2\right)} \times \frac{\left(1+\frac{w_1}{2}\right)\left(1+\frac{w_2}{2}\right)\left(1+\frac{w_3}{2}\right)}{\left(1+\frac{z_1}{2}\right)\left(1+\frac{z_2}{2}\right)} \times$$

$$\times \frac{\left(1+\frac{w_1}{3}\right)\left(1+\frac{w_2}{3}\right)\left(1+\frac{w_3}{3}\right)}{\left(1+\frac{z_1}{3}\right)\left(1+\frac{z_2}{3}\right)} \times \ldots$$

More generally,

16.4 If $z_1 + z_2 + \ldots + z_k = w_1 + w_2 + \ldots + w_l$,

$\dfrac{\Gamma(1+z_1)\Gamma(1+z_2)\ldots\Gamma(1+z_k)}{\Gamma(1+w_1)\Gamma(1+w_2)\ldots.\Gamma(1+w_l)}$ may be simplified

A weaker result that requires that $k = l$, is stated in [Melz, p. 101], and in [Rain, p. 249].

17

Product Calculus of $J_\nu(x)$

For a complex number ν, the Bessel function $J_\nu(z)$ solves Bessel's differential equation

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - \nu^2) w = 0.$$

For ν real, $J_\nu(z)$ has infinitely many real zeros, all simple with the possible exception of $z = 0$.

For $\nu \geq 0$, the positive zeros $j_{\nu,k}$ are a monotonic increasing sequence

$$j_{\nu,1} < j_{\nu,2} < j_{\nu,3} < \ldots$$

17.1 Product Formula for $J_\nu(z)$ [Abram, p.370]

$$\boxed{J_\nu(z) = \left(\frac{z}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)} \left(1 - \frac{z^2}{j_{\nu,1}^2}\right)\left(1 - \frac{z^2}{j_{\nu,2}^2}\right)\left(1 - \frac{z^2}{j_{\nu,3}^2}\right)\ldots}$$

17.2 Geometric Mean Derivative of $J_\nu(z)$

$$\boxed{\frac{DJ_\nu(x)}{J_\nu(x)} = \frac{\nu}{x} - \frac{2x}{j_{\nu,1}^2 - x^2} - \frac{2x}{j_{\nu,2}^2 - x^2} - \frac{2x}{j_{\nu,3}^2 - x^2} - \ldots}$$

Proof: $D^{(0)}J_{\nu}(x) = D^{(0)}\dfrac{1}{2^{\nu}\Gamma(\nu+1)}D^{(0)}x^{\nu} \times$

$$\times D^{(0)}\left(1-\frac{x^2}{j_{\nu,1}^2}\right)D^{(0)}\left(1-\frac{x^2}{j_{\nu,2}^2}\right)D^{(0)}\left(1-\frac{x^2}{j_{\nu,3}^2}\right)...$$

$$= e^0 e^{\frac{Dx^{\nu}}{x^{\nu}}} e^{\frac{D(1-x^2/j_{\nu,1}^2)}{1-x^2/j_{\nu,1}^2}} e^{\frac{D(1-x^2/j_{\nu,2}^2)}{1-x^2/j_{\nu,2}^2}} e^{\frac{D(1-x^2/j_{\nu,3}^2)}{1-x^2/j_{\nu,3}^2}} ...$$

$$= e^{\frac{\nu}{x}} e^{\frac{-2x}{j_{\nu,1}^2-x^2}} e^{\frac{-2x}{j_{\nu,2}^2-x^2}} e^{\frac{-2x}{j_{\nu,3}^2-x^2}} ...$$

$$= e^{\frac{\nu}{x}} e^{-\frac{2x}{j_{\nu,1}^2-x^2}-\frac{2x}{j_{\nu,1}^2-x^2}-\frac{2x}{j_{\nu,2}^2-x^2}} ...$$

Thus,

$$\frac{DJ_{\nu}(x)}{J_{\nu}(x)} = \frac{\nu}{x} - \frac{2x}{j_{\nu,1}^2 - x^2} - \frac{2x}{j_{\nu,2}^2 - x^2} - \frac{2x}{j_{\nu,3}^2 - x^2} - ... \square$$

18

Product Calculus of Trigonometric Series

If

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

the Geometric Mean Derivative can be applied to

$$e^{\left(f(x)-\frac{1}{2}a_0\right)} = e^{\left(a_1 \cos\frac{\pi x}{L} + b_1 \sin\frac{\pi x}{L}\right)} e^{\left(a_2 \cos\frac{2\pi x}{L} + b_2 \sin\frac{2\pi x}{L}\right)} \dots$$

18.1 Product Integral of a Trigonometric Series

on $[0,2\pi]$,

$$x = \pi - 2\left(\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots\right)$$

Therefore,

$$e^{x-\pi} = e^{-2\sin x} e^{-2\frac{\sin 2x}{2}} e^{-2\frac{\sin 3x}{3}} \dots .$$

Product Integrating both sides,

$$e^{\left(\frac{1}{2}x^2 - \pi x\right)} = e^{2\cos x} e^{\left(2\frac{\cos 2x}{2^2}\right)} e^{\left(2\frac{\cos 3x}{3^2}\right)} \dots$$

Hence,

$$\frac{1}{2}x^2 - \pi x = 2\cos x + 2\frac{\cos 2x}{2^2} + 2\frac{\cos 3x}{3^2} + \dots$$

19

Infinite Functional Products

Euler represented Analytic functions by infinite products, [Saks].

to which the Geometric Mean derivative may be applied.

19.1 Geometric Mean Derivative of an Euler Product

Consider Euler's product

$$\frac{1}{1-x} = \left(1+x\right)\left(1+x^{2}\right)\left(1+x^{2^{2}}\right)\left(1+x^{2^{3}}\right).....$$

Applying the Geometric Mean Derivative to both sides,

$$e^{\left(\frac{1}{1-x}\right)} = e^{\frac{1}{1+x}}e^{\frac{2x^{2-1}}{1+x^{2}}}e^{\frac{2^{2}x^{2^{2}-1}}{1+x^{2^{2}}}}e^{\frac{2^{3}x^{2^{3}-1}}{1+x^{2^{3}}}}....e^{\frac{2^{n}x^{2^{n}-1}}{1+x^{2^{n}}}}...$$

Hence,

$$\frac{1}{1-x} = \frac{1}{1+x} + \frac{2^{1}x^{2^{1}-1}}{1+x^{2^{1}}} + \frac{2^{2}x^{2^{2}-1}}{1+x^{2^{2}}} + ...\frac{2^{n}x^{2^{n}-1}}{1+x^{2^{n}}} + ...$$

20

Path Product Integral

20.1 Path Product Integral in the plane

Let $P(x,y)$, and $Q(x,y)$ be positive, and smooth so that

$\log P(x,y)$ is integrable with respect to x,

and

$\log Q(x,y)$ is integrable with respect to y,

along the path γ in the x,y Plane, from (x_1,y_1) to (x_2,y_2).

We define the Product Integral along the path γ by

$$\prod_{\gamma}\big(P(x,y)\big)^{dx}\big(Q(x,y)\big)^{dy} \equiv \prod_{\gamma}\big(P(x,y)\big)^{dx}\prod_{\gamma}\big(Q(x,y)\big)^{dy}$$

$$= e^{\int_{\gamma}\left(\log P(x,y)\right)dx}\, e^{\int_{\gamma}\left(\log Q(x,y)\right)dy}$$

$$\boxed{= e^{\int_{\gamma}\left(\log P(x,y)\right)dx+\left(\log Q(x,y)\right)dy}}$$

20.2 Green's Theorem for the Path Product Integral

Path Product Integral over a loop equals

$$\boxed{e^{\iint_{inerior(\gamma)}\left(\frac{\partial_y P}{P}-\frac{\partial_x Q}{Q}\right)dxdy}}$$

Proof: By Green's Theorem.□

20.3 Path Product Integral in E^3

If γ is a path on a smooth surface in three dimensional space, we define the path product Integral along γ by

$$\boxed{\prod_{\gamma}\left(P(x,y,z)\right)^{dx}\left(Q(x,y,z)\right)^{dy}\left(R(x,y,z)\right)^{dz} \equiv e^{\int_{\gamma}\left(\log P(x,y,z)\right)dx+\left(\log Q(x,y,z)\right)dy+\left(\log R(x,y,z)\right)dz}}$$

20.4 Stokes Theorem for the Path Product Integral

path product Integral over a loop in a smooth surface in E^3 equals

$$\boxed{e^{\iint \nabla\times\begin{bmatrix}\log P(x,y,z)\\ \log Q(x,y,z)\\ \log R(x,y,z)\end{bmatrix}\bullet\begin{pmatrix}dydz\\ dxdz\\ dxdy\end{pmatrix}}}.$$

Proof: By Stokes Theorem. □

21

Iterative Product Integral

21.1 Iterative Product Integral of $f(x,t)$

Let $f(x,t)$ be positive in the rectangle

$$[x_0, x_1] \times [t_0, t_1]$$

so that

$$\log f(x,t)$$

is integrable on the rectangle.

Then, the double product integral is defined iteratively by

$$\prod_{t=t_0}^{t=t_1}\left(\prod_{x=x_0}^{x=x_1} f(x,t)^{dx}\right)^{dt} = \prod_{t=t_0}^{t=t_1}\left(e^{\int_{x=x_0}^{x=x_1}\log f(x,t)dx}\right)^{dt}$$

$$\boxed{= e^{\int_{t=t_0}^{t=t_1}\int_{x=x_0}^{x=x_1}\log f(x,t)dxdt}}$$

21.2 Iterative Product Integral of $e^{r(x,t)dx}$

$$\prod_{t=t_0}^{t=t_1}\left(\prod_{x=x_0}^{x=x_1} e^{r(x,t)dx}\right)^{dt} = e^{\int_{t=t_0}^{t=t_1}\int_{x=x_0}^{x=x_1} r(x,t)dxdt}$$

22

Harmonic Mean Integral

22.1 Harmonic Mean Integral

The electrical voltage on a capacitance $C(q)$ due to a charge dq at x is

$$\frac{dq}{C(q)} = \frac{dq(x)}{C(q(x))} \equiv \frac{dx}{f(x)},$$

where the function $f(x)$ is real and non-vanishing.

Over n equal sub-intervals of the interval $[a,b]$, $\Delta x = \dfrac{b-a}{n}$,
we obtain the sequence of voltages

$$\frac{1}{f(x_1)}\Delta x + \frac{1}{f(x_2)}\Delta x + ... + \frac{1}{f(x_n)}\Delta x = \left(\frac{1}{f(x_1)} + \frac{1}{f(x_2)} + ... + \frac{1}{f(x_n)}\right)\Delta x .$$

As $n \to \infty$, the sequence converges to

$$\boxed{\int_{x=a}^{x=b} \frac{1}{f(x)}\, dx}.$$

We call the limit

the Harmonic Mean integral of $f(x)$ *over the interval* $[a,b]$,

and denote it by

$$\underset{x=a}{\overset{x=b}{I^{(-1)}}} f(x).$$

The inverse operation to Harmonic Mean integration is the Harmonic Mean Derivative

23

Harmonic Mean, and Harmonic Mean Derivative

23.1 The Harmonic Mean of $f(x)$ over $[a,b]$

Given an non-vanishing integrable function $f(x)$ over the interval $[a,b]$, partition the interval into n sub-intervals, of equal length

$$\Delta x = \frac{b-a}{n},$$

choose in each subinterval a point

$$c_i,$$

and consider the Harmonic Mean of $f(x)$

$$n\frac{1}{\frac{1}{f(c_1)}+\frac{1}{f(c_2)}+...+\frac{1}{f(c_n)}} = \frac{b-a}{\left(\frac{1}{f(c_1)}+\frac{1}{f(c_2)}+...+\frac{1}{f(c_n)}\right)\Delta x}$$

As $n \to \infty$, the sequence of Harmonic Means converges to

$$\frac{b-a}{\int\limits_{x=a}^{x=b}\frac{1}{f(x)}dx} = \frac{b-a}{H(b)-H(a)},$$

where

$$H(x) = \int_{t=0}^{t=x} \frac{1}{f(t)} dt .$$

Therefore,

$$\boxed{\frac{b-a}{\int_{x=a}^{x=b} \frac{1}{f(x)} dx}}$$

is defined as the *Harmonic Mean of* $f(x)$ *over* $[a,b]$.$\square$

23.2 Mean Value Theorem for the Harmonic Mean

There is a point $a < c < b$*, so that* $\boxed{\frac{b-a}{\int_{x=a}^{x=b} \frac{1}{f(x)} dx} = f(c)}$

23.3 The Harmonic Mean of $f(x)$ over $[x, x+dx]$

The Harmonic Mean at x *is the Inverse operation to Harmonic Mean Integration*

Proof: The Harmonic Mean of $f(x)$ over the interval $[x, x+\Delta x]$, is

$$\frac{\Delta x}{\int_{t=x}^{t=x+\Delta x} \frac{1}{f(t)} dt}.$$

Letting $\Delta x \to 0$, the Harmonic Mean of $f(x)$ at x equals $f(x)$.

$$\lim_{\Delta x \to 0} \frac{\Delta x}{\int\limits_{t=x}^{t=x+\Delta x} \frac{1}{f(t)} dt} = f(x).$$

Thus, the operation of finding the Harmonic Mean of $f(x)$ at the point x, is inverse to Harmonic Mean Integration.□

This leads to the definition of the Harmonic Mean Derivative

23.4 Harmonic Mean Derivative

The Harmonic Mean Derivative of

$$H(x) = \int\limits_{t=0}^{t=x} \frac{1}{f(t)} dt$$

at x *is defined as the Harmonic Mean of* $f(x)$ *at* x

$$\boxed{D^{(-1)}H(x) \equiv \lim_{\Delta x \to 0} \frac{\Delta x}{\int\limits_{t=x}^{t=x+\Delta x} \frac{1}{f(t)} dt}}$$

23.5

$$\boxed{D^{(-1)}H(x) = \frac{1}{D_x H(x)}}$$

Proof:

$$D^{(-1)}H(x) = \text{Standard Part of } \frac{dx}{H(x+dx) - H(x)}$$

$$= \frac{1}{D_x H(x)}.\square$$

24

Harmonic Mean Calculus

24.1 The Fundamental Theorem of the Harmonic Mean Calculus

$$\boxed{D_x^{(-1)}\left(\underset{t=a}{\overset{t=x}{I^{(-1)}}} f(t)\right) = f(x)}.$$

Proof: $$D_x^{(-1)}\left(\underset{t=a}{\overset{t=x}{I^{(-1)}}} f(t)\right) = D_x^{(-1)}\left(\int_{t=a}^{t=x} \frac{1}{f(t)}dt\right)$$

$$= \frac{1}{D_x \int_{t=a}^{t=x} \frac{1}{f(t)}dt}$$

$$= f(x).\square$$

24.2 Table of Harmonic Mean Derivatives and integrals

We list some Harmonic Mean Derivatives, and Integrals.

$f(x)$	$D^{(-1)}f(x)$	$I^{(-1)}f(x)$
x	1	$\log x$
x^2	$\frac{1}{2x}$	$-\frac{1}{x}$
x^a	$\frac{1}{ax^{a-1}}$	$\frac{1}{(a-1)x^{a-1}}$
$\frac{1}{x}$	$-x^2$	$\frac{1}{2}x^2$
$\log x$	x	$\int \frac{1}{\log x} dx$
e^x	e^{-x}	$-e^{-x}$
e^{2x}	$\frac{1}{2}e^{-2x}$	$-\frac{1}{2}e^{-2x}$
x^x	$\frac{1}{x^x(\log x+1)}$	$\int \frac{1}{x^x} dx$
$\sin x$	$\frac{1}{\cos x}$	$\int \frac{dx}{\sin x}$
$\cos x$	$\frac{1}{\sin x}$	$\int \frac{dx}{\cos x}$
$\tan x$	$\sin^2 x$	$\log \sin x$
e^{e^x}	$e^{-x}e^{-e^x}$	$\int \frac{1}{e^{e^x}} dx$
$e^{\sin x}$	$\cos x e^{\sin x}$	$\int e^{-\sin x} dx$
$e^{\cos x}$	$-\sin x e^{\cos x}$	$\int e^{-\cos x} dx$

25

Quadratic Mean Integral

25.1 Quadratic Mean Integral

Given a Riemann integrable, positive $f(x)$ on $[a,b]$, partition the interval into n sub-intervals, of equal length

$$\Delta x = \frac{b-a}{n},$$

choose in each subinterval a point

$$c_i,$$

and consider the finite products,

$$\left(f^2(c_1) + f^2(c_2) + ... + f^2(c_n)\right)\Delta x$$

As $n \to \infty$, the sequence converges to

$$\boxed{\int_{x=a}^{x=b} f^2(x)dx}.$$

We call this limit

the Quadratic Mean Integral of $f(x)$ over the interval $[a,b]$,

and denote it by

$$\boxed{\underset{x=a}{\overset{x=b}{I^{(2)}}} f(x) = \left\| f \right\|^2_{L^2[a,b]}}$$

Thus, Quadratic Mean integration transforms a function to its L^2 norm squared.

25.2 Cauchy-Schwartz inequality for Quadratic Mean Integrals

$$\left(\underset{x=a}{\overset{x=b}{I^{(2)}}}\big[f(x)+g(x)\big]\right)^{\frac{1}{2}} \leq \left(\underset{x=a}{\overset{x=b}{I^{(2)}}} f(x)\right)^{\frac{1}{2}} + \left(\underset{x=a}{\overset{x=b}{I^{(2)}}} g(x)\right)^{\frac{1}{2}}$$

Proof:

$$\left(\underset{x=a}{\overset{x=b}{I^{(2)}}}\big[f(x)+g(x)\big]\right)^{\frac{1}{2}} = \left(\int_{x=a}^{x=b} \big(f(x)+g(x)\big)^2\,dx\right)^{\frac{1}{2}}$$

By Cauchy-Schwartz Inequality,

$$\leq \left(\int_{x=a}^{x=b} \big(f(x)\big)^2\,dx\right)^{1/2} + \left(\int_{x=a}^{x=b} \big(g(x)\big)^2\,dx\right)^{1/2}$$

$$= \left(\underset{x=a}{\overset{x=b}{I^{(2)}}} f(x)\right)^{\frac{1}{2}} + \left(\underset{x=a}{\overset{x=b}{I^{(2)}}} g(x)\right)^{\frac{1}{2}}. \square$$

25.3 Holder Inequality for Quadratic Mean Integrals

$$\int_{x=a}^{x=b} \big|f(x)g(x)\big|dx \leq \left(\underset{x=a}{\overset{x=b}{I^{(2)}}} f(x)\right)^{\frac{1}{2}} \left(\underset{x=a}{\overset{x=b}{I^{(2)}}} g(x)\right)^{\frac{1}{2}}$$

Proof: By Holder's inequality,

$$\int_{x=a}^{x=b} \big|f(x)g(x)\big|dx \leq \left(\int_{x=a}^{x=b} f^2(x)dx\right)^{1/2} \left(\int_{x=a}^{x=b} g^2(x)dx\right)^{1/2}$$

$$\leq \left(\underset{x=a}{\overset{x=b}{I^{(2)}}} f(x) \right)^{\frac{1}{2}} \left(\underset{x=a}{\overset{x=b}{I^{(2)}}} g(x) \right)^{\frac{1}{2}}. \square$$

26

Quadratic Mean and Quadratic Mean Derivative

26.1 Quadratic Mean of $f(x)$ over $[a,b]$

Given an integrable positive function $f(x)$ on $[a,b]$, partition the interval into n sub-intervals, of equal length $\Delta x = \frac{b-a}{n}$,

choose in each subinterval a point c_i,

and consider the Quadratic Means of $f(x)$

$$\left(\frac{f^2(c_1)+f^2(c_2)+...+f^2(c_n)}{n}\right)^{1/2} = \left(\frac{1}{b-a}\Big(f^2(c_1)+f^2(c_2)+...+f^2(c_n)\Big)\Delta x\right)^{1/2}$$

As $n \to \infty$, the sequence of Quadratic Means converges to

$$\left(\frac{1}{b-a}\int_{x=a}^{x=b} f^2(x)dx\right)^{1/2} = \left(\frac{Q(b)-Q(a)}{b-a}\right)^{1/2},$$

where

$$Q(x) = \int_{t=0}^{t=x} f^2(t)dt.$$

$$\boxed{\left(\frac{1}{b-a}\int_{x=a}^{x=b} f^2(x)dx\right)^{1/2}}$$

is defined as the *Quadratic Mean of* $f(x)$ *over* $[a,b]$.□

26.2 Mean Value theorem for the Quadratic Mean

There is a point $a < c < b$*, so that* $\boxed{\left(\frac{1}{b-a}\int\limits_{x=a}^{x=b} f^2(x)dx\right)^{\frac{1}{2}} = f(c)}$

26.3 The Quadratic Mean of $f(x)$ over $[x, x+dx]$

The Quadratic Mean at x*, is the Inverse operation to Quadratic Mean Integration*

Proof: By 26.2, there is

$$x < c < x + \Delta x,$$

so that

$$\left(\frac{1}{\Delta x}\int\limits_{t=x}^{t=x+\Delta x} f^2(t)dt\right)^{1/2} = f(c).$$

Letting $\Delta x \to 0$, the Quadratic Mean of $f(x)$ at x equals $f(x)$.

$$\lim_{\Delta x \to 0}\left(\frac{1}{\Delta x}\int\limits_{t=x}^{t=x+\Delta x} f^2(t)dt\right)^{1/2} = f(x).$$

Thus, the operation of finding the Quadratic Mean of $f(x)$ at the point x, is inverse to Quadratic Mean Integration.□

This leads to the definition of Quadratic Mean Derivative

26.4 Quadratic Mean Derivative

The Quadratic Mean Derivative of

$$Q(x) = \int_{t=0}^{t=x} f^2(t)dt$$

at x *is defined as the Quadratic Mean of* $f(x)$ *at* x

$$\boxed{D^{(2)}Q(x) \equiv \lim_{\Delta x \to 0}\left(\frac{1}{\Delta x}\int_{t=x}^{t=x+\Delta x} f^2(t)dt\right)^{1/2}}$$

26.5 $$\boxed{D^{(2)}Q(x) = \left(D_x Q(x)\right)^{1/2}}$$

Proof:

$$D^{(2)}Q(x) = \text{Standard Part of }\left(\frac{Q(x+dx)-Q(x)}{dx}\right)^{1/2}$$

$$= \left(D_x Q(x)\right)^{1/2} \square$$

27

Quadratic Mean Calculus

27.1 The Fundamental Theorem of the Quadratic Mean Calculus

$$\boxed{D_x^{(2)}\left(\underset{t=a}{\overset{t=x}{I^{(2)}}} f(t)\right) = f(x)}$$

Proof:

$$D_x^{(2)}\left(\underset{t=a}{\overset{t=x}{I^{(2)}}} f(t)\right) = D_x^{(2)}\left(\int_{t=a}^{t=x} \left(f(t)\right)^2 dt\right)$$

$$= \left[D_x\left(\int_{t=a}^{t=x} \left(f(t)\right)^2 dt\right)\right]^{\frac{1}{2}}$$

$$= f(x).\square$$

27.2 Table of Quadratic Mean Derivatives and integrals

We list some Quadratic Mean Derivatives, and Integrals.

$f(x)$	$D^{(2)}f(x)$	$I^{(2)}f(x)$
1	0	x
2	0	$4x$
a	0	a^2x
x	1	$\frac{1}{3}x^3$
x^2	$\sqrt{2\lvert x\rvert}$	$\frac{1}{5}x^5$
x^a	$\sqrt{\lvert ax^{a-1}\rvert}$	$\frac{1}{2a+1}x^{2a+1}$
x^{-1}	x^{-1}	$-x^{-1}$
$\ln x$	$\lvert x\rvert^{-1/2}$	$\int(\ln x)^2\,dx$
e^x	$e^{x/2}$	$\frac{1}{2}e^{2x}$
e^{2x}	$\sqrt{2}e^x$	$\frac{1}{4}e^{4x}$
e^{ax}	$\sqrt{a}e^{ax/2}$	$\frac{1}{2a}e^{2ax}$
$\sin x$	$\sqrt{\lvert\cos x\rvert}$	$\int(\sin^2 x)dx$
$\cos x$	$\sqrt{\lvert\sin x\rvert}$	$\int(\cos^2 x)dx$
$\tan x$	$\left\lvert\frac{1}{\sin x}\right\rvert$	$\int(\tan^2 x)dx$
$e^{\sin x}$	$(\cos x)^{1/2}e^{(\sin x)/2}$	$\int e^{2\sin x}dx$
$e^{\cos x}$	$(-\sin x)^{1/2}e^{(\cos x)/2}$	$\int e^{2\cos x}dx$

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