Zero Point Energy: Thermodynamic Equilibrium and Planck Radiation Law

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Abstract: In a recent paper, we proved that Planck's radiation law with zero point energy is equivalent to the combined assumptions of zero point energy hypothesis, the quantum law and the approximated Boson statistics.

Here, we apply the principle of maximal entropy to show that Planck's radiation law with zero point energy represents a state of thermodynamic equilibrium.

Introduction In a recent paper [1], we proved that Planck's radiation law with zero point energy [2] is equivalent to the combined assumptions of the zero point energy hypothesis, the quantum law and the approximated Boson statistics.

Planck applied the principle of maximal entropy to a volume in space, to obtain the state of thermodynamic equilibrium for ideal mono-atomic gases [3].

Bose, in a 1924 paper, translated and published by Einstein [4], applied the principle of maximal entropy to a volume in phase space to obtain Planck's radiation law at a state of thermodynamic equilibrium.

We apply the principle of maximal entropy in phase space to show that Planck's radiation law with zero point energy is obtained at a state of thermodynamic equilibrium.

Here, the maximal entropy principle is only a tool, that does not replace any of the three combined assumptions that we proved equivalent to the radiation law with zero point energy. Under the three assumptions, the principle of maximal entropy implies that the radiation law with zero point energy is obtained at a state of thermodynamic equilibrium.

Maximal Entropy and ZPE radiation law:

Following Bose approach in [4], elaborated by Einstein in [5], the momentum of a radiation quantum is

$$p = |\bar{p}| = \frac{h\nu}{c}.$$

That is, radiation quanta with the same momentum are on the surface of the sphere

$$p_x^2 + p_y^2 + p_z^2 = \frac{h^2\nu^2}{c^2}.$$

Then, for each of its two polarizations, the momentum of the radiation per volume element, in the spherical shell between ν, and $\nu + d\nu$, is

$$4\pi\, p^2 dp = 4\pi \frac{h^2\nu^2}{c^2}\frac{h}{c}d\nu = 4\pi \frac{\nu^2 d\nu}{c^3}h^3.$$

Therefore, for both polarizations, there are $8\pi\frac{\nu^2 d\nu}{c^3}$ momentum cells of size h^3 per volume element in the spherical shell between ν, and $\nu + d\nu$.

For the possible frequencies ν_l, $l = 0,1,2,...$ there are

$$n_l = 8\pi\frac{\nu_l^2 d\nu_l}{c^3}$$

momentum cells of size h^3, per volume element, in the spherical shell between ν_l, and $\nu_l + d\nu_l$.

Then,

$$n_l = n_l^{(0)} + n_l^{(1)} + n_l^{(2)} + ... = \sum_{j=0}^{\infty} n_l^{(j)}, \qquad (1)_l$$

where

$n_l^{(0)}$ is the number of cells that have no quanta,

$n_l^{(1)}$ is the number of cells that have one quanta,

$n_l^{(2)}$ is the number of cells that have two quanta,

....

The n_l momentum cells of size h^3, per volume element, in the spherical shell between ν_l and $\nu_l + d\nu_l$, can be arranged in

$$\frac{n_l!}{n_l^{(0)}!n_l^{(1)}!n_l^{(2)}!...} = n_l!\prod_{j=0}^{\infty}\frac{1}{n_l^{(j)}!}$$

ways.

Thus, all the momentum cells of size h^3, per volume element, can be arranged in

$$w = \prod_{l=0}^{\infty} n_l ! \prod_{j=0}^{\infty} \frac{1}{n_l^{(j)} !}$$

ways.

The entropy per volume element is

$$s = k_B \log w = k_B \sum_{l=0}^{\infty} \log(n_l !) - k_B \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} \log(n_l^{(j)} !),$$

where k_B is Boltzman constant.

By Sterling's formula $\ln M! \approx M \ln M - M$,

$$s \approx k_B \sum_{l=0}^{\infty} (n_l \log n_l - n_l) - k_B \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} (n_l^{(j)} \log n_l^{(j)} - n_l^{(j)})$$

$$= k_B \sum_{l=0}^{\infty} n_l \log n_l - k_B \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} n_l^{(j)} \log n_l^{(j)} .$$

The number of radiation quanta $h\nu_l$ between ν_l, and $\nu_l + d\nu_l$ is

$$n_l^{(1)} + 2n_l^{(2)} + 3n_l^{(3)} + ... = \sum_{j=0}^{\infty} j n_l^{(j)} .$$

Following Planck's assumption of Zero Point Energy [2], we assume zero point energy of $\frac{1}{2} h\nu_l$ in each of the n_l momentum cells of size h^3, per volume element, between ν_l and $\nu_l + d\nu_l$,

Therefore, the radiation energy per volume element is

$$u = \sum_{l=0}^{\infty} h\nu_l \sum_{j=0}^{\infty} j n_l^{(j)} + \sum_{l=0}^{\infty} \tfrac{1}{2} h\nu_l n_l = \sum_{l=0}^{\infty} h\nu_l (\sum_{j=0}^{\infty} j n_l^{(j)} + \tfrac{1}{2} n_l) . \qquad (2)$$

At thermal equilibrium, the entropy has a maximum under the constraints $(1)_l$ $l = 0,1,2,,...$, on the number of momentum cells, and the constraint (2), on the radiation energy.

To find that maximum, we apply the Lagrange multiplier method with multipliers λ_l, $l = 0,1,2,,...$, for the constraints $(1)_l$ $l = 0,1,2,,...$, and $\beta > 0$, for the constraint (2), to the auxiliary function

$$F(n_l^{(i)},\lambda_l,\beta) = \sum_{l=0}^{\infty} n_l \log n_l - \sum_{l=0}^{\infty}\sum_{j=0}^{\infty} n_l^{(i)} \log n_l^{(i)}$$

$$+\sum_{l=0}^{\infty}\lambda_l (n_l - \sum_{j=0}^{\infty} n_l^{(i)}) + \beta(u - \sum_{l=0}^{\infty} h\nu_l \sum_{j=0}^{\infty}(j+\tfrac{1}{2})n_l^{(i)}) .$$

The critical points are at

$$0 = \frac{\partial F}{\partial n_l^{(j)}} = -\log n_l^{(j)} - 1 - \lambda_l - \beta(j+\tfrac{1}{2})h\nu_l , \quad (3)$$

$$0 = \frac{\partial F}{\partial \lambda_l} = n_l - \sum_{j=0}^{\infty} n_l^{(j)} , \quad (4)$$

$$0 = \frac{\partial F}{\partial \beta} = u - \sum_{l=0}^{\infty} h\nu_l \sum_{j=0}^{\infty}(j+\tfrac{1}{2})n_l^{(j)} . \quad (5)$$

From (3),

$$\log n_l^{(j)} = -1 - \lambda_l - \beta(j+\tfrac{1}{2})h\nu_l , \quad (6)$$

or,

$$n_l^{(j)} = e^{-1-\lambda_l-\frac{1}{2}\beta h\nu_l} e^{-i\beta h\nu_l} . \quad (7)$$

By (5), and (6),

$$n_l = \sum_{j=0}^{\infty} n_l^{(j)} = e^{-1-\lambda_l-\frac{1}{2}\beta h\nu_l} \sum_{j=0}^{\infty} e^{-j\beta h\nu_l} = \frac{e^{-1-\lambda_l-\frac{1}{2}\beta h\nu_l}}{1-e^{-\beta h\nu_l}} = \frac{e^{-1-\lambda_l+\frac{1}{2}\beta h\nu_l}}{e^{\beta h\nu_l}-1} . \quad (8)$$

Therefore,

$$1 + \lambda_l + \log n_l = \log \frac{e^{\frac{1}{2}\beta h\nu_l}}{e^{\beta h\nu_l}-1} . \quad (9)$$

The quotient

$$\frac{n_l^{(j)}}{\sum_{j=0}^{\infty} n_l^{(j)}} = e^{-(j+1)\beta h\nu_l} \frac{1}{e^{\beta h\nu_l} - 1}$$

is the Bose-Einstein distribution function.

By (8),

$$\sum_{j=0}^{\infty} j n_l^{(j)} = e^{-1-\lambda_l - \frac{1}{2}\beta h\nu_l} \sum_{j=0}^{\infty} j e^{-j\beta h\nu_l} = n_l (1 - e^{-\beta h\nu_l}) \sum_{j=1}^{\infty} (-\frac{d}{d(\beta h\nu_l)} e^{-j\beta h\nu_l}).$$

The series $\sum_{j=1}^{\infty} e^{-j\beta h\nu_l}$ converges uniformly to $\frac{1}{1-\mathrm{e}^{-\beta h\nu_l}}$, and can be differentiated term by term with respect to $\xi \equiv \beta h\nu_l$. Thus,

$$\sum_{j=1}^{\infty} (-\frac{d}{d(\beta h\nu_l)} e^{-j\beta h\nu_l}) = -\frac{d}{d\xi} \sum_{j=0}^{\infty} e^{-j\xi} = -\frac{d}{d\xi} \frac{1}{1-e^{-\xi}} = \frac{e^{-\xi}}{(1-e^{-\xi})^2}.$$

Therefore,

$$\sum_{j=0}^{\infty} j n_l^{(j)} = n_l (1 - e^{-\beta h\nu_l}) \frac{e^{-\beta h\nu_l}}{(1-e^{-\beta h\nu_l})^2} = \frac{n_l}{e^{\beta h\nu_l} - 1}. \qquad (10)$$

By (5), and (4), and (10),

$$u = \sum_{l=0}^{\infty} h\nu_l \sum_{j=0}^{\infty} (j + \tfrac{1}{2}) n_l^{(j)} = \sum_{l=0}^{\infty} n_l h\nu_l (\frac{1}{e^{\beta h\nu_l} - 1} + \tfrac{1}{2}). \quad (11)$$

By (6),

$$s \approx k_B \sum_{l=0}^{\infty} n_l \log n_l - k_B \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} n_l^{(j)} (-1 - \lambda_l - \beta(j + \tfrac{1}{2}) h\nu_l).$$

$$= k_B \sum_{l=0}^{\infty} n_l (1 + \lambda_l + \log n_l) + k_B \beta \sum_{l=0}^{\infty} h\nu_l \sum_{j=0}^{\infty} (j + \tfrac{1}{2}) n_l^{(j)}.$$

Substituting (9), and (10), and (4),

$$s(\beta) = k_B \sum_{l=0}^{\infty} n_l \log \frac{e^{\frac{1}{2}\beta h\nu_l}}{e^{\beta h\nu_l} - 1} + k_B \beta \sum_{l=0}^{\infty} n_l h\nu_l (\frac{1}{e^{\beta h\nu_l} - 1} + \tfrac{1}{2})$$

Using (11),

$$s(\beta) = k_B \sum_{l=0}^{\infty} n_l \log \frac{e^{\frac{1}{2}\beta h\nu_l}}{e^{\beta h\nu_l} - 1} + k_B \beta u \ . \quad (12)$$

Hence,

$$\frac{1}{T} = \frac{\partial s}{\partial u} = k_B \beta \ .$$

That is,

$$\beta_{\max} = \frac{1}{k_B T},$$

and $\beta_{\max}$ is positive as required.

Then,

$$s(\beta_{\max}) = k_B \sum_{l=0}^{\infty} n_l \log \frac{e^{\frac{1}{2}h\nu_l / k_B T}}{e^{h\nu_l / k_B T} - 1} + \frac{u}{T} \qquad (13)$$

is the maximal entropy.

To confirm that the entropy is maximal, we recall that the Lagrange multiplier method produces either a maximum or a minimum for the function in question, which here is the entropy. Thus, to confirm a maximum, it is enough to check the value of the entropy at any other point. For instance, take

$$\beta = 1 \ .$$

Then,

$$\frac{e^{h\nu_l/2}}{e^{h\nu_l} - 1} < \frac{e^{h\nu_l/2k_B T}}{e^{h\nu_l/k_B T} - 1},$$

and for $k_B T < 1$,

$$s(1) = k_B \sum_{l=0}^{\infty} n_l \log \frac{e^{\frac{1}{2}h\nu_l}}{e^{h\nu_l} - 1} + k_B u < k_B \sum_{l=0}^{\infty} n_l \log \frac{e^{\frac{1}{2}h\nu_l / k_B T}}{e^{h\nu_l / k_B T} - 1} + \frac{u}{T} = s(\beta_{\max}) \ .$$

Therefore, at $\beta = \beta_{\max}$, the entropy is maximal, and at the attained thermodynamic equilibrium

$$u(\beta_{\max}) = \sum_{l=0}^{\infty} n_l h\nu_l \left(\frac{1}{e^{h\nu_l / k_B T} - 1} + \tfrac{1}{2}\right) = \sum_{l=0}^{\infty} n_l \overline{\varepsilon_\nu} \, .$$

Then,

$$\overline{\varepsilon_\nu} = h\nu_l \left(\frac{1}{e^{h\nu_l / k_B T} - 1} + \tfrac{1}{2}\right)$$

is Planck's radiation law with zero point energy.

REFERENCES

1. Dannon, H. Vic, *Zero Point Energy: Planck Radiation Law*, Gauge Institute Journal of Math and Physics Vol.1 No 3, August 2005.

2. Planck, M., {Annalen der physik 37 (1912):p.642}.

3. Planck, M. *The Theory of Heat Radiation*, Dover 1959. Page 129.

4. Bose, S., {Zeitschrift fur Physik 26, 178 (1924)} translated into *Planck's Law and the Light quantum Hypothesis* in "Pauli and the Spin-Statistics Theorem" by Ian Duck, and E C G Sudarshan, World Scientific 1997, page 78.

5. Einstein, A. {S. B. d. Preuss. Akad. Wiss. Ber. 22, 261 (1924)} translated into *Quantum Theory of Mono-Atomic Ideal Gas* in "Pauli and the Spin-Statistics Theorem" by Ian Duck, and E C G Sudarshan, World Scientific 1997, page 82.