Feynman Integral and Convolution Products

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Abstract Feynman tried to express the Propagator of the Schrödinger equation as an Infinite Convolution Product that became known as the Feynman Path Integral.

Feynman was not aware of Polya-Laguerre's Infinite Convolution Products, and did not know that the Schrodinger Propagator cannot be written as an Infinite Convolution Product.

Feynman's construction is the same whether we use one iteration or infinitely many iterations, the limit is superfluous, Schrodinger's Propagator cannot be decomposed, and Feynman's Integral remains, as defined, Schrodinger's Propagator.

The Feynman Integral, and approach to Quantum Mechanics, are in fact, the Schrodinger Propagator, and the wave function approach to Quantum Mechanics, renamed after Feynman.

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Introduction

De-Broglie associated with a moving particle a wave, and Schrodinger derived for a particle of mass m moving in the direction of x, free of forces, the wave equation

$$\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2}.$$

Born suggested that $\left|\psi(x,t)\right|^2$ is the probability to find the particle in the time interval $[t, t+dt]$, and in the length interval $\left[x, x+dx\right]$.

If the particle is at x_a, at time t_a, and at x_b, at time t_b, its Schrodinger's wave function is given by

$$\psi(x_b, t_b) = \int_{x_a=-\infty}^{x_a=\infty} K(x_b, t_b; x_a, t_a)\psi(x_a, t_a)dx_a\,,$$

where

$$K(x_b, t_b; x_a, t_a) = \left[\frac{m}{2\pi i\hbar(t_b - t_a)}\right]^{1/2} \exp\frac{i}{\hbar}\int_{t=t_a}^{t=t_b} \mathcal{L}(x_{\min}, \dot{x}_{\min}, t)dt\,,$$

is Schrodinger's Propagator, and $x_{\min}(t)$ is the least-energy-path of the particle.

Feynman attempted to express $K(x_b, t_b; x_a, t_a)$ by what is known as an Infinite Convolution Product.

He showed that if the path is partitioned into the segments

$$a = x_0, x_1, x_2 ... x_N = b$$

so that

$$t_a = t_0 < t_1 < t_2 < ... < t_N = t_b,$$

then

$$K(x_b, t_b; x_{N-1}, t_{N-1}) * K(x_{N-1}, t_{N-1}; x_{N-2}, t_{N-2}) * ... * K(x_1, t_1; x_a, t_a) = K(x_a, t_a; x_b, t_b),$$

where the symbol $*$ denotes the Convolution Product.

This equality fails to yield an Infinite Convolution Product.

To obtain a genuine Infinite Convolution Product, the path partition points must be determined by zeros of the Laplace Transform of the Kernel $K(x_a, t_a; x_b, t_b)$, and the Kernel has to be generated by basic Non-Schrodinger kernels that depend on those zeros.

Feynman's taking the limit $N \to \infty$ makes no difference, because the equality holds for any natural number N, and the limit process does not obtain any new result.

Feynman's Path Integral amounts to impossible expectations of the Schrodinger Propagator.

In the following, we derive the Schrodinger Propagator for a particle in a force-free motion, and for a harmonic oscillator, in order to understand Feynman's failed attempt to express these Propagators as Infinite Convolution Products.

Then, we use Polya-Laguerre theory of Infinite Convolution Products to explain why Feynman's Path Integral does not exist, why Schrodinger's Propagator need not be renamed after Feynman, and why Feynman's New approach to Quantum Mechanics is Schrodinger's Wave Mechanics.

1.

Schrodinger's Propagator of a Free Particle

Consider a particle of mass m, at the point a, at time t_a, moving in a force-free zone, in the direction of x at speed $\dot{x} = \dfrac{dx}{dt}$, and arriving at the point b, at time t_b.

Schrodinger's wave function is given by

$$\psi(x_b, t_b) = \int_{x_a=-\infty}^{x_a=\infty} K(x_b, t_b; x_a, t_a)\psi(x_a, t_a)dx_a,$$

where $K(x_b, t_b; x_a, t_a)$ is Schrodinger's Propagator, obtained along the particle's least-energy-path.

We aim to derive the known formula stated in the introduction:

1.1 The Form of Schrodinger's Operator

There is a constant A *so that* $K(x_b, t_b; x_a, t_a) = \dfrac{1}{A} e^{i\frac{m}{2\hbar(t_b - t_a)}(x_b - x_a)^2}$.

Proof: For a particle of mass m moving in the direction of x at speed $\dot{x} = \frac{dx}{dt}$, in a force-free zone, Lagrange energy function is

$$\mathcal{L}_{free}(x, \dot{x}, t) = \tfrac{1}{2} m \dot{x}^2 .$$

The path of minimal energy, solves the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = 0 .$$

Here,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{d}{dt} m\dot{x} = m\ddot{x} ,$$

$$\frac{\partial \mathcal{L}}{\partial x} = 0 ,$$

and the Euler-Lagrange equation is

$$m\ddot{x} = 0 .$$

Thus, the least-energy-path

$$x_{\min}(t) ,$$

has constant velocity

$$\dot{x}_{\min}(t) = const.$$

Since the particle is at the point a, at time t_a, and at the point b, at time t_b,

$$\dot{x}_{\min}(t) = \frac{x_b - x_a}{t_b - t_a}$$

Hence, the Lagrangian along the path of least energy is

$$\mathcal{L}_{free}(x_{\min}, \dot{x}_{\min}, t) = \frac{1}{2} m \dot{x}_{\min}^2$$

$$= \frac{1}{2} m \left(\frac{x_b - x_a}{t_b - t_a} \right)^2$$

Therefore,

$$\int_{t=t_a}^{t=t_b} \mathcal{L}_{free}(x_{\min}, \dot{x}_{\min}, t)dt = \int_{t=t_a}^{t=t_b} \frac{1}{2} m \left(\frac{x_b - x_a}{t_b - t_a} \right)^2 dt$$

$$= \frac{1}{2} m \frac{\left(x_b - x_a \right)^2}{t_b - t_a}$$

Thus, Schrodinger's Propagator is

$$K_{free}(,a) = \frac{1}{A} e^{\frac{i}{\hbar} \int_{t=t_a}^{t=t_b} \mathcal{L}_{free}(x_{\min}, \dot{x}_{\min}, t)dt}$$

$$= \frac{1}{A} e^{i \frac{m}{2\hbar(t_b - t_a)}(x_b - x_a)^2} . \square$$

We proceed to determine the constant A.

1.2 The Constant for Schrodinger's Propagator

$$\frac{1}{A} = \left[\frac{m}{2\pi i \hbar (t_b - t_a)} \right]^{\frac{1}{2}}$$

Proof: Schrodinger's wave function is

$$\psi(x_b, t_b) = \frac{1}{A} \int_{x_a=-\infty}^{x_a=\infty} K_{free}(x_b, t_b; x_a, t_a) \psi(x_a, t_a) dx_a$$

$$= \frac{1}{A} \int_{x_a=-\infty}^{x_a=\infty} e^{i\frac{m}{2\hbar(t_b-t_a)}(x_b-x_a)^2} \psi(x_a, t_a) dx_a \,,$$

Letting

$$t_b = t_a + \delta t \,,$$
$$x_b = x_a + \delta x \,,$$

we have

$$\psi(x_b, t_a + \delta t) = \frac{1}{A} \int_{x_a=-\infty}^{x_a=\infty} e^{i\frac{m}{2\hbar(t_b-t_a)}(x_b-x_a)^2} \psi(x_b - \delta x, t_a) dx_a \,.$$

Hence, to first order,

$$\psi(x_b, t_a) + \frac{\partial \psi}{\partial t} \delta t \approx \frac{1}{A} \int_{x_a=-\infty}^{x_a=\infty} e^{i\frac{m}{2\hbar(t_b-t_a)}(x_b-x_a)^2} \left(\psi(x_b, t_a) - \frac{\partial \psi}{\partial x} \delta x \right) dx_a \,.$$

Equating coefficients of $\psi(x_b, t_a)$ on both sides, we conclude that

$$A = \int_{x_a=-\infty}^{x_a=\infty} e^{i\frac{m}{2\hbar(t_b-t_a)}(x_b-x_a)^2} dx_a$$

$$= \int_{x_a=-\infty}^{x_a=\infty} e^{-\frac{m}{2i\hbar(t_b-t_a)}(x_b-x_a)^2} dx_a \,.$$

Let $\xi = \sqrt{\frac{m}{2i\hbar(t_b-t_a)}}(x_a - x_b)$. Then, $d\xi = \sqrt{\frac{m}{2i\hbar(t_b-t_a)}} dx_a$, and

$$A = \frac{1}{\sqrt{\frac{m}{2i\hbar(t_b-t_a)}}} \int_{\xi=-\infty}^{\xi=\infty} e^{-\xi^2} d\xi$$

$$= \frac{1}{\sqrt{\frac{m}{2i\hbar(t_b-t_a)}}} \sqrt{\pi} \,. \square$$

Substituting the constant A, we obtain Schrodinger's Propagator, and Born's probability for the free particle:

1.3 Schrodinger's Propagator

$$K(x_b, t_b; x_a, t_a) = \left[\frac{m}{2\pi i\hbar(t_b - t_a)}\right]^{\frac{1}{2}} e^{i\frac{m}{2\hbar(t_b - t_a)}(x_b - x_a)^2}.$$

1.4 Born's Probability for the Free Particle

$$P(b,a) = \frac{m}{2\pi\hbar(t_b - t_a)}.$$

2.

Feynman Path Integral for a Free Particle

Feynman attempted to write $K(x_b, t_b; x_a, t_a)$ as an Infinite Convolution Product, over the least-energy-path of the free particle.

We examine his derivation.

Divide the time interval $[t_a, t_b]$ into N equal sub-intervals of length

$$\Delta t = \frac{t_b - t_a}{N},$$

with Partition points

$$t_a = t_0 < t_1 < t_2 < ..... < t_{N-1} < t_N = t_b.$$

Consider

$$K(x_1, t_1; x_0, t_0) = \left[\frac{m}{2\pi i\hbar(t_1 - t_0)}\right]^{\frac{1}{2}} e^{\left(\frac{m}{2i\hbar(t_1 - t_0)}(x_1 - x_0)^2\right)},$$

$$K(x_2, t_2; x_1, t_1) = \left[\frac{m}{2\pi i\hbar(t_2 - t_1)}\right]^{\frac{1}{2}} e^{\left(\frac{m}{2i\hbar(t_2 - t_1)}(x_2 - x_1)^2\right)},$$

$$K(x_3, t_3; x_2, t_2) = \left[\frac{m}{2\pi i\hbar(t_3 - t_2)}\right]^{\frac{1}{2}} e^{\left(\frac{m}{2i\hbar(t_3 - t_2)}(x_3 - x_2)^2\right)},$$

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$$K(x_{N-1},t_{N-1};x_{N-2},t_{N-2}) = \left[\frac{m}{2\pi i\hbar(t_{N-1}-t_{N-2})}\right]^{\frac{1}{2}} e^{\left(\frac{m}{2i\hbar(t_{N-1}-t_{N-2})}(x_{N-1}-x_{N-2})^2\right)},$$

$$K(x_N,t_N;x_{N-1},t_{N-1}) = \left[\frac{m}{2\pi i\hbar(t_N-t_{N-1})}\right]^{\frac{1}{2}} e^{\left(\frac{m}{2i\hbar(t_N-t_{N-1})}(x_N-x_{N-1})^2\right)}.$$

Then,

$$K(x_2,t_2;x_1,t_1) * K(x_1,t_1;x_0,t_0)$$

$$= \int_{x_1=-\infty}^{x_1=\infty} K(x_2,t_2;x_1,t_1)K(x_1,t_1;x_0,t_0)dx_1$$

$$= \frac{m}{2\pi i\hbar(t_1-t_0)} \int_{x_1=-\infty}^{x_1=\infty} e^{\left(\frac{m}{2i\hbar(t_2-t_1)}(x_2-x_1)^2\right)+\left(\frac{m}{2i\hbar(t_1-t_0)}(x_1-x_0)^2\right)} dx_1 .$$

By [Feyn1, p.357]

$$\int_{x=-\infty}^{x=\infty} e^{\alpha(x_2-\xi)^2+\beta(\xi-x_1)^2} d\xi = \sqrt{\frac{-\pi}{\alpha+\beta}} e^{\frac{\alpha\beta}{\alpha+\beta}(x_2-x_1)^2}$$

Here,

$$\sqrt{\frac{-\pi}{\alpha+\beta}} = \sqrt{\frac{-\pi 2i\hbar}{m}} \left(\frac{1}{\frac{1}{t_2-t_1}+\frac{1}{t_1-t_0}}\right)^{\frac{1}{2}} = \sqrt{\frac{-\pi 2i\hbar}{m}} \left(\frac{(t_2-t_1)(t_1-t_0)}{t_2-t_0}\right)^{\frac{1}{2}}.$$

$$\frac{\alpha\beta}{\alpha+\beta} = \frac{\dfrac{m}{2i\hbar(t_2-t_1)}\dfrac{m}{2i\hbar(t_1-t_0)}}{\dfrac{m}{2i\hbar(t_2-t_1)}+\dfrac{m}{2i\hbar(t_1-t_0)}} = \frac{m}{2i\hbar(t_2-t_0)}$$

Substituting in $K(x_2,t_2;x_1,t_1) * K(x_1,t_1;x_0,t_0)$,

$$= \frac{m}{2\pi i\hbar(t_1-t_0)}\sqrt{\frac{-\pi 2i\hbar}{m}}\left(\frac{(t_2-t_1)(t_1-t_0)}{t_2-t_0}\right)^{\frac{1}{2}} e^{\frac{m}{2i\hbar(t_2-t_0)}(x_2-x_0)^2}$$

$$= \left[\frac{m}{2\pi i\hbar(t_2-t_0)}\right]^{\frac{1}{2}} e^{\left(\frac{m}{2i\hbar(t_2-t_0)}(x_2-x_0)^2\right)}$$

$$= K(x_2,t_2;x_0,t_0).$$

Therefore,

$$K(x_3,t_3;x_2,t_2) * K(x_2,t_2;x_1,t_1) * K(x_1,t_1;x_0,t_0)$$

$$= K(x_3,t_3;x_2,t_2) * K(x_2,t_2;x_0,t_0)$$

$$= K(x_3,t_3;x_0,t_0)$$

Indunctively,

$$K(x_b,t_b;x_{N-1},t_{N-1}) * K(x_{N-1},t_{N-1};x_{N-2},t_{N-2}) * \ldots * K(x_1,t_1;x_a,t_a) = K(x_b,t_b;x_a,t_a)$$

Letting $N \to \infty$,

$$\lim_{N\to\infty} K(x_b,t_b;x_{N-1},t_{N-1}) * K(x_{N-1},t_{N-1};x_{N-2},t_{N-2}) * \ldots * K(x_1,t_1;x_a,t_a)$$

$$= K(x_b,t_b;x_a,t_a).$$

Feynman defined the limit, that is, *the already known Schrodinger Propagator*, $K(x_b, t_b; x_a, t_a)$, as his Path Integral for the free particle.

But since equality holds for any natural number N, the limit process makes no difference, and serves no purpose.

Therefore, Feynman Path Integral is the Schrodinger Propagator renamed after Feynman.

3.

Schrodinger's Propagator for the Harmonic Oscillator

For a particle of mass m oscillating harmonically along x, at angular frequency ω, Schrodinger wave function satisfies the equation

$$\frac{\partial\psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{m}{2i\hbar}\omega^2 x^2\psi\,,$$

and is given by

$$\psi(x_b, t_b) = \int_{x_a=-\infty}^{x_a=\infty} K_{harm}(x_b, t_b; x_a, t_a)\psi(x_a, t_a)dx_a\,,$$

where $K_{harm}(x_b, t_b; x_a, t_a)$ is Schrodinger's Propagator, obtained along the particle's least-energy-path.

We aim to derive the known formula [Feyn1,p.198] for it.

3.1 The Form of Schrodinger's Propagator

There is a constant A *so that*

$$K_{harm}(x_b, t_b; x_a, t_a) = \frac{1}{A} e^{\frac{m\omega}{2i\hbar\sin\omega(t_b - t_a)}\left((x_b^2 + x_a^2)\cos\omega(t_b - t_a) - 2x_b x_a\right)}.$$

Proof: For a particle of mass m oscillating harmonically along x, at angular frequency ω, Lagrange energy function is

$$\mathcal{L}_{harm}(x,\dot{x},t)=\frac{1}{2}m\dot{x}^2-\frac{1}{2}m\omega^2x^2\,.$$

The path of minimal energy, solves the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}}-\frac{\partial\mathcal{L}}{\partial x}=0\,.$$

Here,

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}}=\frac{d}{dt}m\dot{x}=m\ddot{x}\,,$$

$$\frac{\partial\mathcal{L}}{\partial x}=-m\omega^2x\,,$$

and the Euler-Lagrange equation is

$$m\ddot{x}+m\omega^2x=0\,.$$

Thus, the least-energy-path is

$$x_{\min}(t)=A\cos\omega t+B\sin\omega t\,,$$

If the particle location is x_a, at time t_a, and x_b, at time t_b,

$$A\cos\omega t_a+B\sin\omega t_a=x_a$$

$$A\cos\omega t_b+B\sin\omega t_b=x_b\,.$$

Solving for A, and B,

$$A=\frac{\begin{vmatrix}x_a & \sin\omega t_a\\ x_b & \sin\omega t_b\end{vmatrix}}{\begin{vmatrix}\cos\omega t_a & \sin\omega t_a\\ \cos\omega t_b & \sin\omega t_b\end{vmatrix}}=\frac{x_a\sin\omega t_b-x_b\sin\omega t_a}{\sin\omega(t_b-t_a)}$$

$$B = \frac{\begin{vmatrix} \cos\omega t_a & x_a \\ \cos\omega t_b & x_b \end{vmatrix}}{\begin{vmatrix} \cos\omega t_a & \sin\omega t_a \\ \cos\omega t_b & \sin\omega t_b \end{vmatrix}} = \frac{-x_a\cos\omega t_b + x_b\cos\omega t_a}{\sin\omega(t_b - t_a)}$$

The Lagrangian along the path of least energy is

$$\mathcal{L}_{harm}(x_{\min}, \dot{x}_{\min}, t) = \frac{1}{2}m\dot{x}_{\min}^2 - \frac{1}{2}m\omega^2 x_{\min}^2$$

$$= \frac{1}{2}m\Big[\big(-A\omega\sin\omega t + B\omega\cos\omega t\big)^2 - \omega^2(A\cos\omega t + B\sin\omega t)^2\Big]$$

$$= \frac{1}{2}m\omega^2\Big[(B^2 - A^2)\cos 2\omega t - 2AB\sin 2\omega t\Big].$$

Hence,

$$\int_{t=t_a}^{t=t_b} \mathcal{L}_{harm}(x_{\min}, \dot{x}_{\min}, t)dt =$$

$$= \frac{1}{2}m\omega^2 \int_{t=t_a}^{t=t_b} \Big[(B^2 - A^2)\cos 2\omega t - 2AB\sin 2\omega t\Big]dt$$

$$= \frac{1}{4}m\omega\Big[(B^2 - A^2)\sin 2\omega t + 2AB\cos 2\omega t\Big]_{t=t_a}^{t=t_b}$$

$$= \frac{1}{4}m\omega\Big[(B^2 - A^2)(\sin 2\omega t_b - \sin 2\omega t_a)$$

$$+2AB(\cos 2\omega t_b - \cos 2\omega t_a)\Big]$$

Substituting

$$B^2 - A^2 = \frac{x_b^2 \cos 2\omega t_a + x_a^2 \cos 2\omega t_b - 2x_a x_b \cos w(t_b + t_a)}{\sin^2 \omega(t_b - t_a)}$$

$$2AB = \frac{-x_b^2 \sin 2\omega t_a - x_a^2 \sin 2\omega t_b + 2x_a x_b \sin w(t_b + t_a)}{\sin^2 \omega(t_b - t_a)}$$

$$= \frac{1}{4}\frac{m\omega}{\sin^2 \omega(t_b - t_a)}\Big[(x_b^2 + x_a^2)\sin 2\omega(t_b - t_a)$$

$$-2x_a x_b \cos w(t_b + t_a)[\sin 2\omega t_b - \sin 2\omega t_a]$$

$$+2x_a x_b \sin w(t_b + t_a)[\cos 2\omega t_b - \cos 2\omega t_a]\Big]$$

$$= \frac{1}{4}\frac{m\omega}{\sin^2 \omega(t_b - t_a)}\Big[(x_b^2 + x_a^2)\sin 2\omega(t_b - t_a)$$

$$-2x_a x_b \cos w(t_b + t_a)[2\sin \omega(t_b - t_a)\cos \omega(t_b + t_a)]$$

$$+2x_a x_b \sin w(t_b + t_a)[-2\sin \omega(t_b + t_a)\sin \omega(t_b - t_a)]\Big]$$

$$= \frac{1}{2}\frac{m\omega}{\sin \omega(t_b - t_a)}\Big[(x_b^2 + x_a^2)\cos \omega(t_b - t_a) - 2x_a x_b\Big]$$

Therefore, Schrodinger's Propagator is

$$K_{harm}(x_b, t_b; x_a, t_a) = \frac{1}{A} e^{i\frac{m\omega}{2\hbar \sin \omega(t_b - t_a)}\left((x_b^2 + x_a^2)\cos \omega(t_b - t_a) - 2x_b x_a\right)}. \square$$

Similarly to the proof of 1.2, we obtain

3.2 The Constant for Schrodinger's Propagator

$$\frac{1}{A} = \left[\frac{m\omega}{2\pi i\hbar \sin\omega(t_b - t_a)}\right]^{\frac{1}{2}}$$

Substituting the constant, we obtain the propagator, and the probability for the harmonic oscillator.

3.3 Schrodinger's Propagator

$$K_{harm}(x_b, t_b; x_a, t_a) =$$

$$= \left[\frac{m}{2\pi i\hbar \sin\omega(t_b - t_a)}\right]^{\frac{1}{2}} e^{i\frac{m\omega}{2\hbar \sin\omega(t_b - t_a)}\left((x_b^2 + x_a^2)\cos\omega(t_b - t_a) - 2x_b x_a\right)}$$

3.4 Conditional Probability for the Harmonic Oscillator

A particle of mass m*, at the point* a*, at time* t_a*, oscillating harmonically along* x*, at angular velocity* ω*, with speed* $\dot{x} = \dfrac{dx}{dt}$*, will arrive at the point* b*, at time* t_b *with the conditional probability*

$$P(b, a) = \frac{m\omega}{2\pi\hbar\left|\sin\omega(t_b - t_a)\right|}.$$

3.5 Feynman Integral for the Harmonic Oscillator

Similarly to section 2, Feynman attempted to express Schrodinger's Propagator for the harmonic oscillator

$$\left[\frac{m\omega}{2\pi i\hbar\sin\omega(t_b - t_a)}\right]^{\frac{1}{2}} e^{\frac{m\omega}{2i\hbar\sin\omega(t_b - t_a)}\left((x_b^2 + x_a^2)\cos\omega(t_b - t_a) - 2x_b x_a\right)},$$

as an Infinite Convolution Product.

Again, applying the limit process to the convolution property for Schrodinger Propagators serves no purpose, and Feynman Path Integral for the harmonic oscillator, is the Schrodinger Propagator renamed after Feynman.

4.

Infinite Convolution Products

The evolution of Schrodinger's wave function from time t_1 to time t_2, is given by the Convolution

$$\psi(x,t_2) = \int_{\xi=-\infty}^{\xi=\infty} G(x,t;\xi,t_1)\psi(\xi,t_1)d\xi\,,$$

where the Schrodinger Propagator $G(x,t_2,\xi,t_1)$ is obtained along the path of least energy.

It is a particular case of the Convolution Transform

$$F(t) = \int_{u=-\infty}^{u=\infty} G(t-u)f(u)du\,,$$

that transforms the function $f(u)$ into the function $F(t)$.

Then, the Kernel function $G(t)$ may be an Infinite Convolution Product of basic kernel functions $g_1(t)$, $g_2(t)$, $g_2(t)\dots$ Namely,

$$G(t) = g_1(t) * g_2(t) * g_3(t) * \dots$$

Is Schrodinger Propagator such kernel $G(t)$?

4.1 The Kernel of an Infinite Convolution Product.

We denote a complex number by

$$z = x + iy\,.$$

Following the notations of [Hirs1], [Hirs2], and [widd1].

$$b\text{, and } a_k \neq 0\,,\ k = 1,2,3,...$$

are real numbers so that

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \frac{1}{a_3^2} + ... < \infty\,.$$

Consider the basic kernel function

$$g_1(t) = \begin{cases} a_1 e^{a_1(t-b)-1}, a_1(t-b) - 1 < 0 \\ 0, \qquad a_1(t-b) - 1 \geq 0 \end{cases}$$

$$= \begin{cases} a_1 e^{a_1(t-b)-1}, t < \dfrac{1}{a_1} + b \\ 0, \qquad t \geq \dfrac{1}{a_1} + b \end{cases}.$$

The Two-Sided Laplace Transform of $g_1(t)$ is

$$\mathfrak{L}\{g_1(t)\} = \int_{t=-\infty}^{t=\infty} g_1(t) e^{-zt} dt$$

$$= \int_{t=-\infty}^{t=\frac{1}{a_1}+b} a_1 e^{a_1(t-b)-1} e^{-zt} dt$$

$$= \frac{a_1}{e^{a_1 b+1}} \int_{t=-\infty}^{t=\frac{1}{a_1}+b} e^{(a_1-z)t} dt$$

$$= \frac{a_1}{e^{a_1 b+1}} \frac{1}{a_1 - iz} e^{(a_1 - iz)(\frac{1}{a_1}+b)}$$

$$= e^{-biz} \frac{1}{1 - \frac{z}{a_1}} e^{-\frac{z}{a_1}}.$$

For $k = 2, 3, 4 \ldots n$, consider the basic kernel functions

$$g_k(t) = \begin{cases} a_2 e^{a_k t - 1}, a_k t - 1 < 0 \\ 0, \qquad a_k t - 1 \geq 0 \end{cases} = \begin{cases} a_2 e^{a_k t - 1}, t < \dfrac{1}{a_k} \\ 0, \qquad t \geq \dfrac{1}{a_k} \end{cases}.$$

The Two-Sided Laplace Transform of $g_k(t)$ is

$$\mathfrak{L}\left\{ g_k(t) \right\} = \int_{t=-\infty}^{t=\infty} g_k(t) e^{-zt} dt$$

$$= \int_{t=-\infty}^{t=\frac{1}{a_k}} a_k e^{a_k t - 1} e^{-zt} dt$$

$$= \frac{a_k}{e} \int_{t=-\infty}^{t=\frac{1}{a_k}} e^{(a_k - z)t} dt$$

$$= \frac{a_k}{e} \frac{1}{a_k - z} e^{(a_k - z)\frac{1}{a_k}}$$

$$= \frac{1}{1 - \dfrac{z}{a_k}} e^{-\frac{z}{a_k}}.$$

The Convolution Product of $g_1(t)$, and $g_2(t)$ is defined by

$$g_1(t) * g_2(t) = \int_{u_1=-\infty}^{u_1=\infty} g_1(t - u_1) g_2(u_1) du_1$$

By the Convolution Theorem, the Two-Sided Laplace Transform of the Convolution Product $g_1(t) * g_2(t)$ is

$$\mathfrak{L}\left\{g_1(t) * g_2(t)\right\} = \mathfrak{L}\left\{g_1(t)\right\} \times \mathfrak{L}\left\{g_2(t)\right\}$$

$$= e^{-bz} \frac{1}{1 - \dfrac{z}{a_1}} e^{-\frac{z}{a_1}} \frac{1}{1 - \dfrac{z}{a_2}} e^{-\frac{z}{a_2}}.$$

Similarly, the Two-Sided Laplace Transform of the convolution product

$$g_1(t) * g_2(t) * ... * g_n(t)$$

is

$$\mathfrak{L}\left\{g_1(t) * g_2(t) * ... * g_n(t)\right\} =$$

$$= \mathfrak{L}\left\{g_1(t)\right\} \times \mathfrak{L}\left\{g_2(t)\right\} \times ... \times \mathfrak{L}\left\{g_n(t)\right\}$$

$$= \frac{1}{e^{bz}} \frac{1}{\left(1 - \frac{z}{a_1}\right) e^{\frac{z}{a_1}}} \frac{1}{\left(1 - \frac{z}{a_2}\right) e^{\frac{z}{a_2}}} \cdots \frac{1}{\left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}}}$$

The denominator functions

$$E_n(z) = e^{bz} \left(1 - \frac{z}{a_1}\right) e^{\frac{z}{a_1}} \left(1 - \frac{z}{a_2}\right) e^{\frac{z}{a_2}} \cdots \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}}$$

are analytic for any complex number z, and converge uniformly [Widd1, p. 173], to the infinite product

$$E(z) = e^{bz} \left(1 - \frac{z}{a_1}\right) e^{\frac{z}{a_1}} \left(1 - \frac{z}{a_2}\right) e^{\frac{z}{a_2}} \left(1 - \frac{z}{a_3}\right) e^{\frac{z}{a_3}} \cdots$$

on every compact set of the z plane.

Thus, $E(z)$ is analytic for any complex number z.

Therefore, for $-\infty < t < \infty$, the Infinite Convolution Product

$$G(t) = g_1(t) * g_2(t) * g_3(t) * \ldots.$$

is the Inverse Two-Sided Laplace Transform

$$\mathfrak{L}^{-1}\left\{\frac{1}{E(z)}\right\} = \frac{1}{2\pi i} \int_{z=-i\infty}^{z=i\infty} \frac{1}{E(z)} e^{zt} d\omega .$$

$$= \frac{1}{2\pi} \int_{y=-\infty}^{y=\infty} \frac{1}{E(iy)} e^{iyt} dy$$

Rather than evaluate directly the Infinite Convolution Product, as Feynman attempted, we evaluate the Two-Sided Laplace Transforms

$$\mathfrak{L}\left\{g_1(t)\right\}, \mathfrak{L}\left\{g_2(t)\right\}, \mathfrak{L}\left\{g_3(t)\right\}...,$$

and invert their product.

This follows from the *Convolution-Kernel Integral Theorem*:

4.2 The Convolution-Kernel Integral Theorem

$$G(t) = g_1(t) * g_2(t) * g_3(t) * ....$$

$$= \mathfrak{L}^{-1}\left\{\mathfrak{L}\left\{g_1(t)\right\}\mathfrak{L}\left\{g_2(t)\right\}\mathfrak{L}\left\{g_3(t)\right\}...\right\}$$

$$= \frac{1}{2\pi}\int_{y=-\infty}^{y=\infty}\left\{\frac{1}{e^{iby}} \times \frac{1}{\left(1-\frac{iy}{a_1}\right)e^{\frac{iy}{a_1}}} \times \frac{1}{\left(1-\frac{iy}{a_2}\right)e^{\frac{iy}{a_2}}} \times ...\right\}e^{iyt}dy . \square$$

4.3 The Kernel $G(t)$ as a Probability Density Function

In the strip $-\left|a_1\right| < x < \left|a_1\right|$,

$$\frac{1}{e^{bz}} \times \frac{1}{\left(1-\frac{z}{a_1}\right)e^{\frac{z}{a_1}}} \times \frac{1}{\left(1-\frac{z}{a_2}\right)e^{\frac{z}{a_2}}} \times \frac{1}{\left(1-\frac{z}{a_3}\right)e^{\frac{z}{a_3}}} \times ... =$$

$$= \mathfrak{L}\left\{ g_1(t) * g_2(t) * g_3(t) * ....\right\}$$

$$= \int_{t=-\infty}^{t=\infty} \left\{ g_1(t) * g_2(t) * g_3(t) * ....\right\} e^{-zt} dt$$

At $z = 0$,

$$\int_{t=-\infty}^{t=\infty} \left\{ g_1(t) * g_2(t) * g_3(t) * ...\right\} dt = \frac{1}{E(0)}$$

$$= 1$$

Also,

$$g_1(t) * g_2(t) * g_3(t) * ... \geq 0,$$

because for any n,

$$g_1(t) * g_2(t) * ... * g_n(t) \geq 0.$$

Thus, $G(t)$ is a probability density function.

It is shown (in [Hirs1], [Widd1]) that its Mean is

$$b,$$

and its Variance is

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \frac{1}{a_3^2} + ...$$

4.4 The Convolution Transform

The convolution of $G(t)$ with an integrable function $f(t) \in L^1$ is the Convolution Transform determined by the sequence $a_1, a_2, a_3, .,$

$$F(t) = G(t) * f(t)$$

$$= \int_{u=-\infty}^{u=\infty} G(t-u)f(u)du$$

$$= \int_{u=-\infty}^{u=\infty} f(t-u)\left[g_1(u) * g_2(u) * g_3(u) * ...\right]du.$$

The following examples were derived from [Hirs1, p.79].

	a_k	$G(t)$	$E(z)$	$\Phi(\xi)$
Laplace	$1,2,3..$	$e^t e^{-e^t}$	$\dfrac{1}{\Gamma(1-z)}$	$\int_{\tau=0}^{\tau=\infty} e^{-\xi\tau}\varphi(\tau)d\tau$
Iterated Laplace		$e^{-e^t} * e^{-e^t}$	$\dfrac{1}{[\Gamma(1-z)]^2}$	$\int_{\theta=0}^{\theta=\infty}\left(\int_{\tau=0}^{\tau=\infty} e^{-\tau/\theta}\varphi(\tau)d\tau\right)\dfrac{1}{\theta}e^{-\xi\theta}d\theta$
Stieltjes $\nu=1$	$\frac{2}{-5},\frac{2}{-3},\frac{2}{-1},\frac{2}{3},\frac{2}{5}$	$\dfrac{1}{e^{t/2}+e^{-t/2}}$	$\dfrac{1}{\Gamma(iz)\Gamma(1-iz)}$	$\int_{\tau=0}^{\tau=\infty}\dfrac{\varphi(\tau)}{\xi+\tau}d\tau$
Iterated Stieltjes		$\dfrac{t/2}{e^{t/2}-e^{-t/2}}$	$\dfrac{1}{[\Gamma(z)\Gamma(1-z)]^2}$	$\int_{\tau=0}^{\tau=\infty}\dfrac{\log\xi-\log\tau}{\xi-\tau}\varphi(\tau)d\tau$
Stieltjes $\nu = 2$	$...-2, -1, 1, 2..$	$\dfrac{1}{(e^{t/2}+e^{-t/2})^2}$	$\dfrac{1}{z\Gamma(z)\Gamma(1-z)}$	$\int_{\tau=0}^{\tau=\infty}\dfrac{\varphi(\tau)}{\left(\xi+\tau\right)^2}d\tau$
Stieltjes $\nu>0$		$\dfrac{1}{(e^{t/2}+e^{-t/2})^\nu}$	$\dfrac{\Gamma(\nu)}{\Gamma\left(\frac{\nu}{2}+z\right)\Gamma\left(\frac{\nu}{2}-z\right)}$	$\int_{\tau=0}^{\tau=\infty}\dfrac{\varphi(\tau)}{\left(\xi+\tau\right)^\nu}d\tau$
Meijer $\nu = 0$		$e^t K_0(e^t)$	$\dfrac{2^{z+1}}{\left[\Gamma\left(\frac{1-z}{2}\right)\right]^2}$	$\int_{\tau=0}^{\tau=\infty}\sqrt{\xi\tau}K_0(\xi\tau)\varphi(\tau)d\tau$
Meijer $-1 < \nu < 1$		$e^t K_\nu(e^t)$	$\dfrac{2^{z+1}}{\Gamma\left(\frac{1+\nu-z}{2}\right)\Gamma\left(\frac{1-\nu-z}{2}\right)}$	$\int_{\tau=0}^{\tau=\infty}\sqrt{\xi\tau}K_\nu(\xi\tau)\varphi(\tau)d\tau$

To see how the Convolution Transform $F(t)$ of $f(u)$ with the Kernel $e^{t}e^{-e^{t}}$ is the Laplace Transform $\Phi(\xi)$ of $\varphi(\tau)$, write

$$F(t) = \int_{u=-\infty}^{u=\infty} e^{t-u}e^{-e^{t-u}} f(u)du .$$

Multiply both sides by e^{-t}. Then,

$$e^{-t}F(t) = \int_{u=-\infty}^{u=\infty} e^{-e^{t-u}} f(u)e^{-u}du .$$

Put $\tau = e^{-u}$. Then,

$$u = \log\frac{1}{\tau},$$

$$d\tau = -e^{-u}du ,$$

and

$$e^{-t}F(t) = \int_{\tau=\infty}^{\tau=0} e^{-e^{t}\tau} f(\log\frac{1}{\tau})\left(-d\tau\right)$$

$$= \int_{\tau=0}^{\tau=\infty} e^{-e^{t}\tau} f(\log\frac{1}{\tau})d\tau .$$

Denote $e^{t} = \xi$. Then,

$$\underbrace{\frac{1}{\xi}F(\log\xi)}_{\Phi(\xi)} = \int_{\tau=0}^{\tau=\infty} e^{-\xi\tau}\, \underbrace{f(\log\frac{1}{\tau})}_{\varphi(\tau)}\, d\tau .$$

Thus,

$$\Phi(\xi) = \int_{\tau=0}^{\tau=\infty} e^{-\xi\tau}\varphi(\tau)d\tau\,.$$

We obtain the Convolution Transform, similarly to the way we obtain the Convolution-Kernel Integral Theorem, by applying the Two-Sided Laplace Integral Theorem,

4.5 The Convolution-Transform Integral Theorem

$$F(t) = \Big(g_1(t) * g_2(t) * g_3(t) * ....\Big) * f(t)$$

$$= \mathfrak{L}^{-1}\Big\{\Big(\mathfrak{L}\Big\{g_1(t)\Big\}\mathfrak{L}\Big\{g_2(t)\Big\}\mathfrak{L}\Big\{g_3(t)\Big\}...\Big)\mathfrak{L}\Big\{f(t)\Big\}\Big\}$$

$$= \frac{1}{2\pi}\int_{y=-\infty}^{y=\infty}\left\{\left(\frac{1}{e^{iby}} \times \frac{1}{\left(1-\frac{iy}{a_1}\right)e^{\frac{iy}{a_1}}} \times \frac{1}{\left(1-\frac{iy}{a_2}\right)e^{\frac{iy}{a_2}}} \times ...\right)\mathfrak{L}\Big\{f(t)\Big\}\right\}e^{iyt}dy \,\square$$

5.

Schrodinger Propagator and Infinite Convolution Product

Feynman attempted to express the Schrodinger Propagator as an Infinite Convolution Product over the least energy path of the particle.

To that end he divided the time interval $[t_a, t_b]$ into N equal subintervals, and by the properties of the Gaussian had

$$K(t_N, t_{N-1}) * K(t_{N-1}, t_{N-2}) * \ldots * K(t_1, t_0) = K(b, a),$$

where, for the force-free particle,

$$K(t_k, t_{k-1}) = \left[\frac{m}{2\pi i\hbar(t_k - t_{k-1})}\right]^{\frac{1}{2}} e^{\left(\frac{m}{2i\hbar(t_k - t_{k-1})}(x_k - x_{k-1})^2\right)}.$$

As $N \to \infty$, the finite convolution product becomes infinite, but its limit is the same $K(b, a)$.

Indeed, Schrodinger's Propagator is not an Infinite Convolution Product.

An Infinite Convolution Product requires a sequence of real numbers

$$a_1, a_2, a_3, \ldots \text{ with } \frac{1}{a_1^2} + \frac{1}{a_2^2} + \frac{1}{a_3^2} + \ldots < \infty,$$

that are roots of the analytic infinite product $E(z)$.

But the Schrodinger Propagator $K(b,a)$, is a kernel $G(\xi)$ with no roots, and its Two-Sided Laplace Transform $\frac{1}{E(z)}$, is such that $E(z)$ has no roots either. Therefore, $G(\xi)$ is not an Infinite Convolution Product.

5.1 Schrodinger's propagator is not an Infinite Convolution Product

Denote $x_b - x_a \equiv \xi$, and $K(b,a) = G(\xi)$

Then, $\frac{1}{\mathfrak{L}\left\{G(\xi)\right\}} = E(z)$ *has no roots.*

Thus, the Schrodinger Propagator is not an Infinite Convolution Product, and the Infinite Convolution Method cannot be applied to the Feynman Integral.

Proof: Denote

$$\frac{m}{2\hbar(t_b - t_a)} = -c.$$

Then,

$$K(b,a) = \left[\frac{m}{2\pi i\hbar(t_b - t_a)}\right]^{\frac{1}{2}} e^{\left(\frac{m}{2i\hbar(t_b - t_a)}(x_b - x_a)^2\right)} = \sqrt{\frac{c}{\pi i}} e^{-\frac{c}{i}\xi^2}$$

Therefore, the Two-Sided Laplace Transform of $K(b,a)$ is

$$\mathfrak{L}\left\{K(b,a)\right\} = \sqrt{\frac{c}{\pi i}} \int_{\xi=-\infty}^{\xi=\infty} e^{-\frac{c}{i}\xi^2} e^{-z\xi} d\xi$$

$$= \sqrt{\frac{c}{\pi i}} \int_{\xi=0}^{\xi=\infty} e^{-(c/i)\xi^2} \left(e^{z\xi} + e^{-z\xi}\right) d\xi$$

By [Haan, Table 26, formula 14],

$$\int_0^\infty e^{-q^2x^2}\left(e^{2px} + e^{-2px}\right)dx = \sqrt{\pi}\,\frac{1}{q} e^{p^2/q^2}.$$

Identifying $q = \sqrt{c / i}$, $p = z / 2$, we have

$$= e^{i\frac{z^2}{4c}}.$$

Hence,

$$E(z) = \frac{1}{\mathfrak{L}\left\{K(b,a)\right\}} = e^{-i\frac{z^2}{4c}} \quad \text{has no zeros.}\square$$

Feynman's construction of his Path Integral is erroneous because the Schrödinger Propagator $K(b,a)$ cannot be decomposed into an Infinite Convolution Product.

The Feynman Integral is precisely the Schrodinger Propagator, and Feynman's New approach to Quantum Mechanics is Schrodinger's Wave Mechanics.

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