The Fundamental Theorem of the Fractional Calculus, and the Meaning of Fractional Derivatives

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Abstract We obtain the Fundamental Theorem of the Fractional Calculus in the Arithmetic Means Calculus, and interpret Fractional Derivatives in the Arithmetic Means Calculus.

Our derivation and interpretation can be extended to Fractional derivatives in any other Power Mean Calculus.

In particular, we define the Fractional Product Integral, the Fractional Product Derivative, and prove the Fundamental Theorem of the Fractional Product Calculus.

Keywords Fractional Calculus, Fractional Derivative, Fractional Integral, Arithmetic Mean Calculus, Product Calculus, Convolution, Tautochrone,

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Contents

Introduction

Since the beginning of the 18th century, attempts were made to generalize the Arithmetic Means Derivative, and obtain a Generalized Arithmetic Means Calculus.

As early as 1812, Lacroix [Lac] observed that for $n = 1...m$,

$$\frac{d^n}{dx^n} x^m = \frac{m!}{(m-n)!} x^{m-n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n},$$

and by formal substitution of $n = \frac{1}{2}$, obtained [Ross]

$$\frac{d^{1/2}}{(dx)^{1/2}} x = \frac{\Gamma(1+1)}{\Gamma(1-\frac{1}{2}+1)} x^{1-1/2} = \frac{1}{\sqrt{\pi}\ /\ 2}\sqrt{x}\,.$$

In 1832, Liouville [Liou] used the Euler Gamma Integral formula

$$\Gamma(a)x^{-a} = \int_{u=0}^{u=\infty} u^{a-1}e^{-xu}du$$

Differentiating both sides to an order ν, he obtained

$$\Gamma(a)\frac{d^\nu}{(dx)^\nu} x^{-a} = \int_{u=0}^{u=\infty} u^{a-1}\frac{d^\nu}{(dx)^\nu} e^{-xu}du\,.$$

Assuming

$$\frac{d^\nu}{(dx)^\nu} e^{-xu} = (-u)^\nu e^{-xu}, \qquad (3)$$

he had

$$\Gamma(a)\frac{d^\nu}{(dx)^\nu} x^{-a} = (-1)^\nu \int_{u=0}^{u=\infty} u^{a+\nu-1}e^{-xu}du$$

$$= (-1)^{\nu}\Gamma(a+\nu)x^{-(a+\nu)},$$

and concluded [Ross] that

$$\frac{d^{\nu}}{(dx)^{\nu}}x^{-a} = (-1)^{\nu}\frac{\Gamma(a+\nu)}{\Gamma(a)}x^{-(a+\nu)}$$

Many years later, the meaning of Fractional Derivatives is still unclear.

The common perception is that the Fractional Derivative Method represents a new kind of Calculus, such as the Product Calculus, that is based on the Geometric Mean [Dan1],

But so far, the Fractional Derivative Method has been developed only as a refinement of the Arithmetic Means Calculus, and is not a new kind of Calculus, such as the Product Calculus is.

Here, We interpret the Fractional Derivative in the context of the Arithmetic Means Calculus in which it was presented. We note that it can be similarly developed, and interpreted in the Product Calculus.

We proceed with the simplest fraction $q = \frac{1}{2}$. The Fractional integral

$$\frac{d^{-\frac{1}{2}}f}{(dx)^{-\frac{1}{2}}} \equiv D^{-\frac{1}{2}}f,$$

and the derivative

$$\frac{d^{\frac{1}{2}}f}{(dx)^{\frac{1}{2}}} \equiv D^{\frac{1}{2}}f.$$

1.

Fractional Integral, and Derivative

1.1 Liouville's Fractional Integral of order $-\frac{1}{2}$

Let the function $f(x)$ be integrable over the interval $[a,b]$.

Liuoville defined the Fractional Integral [Old, p.49]

$$\frac{d^{-\frac{1}{2}}f}{\left(d(x-a)\right)^{-\frac{1}{2}}} \equiv D^{-\frac{1}{2}}f$$

as the convolution of the function $f(x)$, with the Kernel function $\frac{x^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}$, over the interval $[a,x]$

$$\frac{1}{\Gamma(\frac{1}{2})}\int_{u=a}^{u=x}(x-u)^{-\frac{1}{2}}f(u)du \equiv \frac{x^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} * f(x) \equiv F^{(-\frac{1}{2})}(x).$$

We use the notations,

$$\boxed{D^{-\frac{1}{2}}f \equiv \frac{d^{-\frac{1}{2}}f}{\left(d(x-a)\right)^{-\frac{1}{2}}} \equiv \frac{1}{\Gamma(\frac{1}{2})}\int_{u=a}^{u=x}(x-u)^{-\frac{1}{2}}f(u)du \equiv F^{(-\frac{1}{2})}(x)}$$

1.2 Liouville's Fractional Derivative of order $\frac{1}{2}$

Let the function $f(x)$ be integrable over the interval $[a,b]$.

Liuoville defined the Fractional Derivative [Old, p.49]

$$\frac{d^{\frac{1}{2}}f}{\left(d(x-a)\right)^{\frac{1}{2}}} \equiv D^{\frac{1}{2}}f$$

as the convolution of the function $f(x)$, with the Kernel function $\frac{x^{-\frac{3}{2}}}{\Gamma(-\frac{1}{2})}$, over the interval $[a,x]$

$$\frac{1}{\Gamma(-\frac{1}{2})}\int_{u=a}^{u=x}(x-u)^{-\frac{3}{2}}f(u)du \equiv \frac{x^{-\frac{3}{2}}}{\Gamma(-\frac{1}{2})} * f(x).$$

We use the notations

$$\boxed{D^{\frac{1}{2}}f \equiv \frac{d^{\frac{1}{2}}f}{\left(d(x-a)\right)^{\frac{1}{2}}} \equiv \frac{1}{\Gamma(-\frac{1}{2})}\int_{u=a}^{u=x}(x-u)^{-\frac{3}{2}}f(u)du}$$

2.

The Fundamental Theorem of the Fractional Calculus

The meaning of the Fractional derivative follows from the Fundamental Theorem of the Fractional Calculus.

This theorem is nowhere stated in the standard literature, such as the books by Miller and Ross [Mil], or Oldham and Spanier [Old], or Podlubny [Pod], and the meaning of the fractional derivative remains there unknown.

To keep it simple, we focus here on the Fractional Calculus of order $\frac{1}{2}$, and later of order $\frac{1}{3}$.

The Fundamental Theorem of the Fractional Calculus of order $\frac{1}{2}$ is

$$\boxed{D^{\frac{1}{2}} f(x) = DF^{\left(-\frac{1}{2}\right)}\left(x\right)}$$

We first outline the proof of this theorem by an operational derivation. Then, we establish it with a direct proof.

2.1 Operational Derivation of the Fundamental Theorem

Proof Outline:

Since the Fractional Derivative, and Fractional Integral operators apply successively [Old] under the rule

$$\frac{d^{\alpha}}{\big(d(x-a)\big)^{\alpha}}\frac{d^{\beta}}{\big(d(x-a)\big)^{\beta}}f(x)=\frac{d^{\alpha+\beta}}{\big(d(x-a)\big)^{\alpha+\beta}}f(x),$$

we have the Fundamental Theorem of the Fractional Calculus

$$\frac{d^{\frac{1}{2}}}{\big(d(x-a)\big)^{\frac{1}{2}}}\frac{d^{-\frac{1}{2}}}{\big(d(x-a)\big)^{-\frac{1}{2}}}f(x)=f(x).$$

Applying to both sides the Fractional Derivative operator

$$\frac{d^{\frac{1}{2}}}{\big(d(x-a)\big)^{\frac{1}{2}}}$$

we have,

$$\frac{d^{\frac{1}{2}}}{\big(d(x-a)\big)^{\frac{1}{2}}}\frac{d^{\frac{1}{2}}}{\big(d(x-a)\big)^{\frac{1}{2}}}\frac{d^{-\frac{1}{2}}}{\big(d(x-a)\big)^{-\frac{1}{2}}}f(x)=\frac{d^{\frac{1}{2}}}{\big(d(x-a)\big)^{\frac{1}{2}}}f(x)$$

Composing the first two operators, we conclude that

$$\frac{d}{\big(d(x-a)\big)}\frac{d^{-\frac{1}{2}}}{\big(d(x-a)\big)^{-\frac{1}{2}}}f(x)=\frac{d^{\frac{1}{2}}}{\big(d(x-a)\big)^{\frac{1}{2}}}f(x)$$

That is,

$$DF^{(-\frac{1}{2})}\big(x\big)=D^{\frac{1}{2}}f(x).\square$$

This outline becomes the proof if we supplement it with the proof of the exponent rule [Old]. Instead, we give a direct derivation of the Fundamental Theorem.

2.2 Direct Derivation of the Fundamental Theorem

Proof:

$$DF^{(-\frac{1}{2})}(x) = \frac{d}{\left(d(x-a)\right)}\frac{d^{-\frac{1}{2}}}{\left(d(x-a)\right)^{-\frac{1}{2}}}f(x)$$

$$= \frac{d}{\left(d(x-a)\right)}\frac{1}{\Gamma(\frac{1}{2})}\int_{u=a}^{u=x}(x-u)^{-\frac{1}{2}}f(u)du$$

$$= \frac{1}{\Gamma(\frac{1}{2})}\frac{d}{dx}\int_{u=a}^{u=x}(x-u)^{-\frac{1}{2}}f(u)du$$

By Leibnitz Rule for Differentiation of Integrals, [Spieg, p. 95],

$$\frac{d}{dx}\int_{u=\varphi(x)}^{u=\psi(x)}G(x,u)du = \int_{u=\varphi(x)}^{u=\psi(x)}\partial_x G(x,u)du + \varphi'(x)G(x,\varphi(x)) - \psi'(x)G(x,\psi(x)).$$

If we take $G(x,u) = (x-u)^{-\frac{1}{2}}f(u)$, then $G(x,x)$ is not defined. To remove the singularity, we integrate by parts

$$\int_{u=a}^{u=x}(x-u)^{-\frac{1}{2}}f(u)du = \left[2(x-u)^{\frac{1}{2}}f(u)\right]_{u=a}^{u=x} - 2\int_{u=a}^{u=x}(x-u)^{\frac{1}{2}}f'(u)du$$

$$= -2(x-a)^{\frac{1}{2}}f(a) - 2\int_{u=a}^{u=x}(x-u)^{\frac{1}{2}}f'(u)du$$

Therefore,

$$DF^{(-\frac{1}{2})}(x) = -\frac{1}{\Gamma(\frac{1}{2})}\frac{d}{dx}\left\{2(x-a)^{\frac{1}{2}}f(a) + 2\int\limits_{u=a}^{u=x}(x-u)^{\frac{1}{2}}f'(u)du\right\}$$

$$= -\frac{1}{\Gamma(\frac{1}{2})}\left\{(x-a)^{-\frac{1}{2}}f(a) + \frac{d}{dx}\int\limits_{u=a}^{u=x}2(x-u)^{\frac{1}{2}}f'(u)du\right\}$$

If we take $G(x,u) = 2(x-u)^{\frac{1}{2}}f(u)$, then by Leibnitz rule

$$= -\frac{1}{\Gamma(\frac{1}{2})}\left\{(x-a)^{-\frac{1}{2}}f(a) + \int\limits_{u=a}^{u=x}(x-u)^{-\frac{1}{2}}f'(u)du\right\} \qquad (\bigstar)$$

Integrating by parts with respect to u

$$\int\limits_{u=a}^{u=x}(x-u)^{-\frac{1}{2}}f'(u)du = \left[(x-u)^{-\frac{1}{2}}f(u)\right]_{u=a}^{u=x} + \frac{1}{2}\int\limits_{u=a}^{u=x}(x-u)^{-\frac{3}{2}}f(u)du$$

$$= -(x-a)^{-\frac{1}{2}}f(a) + \frac{1}{2}\int\limits_{u=a}^{u=x}(x-u)^{-\frac{3}{2}}f(u)du$$

Substituting into $(\bigstar)$,

$$DF^{(-\frac{1}{2})}(x) = -\frac{1}{\Gamma(\frac{1}{2})}\frac{1}{2}\int\limits_{u=a}^{u=x}(x-u)^{-\frac{3}{2}}f(u)du\,.$$

Since $\Gamma(-\frac{1}{2}+1) = -\frac{1}{2}\Gamma(-\frac{1}{2})$,

$$DF^{(-\frac{1}{2})}(x) = \frac{1}{\Gamma(-\frac{1}{2})}\int\limits_{u=a}^{u=x}(x-u)^{-\frac{3}{2}}f(u)du$$

$$= D^{\frac{1}{2}}f(x)\,.\square$$

3.

The Meaning of the Fractional Derivative

The Fundamental Theorem of the Fractional Calculus of order $\frac{1}{2}$ is the key to the meaning of the Fractional Derivative of order $\frac{1}{2}$.

The following are interpretations of the Fundamental Theorem of the Fractional Calculus of order $\frac{1}{2}$, stated as

$$D^{\frac{1}{2}}f(x) = DF^{(-\frac{1}{2})}(x).$$

3.1 Slope of the Convolution Transform

The Fractional Derivative of order $\frac{1}{2}$ *of* $f(x)$ *at* x *is the slope of the tangent to the curve of the Convolution Transform* $F^{(-\frac{1}{2})}(x)$ *at* x.

3.2 Rate of Change of the Convolution Transform

The Fractional Derivative of order $\frac{1}{2}$ *of* $f(x)$ *at* x *is the rate of Change of* $F^{(-\frac{1}{2})}(x)$ *at* x.

In [Dan1], we presented the Arithmetic Mean Calculus, and interpreted the Fermat-Newton-Leibnitz Derivative in it.

In terms of the Arithmetic Mean Calculus, we have

3.3 Arithmetic Mean of the Convolution Transform

The Fractional Derivative of order $\frac{1}{2}$ *of* $f(x)$ *at* x *is the Arithmetic Mean of the Convolution Transform* $F^{(-\frac{1}{2})}(x)$ *over* $[x, x+dx]$.

More generally,

3.4 Fractional Calculus, and Arithmetic Mean Calculus

The Fractional Calculus of order $\frac{1}{2}$ *of* $f(x)$ *is the Arithmetic Mean Calculus of the Convolution Transform* $F^{(-\frac{1}{2})}(x)$.

3.5 The Meaning of Higher Derivatives of order $\frac{1}{2}$

The Fundamental Theorem of the Fractional Calculus of order $\frac{1}{2}$ enables us to interpret Fractional Derivatives of order $n+\frac{1}{2}$ as higher derivatives of the Convolution Transform $F^{(-\frac{1}{2})}(x)$. In particular,

$$D^{\frac{3}{2}}f(x) = DD^{\frac{1}{2}}f(x) = D^2F^{(-\frac{1}{2})}(x)$$

$$D^{\frac{5}{2}}f(x) = D^2D^{\frac{1}{2}}f(x) = D^3F^{(-\frac{1}{2})}(x)$$

The meaning of order $n+\frac{1}{2}$ Fractional Derivative follows from

$$D^{n+\frac{1}{2}}f(x)=D^{n}D^{\frac{1}{2}}f(x)=D^{n+1}F^{(-\frac{1}{2})}(x).$$

3.6 The Applicability of the Fractional Calculus

The meaning of the Fractional Derivative enables us to evaluate the applicability of the Fractional Calculus in simple terms:

The Fractional Derivative of order $\frac{1}{2}$ may be useful in a model that involves the Convolution Transform $F^{(-\frac{1}{2})}(x)$. That is,

If you can see the Convolution Transform underlying your model, you may be well served by the Fractional Derivative.

The solution of the Tautochrone Problem in the next section with the aid of the Fundamental Theorem, shows that

the required Convolution Transform modeling and identification is far from self-evident.

4.

The Tautochrone Problem and the Fundamental Theorem

The Tautochrone Problem is associated with Abel [Abel]. Its solution by Abel involves the use of the Fundamental Theorem.

4.1 The Tautochrone Problem

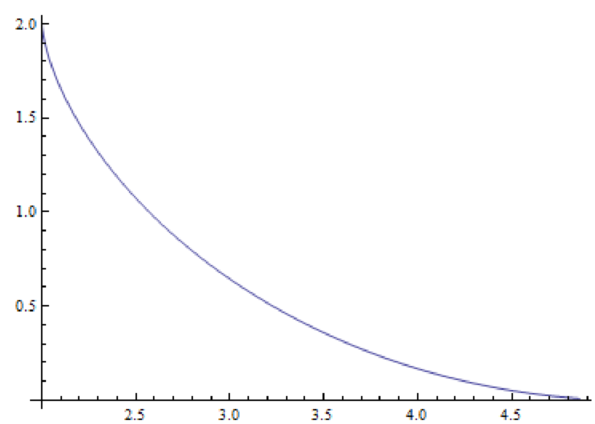
A bead of mass m slips along a frictionless wire,

from height

$$\eta = y, \text{ at time } t = 0$$

to height

$$\eta = 0, \text{ at time } t = T.$$



We want to find the smooth curve -if there is such curve- along which the bead's falling time is the same for any y. To that end, we look for a differential equation that will describe such motion.

4.2 The Tautochrone Differential Equation

$$\frac{ds}{\sqrt{(y-\eta)}} = -\sqrt{2g}dt$$

Proof:

The potential energy

$$mg(y-\eta)$$

converts into the kinetic energy

$$\tfrac{1}{2}m\left(\frac{ds}{dt}\right)^2,$$

where

$$s = s(\eta)$$

is the arc-length of the curve. Therefore,

$$\frac{ds}{dt} = \pm\sqrt{2g(y-\eta)}.$$

Since the particle speed is in the direction of $-y$, we have

$$\frac{ds}{dt} = -\sqrt{2g(y-\eta)}$$

$$\frac{s'(\eta)d\eta}{\sqrt{(y-\eta)}} = -\sqrt{2g}dt. \square$$

Integrating the right side over the time interval, and the left over the height interval we obtain an integral equation for the curve of the bead's motion:

4.3 The Tautochrone Integral Equation

$$\int_{\eta=0}^{\eta=y} \frac{1}{\sqrt{(y-\eta)}} s'(\eta) d\eta = \sqrt{2g}T \equiv C .$$

Now, the Convolution Transform on the left hand side of the integral equation, can be written as a Fractional Integral of order $\frac{1}{2}$, and we obtain the Tautochrone Fractional Equation

4.4 The Tautochrone Fractional Equation

$$D^{-\frac{1}{2}} s'(y) = \frac{C}{\Gamma(\frac{1}{2})}$$

This equation is attributed to Abel [Abel]

To solve this Fractional Equation, we apply the Fundamental Theorem of the Fractional calculus of order $\frac{1}{2}$:

4.5 The Tautochrone and the Fundamental Theorem

$$s'(y) = \frac{C}{\pi\sqrt{y}}$$

Proof:

Applying $D^{\frac{1}{2}}$ to both sides of 4.4,

$$D^{\frac{1}{2}} D^{-\frac{1}{2}} s'(y) = D^{\frac{1}{2}} \frac{C}{\Gamma(\frac{1}{2})}$$

By the Fundamental Theorem,

$$s'(y) = D^{\frac{1}{2}} \frac{C}{\Gamma(\frac{1}{2})}.$$

Since $D^{\frac{1}{2}}$ is a linear operator,

$$s'(y) = \frac{C}{\Gamma(\frac{1}{2})} D^{\frac{1}{2}} y^0 .$$

By Lacroix,

$$D^{\frac{1}{2}} y^0 = \frac{\Gamma(0+1)}{\Gamma(0-\frac{1}{2}+1)} y^{-\frac{1}{2}} = \frac{1}{\Gamma(\frac{1}{2})} y^{-\frac{1}{2}}.$$

Hence,

$$s'(y) = \frac{C}{\Gamma(\frac{1}{2})} \frac{1}{\Gamma(\frac{1}{2})} y^{-\frac{1}{2}} = \frac{C}{\pi\sqrt{y}}.\square$$

Alternatively, using Liouville's definition,

$$D^{\frac{1}{2}} y^0 = \frac{1}{\Gamma(-\frac{1}{2})} \int_{\eta=0}^{\eta=y} (y-\eta)^{\frac{3}{2}} d\eta .$$

But the integral diverges.

To resolve that, we apply the Fundamental Theorem,

$$DD^{-\frac{1}{2}} y^0 = D \frac{1}{\Gamma(\frac{1}{2})} \int_{\eta=0}^{\eta=y} (y-\eta)^{-\frac{1}{2}} d\eta$$

$$= \frac{1}{\Gamma(\frac{1}{2})} D \left\{ \left[-2(y-\eta)^{\frac{1}{2}} \right]_{\eta=0}^{\eta=y} \right\}$$

$$= \frac{1}{\Gamma(\frac{1}{2})} D\left\{2y^{\frac{1}{2}}\right\}$$

$$= \frac{1}{\Gamma(\frac{1}{2})} y^{-\frac{1}{2}}.\square$$

Next, we see that this is the differential equation of a cycloid. Thus, the motion is along a cycloid.

4.6 The Cycloid Differential Equation

$$dx = \sqrt{\frac{a}{y} - 1}dy\,, \qquad \text{where } a = \frac{C^2}{\pi^2}$$

Proof:

By 4.5,

$$\left(\frac{ds}{dy}\right)^2 = \frac{C^2}{\pi^2}\frac{1}{y} \equiv \frac{a}{y}$$

$$\frac{(dx)^2 + (dy)^2}{(dy)^2} = \frac{a}{y}$$

$$dx = \sqrt{\frac{a}{y} - 1}dy\,.\square$$

The cycloid may be given by its parametric equations:

4.7 The Cycloid Parametric Equations

$$x(\alpha) = \tfrac{a}{2}(2\alpha + \sin 2\alpha),$$

$$y(\alpha) = \tfrac{a}{2}(1 - \cos 2\alpha),$$

Proof:

By 5.6,

$$x = \int_{\eta=0}^{\eta=y} \sqrt{\frac{a}{\eta} - 1}\, d\eta.$$

Substituting

$$\eta = a\sin^2\theta,$$

$$y = a\sin^2\alpha,$$

we have,

$$\sqrt{\frac{a}{\eta} - 1} = \sqrt{\frac{1}{\sin^2\theta} - 1} = \frac{\cos\theta}{\sin\theta},$$

$$d\eta = 2a\sin\theta\cos\theta d\theta,$$

$$x = 2a\int_{\theta=0}^{\theta=\alpha} \cos^2\theta d\theta$$

$$= a\int_{\theta=0}^{\theta=\alpha} (1 + \cos 2\theta) d\theta$$

$$= \tfrac{a}{2}(2\alpha + \sin 2\alpha).$$

$$y = a\sin^2\alpha$$

$$= \tfrac{a}{2}(1 - \cos 2\alpha).\square$$

The Tautochrone problem is treated by Nahin [Nahin] by traditional methods.

5.

The Fractional Calculus of order $\frac{1}{3}$

We outline the interpretation of the Fractional Derivative of order $\frac{1}{3}$ in the context of the Arithmetic Mean Calculus.

5.1 Liouville's Fractional Integral of order $-\frac{1}{3}$

$$\boxed{D^{-\frac{1}{3}}f(x) \equiv \frac{d^{-\frac{1}{3}}f(x)}{\big(d(x-a)\big)^{-\frac{1}{3}}} \equiv \frac{1}{\Gamma(\frac{1}{3})}\int_{u=a}^{u=x}(x-u)^{-\frac{2}{3}}f(u)du \equiv F^{(-\frac{1}{3})}(x)}$$

5.2 Liouville's Fractional Integral of order $-\frac{2}{3}$

$$\boxed{D^{-\frac{2}{3}}f(x) \equiv \frac{d^{-\frac{2}{3}}f}{\big(d(x-a)\big)^{-\frac{2}{3}}} \equiv \frac{1}{\Gamma(\frac{2}{3})}\int_{u=a}^{u=x}(x-u)^{-\frac{1}{3}}f(u)du \equiv F^{(-\frac{2}{3})}(x)}$$

5.3 Liouville's Fractional Derivative of order $\frac{1}{3}$

$$\boxed{D^{\frac{1}{3}}f(x) \equiv \frac{d^{\frac{1}{3}}f}{\big(d(x-a)\big)^{\frac{1}{3}}} \equiv \frac{1}{\Gamma(-\frac{1}{3})}\int_{u=a}^{u=x}(x-u)^{-\frac{4}{3}}f(u)du}$$

5.4 Liouville's Fractional Derivative of order $\frac{2}{3}$

$$\boxed{D^{\frac{2}{3}}f(x) \equiv \frac{d^{\frac{2}{3}}f}{\big(d(x-a)\big)^{\frac{2}{3}}} \equiv \frac{1}{\Gamma(-\frac{2}{3})}\int_{u=a}^{u=x}(x-u)^{-\frac{5}{3}}f(u)du}$$

5.5 The Fundamental Theorems of the Fractional Calculus of order $\frac{1}{3}$

The Fundamental Theorem of the Fractional Calculus of order $\frac{1}{3}$ is

$$\boxed{D^{\frac{1}{3}}f(x) = DD^{-\frac{2}{3}}f(x) = DF^{(-\frac{2}{3})}\big(x\big)}$$

This statement is equivalent to

$$\boxed{D^{\frac{2}{3}}f(x) = DD^{-\frac{1}{3}}f(x) = DF^{(-\frac{1}{3})}\big(x\big)}$$

5.6 The Meaning of Higher Derivatives of order $\frac{1}{3}$

The Fundamental Theorem of the Fractional Calculus of order $\frac{1}{3}$ enables us to interpret Fractional Derivatives of order $n+\frac{1}{3}$, as higher derivatives of the Convolution Transform $F^{(-\frac{2}{3})}(x)$, and Fractional Derivatives of order $n+\frac{2}{3}$ as higher derivatives of $F^{(-\frac{1}{3})}(x)$. In particular,

$$D^{\frac{4}{3}}f(x) = DD^{\frac{1}{3}}f(x) = D^2F^{(-\frac{2}{3})}(x)$$

In general,

$$D^{n+\frac{1}{3}}f(x) = D^n D^{\frac{1}{3}}f(x) = D^{n+1}F^{(-\frac{2}{3})}(x)$$

Similarly,

$$D^{\frac{5}{3}}f(x) = DD^{\frac{2}{3}}f(x) = D^2F^{(-\frac{1}{3})}(x),$$

and

$$D^{n+\frac{2}{3}}f(x) = D^n D^{\frac{2}{3}}f(x) = D^{n+1}F^{(-\frac{1}{3})}(x).$$

6.

The Fractional Product Calculus

We developed the Product Calculus, or Geometric Mean Calculus in [Dan1]. The Product Derivative and the Product Integral of that Calculus are related by the Fundamental Theorem of the Product Calculus [Dan1]

The Fractional Derivative, and the Fractional Integral can be defined in the context of the Product Calculus to yield a Fractional Product Calculus.

We outline here the development of the Fractional Product Calculus of order $\frac{1}{2}$, with its Fundamental Theorem. We keep the notations of [Dan1]

6.1 Definition of Fractional Product Integral of Order $\frac{1}{2}$

$$\boxed{(D^{(0)})^{-\frac{1}{2}} f(x) \equiv e^{D^{-\frac{1}{2}} \ln f(x)} \equiv G^{(-\frac{1}{2})}(x)}$$

6.2 Definition of Fractional Product Derivative of Order $\frac{1}{2}$

$$\boxed{(D^{(0)})^{\frac{1}{2}} f(x) \equiv e^{D^{\frac{1}{2}} \ln f(x)}}$$

6.3 The Fundamental Theorem of the Fractional Product Calculus of Order $\frac{1}{2}$

$$\boxed{(D^{(0)})^{\frac{1}{2}} f(x) = D^{(0)}\left\{(D^{(0)})^{-\frac{1}{2}} f(x)\right\}}$$

Proof:

$$D^{(0)}(D^{(0)})^{-\frac{1}{2}} f(x) = e^{D\log\left\{(D^{(0)})^{-\frac{1}{2}} f(x)\right\}}$$

$$= e^{D\log\left\{e^{D^{-\frac{1}{2}}\log f(x)}\right\}}$$

$$= e^{DD^{-\frac{1}{2}}\log f(x)}$$

$$= e^{D^{\frac{1}{2}}\log f(x)}$$

$$= (D^{(0)})^{\frac{1}{2}} f(x)$$

This leads to the interpretation of the Fractional Product derivative of order $\frac{1}{2}$.

6.4 The Geometric Mean of the Convolution Transform

The Fractional Product Derivative of order $\frac{1}{2}$ *of* $f(x)$ *at* x *is the Geometric Mean of* $G^{(-\frac{1}{2})}(x)$ *over* $[x, x + dx]$.

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