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Fractional Calculus

The Fundamental Theorem of the Fractional Calculus, and the Meaning of Fractional Derivatives

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Abstract: Since the beginning of the 18th century, attempts were made to generalize the Arithmetic Means Derivative, and obtain a Generalized Arithmetic Means Calculus.

As early as 1812, Lacroix [Lac] observed that for $n = 1 ... m$,

$$\frac{d^n}{dx^n} x^m = \frac{m!}{(m-n)!} x^{m-n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n},$$

and by formal substitution of $n = \frac{1}{2}$, obtained [Ross]

$$\frac{d^{1/2}}{(dx)^{1/2}} x = \frac{\Gamma(1+1)}{\Gamma(1-\frac{1}{2}+1)} x^{1-1/2} = \frac{1}{\sqrt{\pi}\ /\ 2}\sqrt{x}\,.$$

In 1832, Liouville [Liou] used the Euler Gamma Integral formula

$$\Gamma(a)x^{-a} = \int_{u=0}^{u=\infty} u^{a-1}e^{-xu}du$$

Differentiating both sides to an order ν, he obtained

$$\Gamma(a)\frac{d^\nu}{(dx)^\nu}x^{-a} = \int\limits_{u=0}^{u=\infty} u^{a-1}\frac{d^\nu}{(dx)^\nu}e^{-xu}du.$$

Assuming

$$\frac{d^\nu}{(dx)^\nu}e^{-xu} = (-u)^\nu e^{-xu}, \qquad (3)$$

he had

$$\Gamma(a)\frac{d^\nu}{(dx)^\nu}x^{-a} = (-1)^\nu \int\limits_{u=0}^{u=\infty} u^{a+\nu-1}e^{-xu}du$$
$$= (-1)^\nu\Gamma(a+\nu)x^{-(a+\nu)},$$

and concluded [Ross] that

$$\frac{d^\nu}{(dx)^\nu}x^{-a} = (-1)^\nu\frac{\Gamma(a+\nu)}{\Gamma(a)}x^{-(a+\nu)}$$

Many years later, the meaning of Fractional Derivatives is still unclear.

The common perception is that the Fractional Derivative Method represents a new kind of Calculus, such as the Product Calculus, that is based on the Geometric Mean [Dan1],

But so far, the Fractional Derivative Method has been developed only as a refinement of the Arithmetic Means Calculus, and is not a new kind of Calculus, such as the Product Calculus is.

Here, We interpret the Fractional Derivative in the context of the Arithmetic Means Calculus in which it was presented. We note that it can be similarly developed, and interpreted in the Product Calculus.

We proceed with the simplest fraction $q = \frac{1}{2}$. The Fractional integral

$$\frac{d^{-\frac{1}{2}} f}{(dx)^{-\frac{1}{2}}} \equiv D^{-\frac{1}{2}} f \,,$$

and the derivative

$$\frac{d^{\frac{1}{2}} f}{(dx)^{\frac{1}{2}}} \equiv D^{\frac{1}{2}} f \,.$$