# Einstein's Zero Point Energy is Inconsistent with Planck's Radiation Law

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**Abstract** Zero Point energy appeared in Planck's Radiation Law, when in the derivation of his Radiation Law, he assumed that the radiation energy of his oscillator includes Zero Point Energy.

Being unaware of his assumption, Planck believed that quantized energy implied the zero point energy.

Actually, Planck's Zero Point Energy is an assumption independent of the well-established fact that energy is quantized.

Einstein-Stern believed that Zero Point Energy is a fact that implies the Radiation Law, with no need for quantized energy.

We show that Einstein-Stern Dipole Model that assumes Zero Point Energy fails to produce Planck's Radiation Law, and is inconsistent with it. In his final treatment of Radiation, Einstein does not mention Zero Point Energy.

Keywords: Planck's radiation law, Zero Point energy, Quantum hypothesis, Bose statistics, Entropy radiation law.

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# Contents

# Zero Point Energy and the Radiation Law

Zero point energy was introduced by Planck in 1912 [Planck1] in his second radiation law. The minimal average energy of a Planck radiator is $\frac{1}{2}\hbar\omega$, even at absolute temperature zero.

Planck used his quantum hypothesis to derive his second radiation law that includes the ZPE, and the ZPE seemed to be a consequence of the quantum hypothesis.

Einstein-Stern examined specific heat measurements by Eucken that convinced them of the existence of Zero Point Energy [EinsteinStern]. They utilized an old model that was earlier examined by Planck, and discarded by him since it was difficult to augment that model with the quantum hypothesis.

They erroneously believed that their model did not use the quantum hypothesis, and hopped that the Radiation Law resulted from the Zero Point Energy.

They speculated that the radiation law solves the differential equation that represents the model, and that Zero Point Energy may replace the quantum hypothesis [EinsteinStern,p145]:

*"The assumption of zero point energy opens a way for deriving Planck's radiation formula without recourse to any kind of discontinuities. Nevertheless, it seems doubtful that the other*

*difficulties can also be overcome without the assumption of quanta."*

The "other difficulties" are the incompatibility of $\hbar\omega$ Zero Point Energy with Eucken's experiments and Planck's derivation that indicate $\frac{1}{2}\hbar\omega$.

In fact, the Einstein-Stern Model is based on a solution with $\frac{1}{2}\hbar\omega$ zero point energy.

However, Einstein-Stern were not aware of their assumption of quanta. Their decomposition of the electromagnetic radiation field into modes [EinsteinHopf] is a step towards the second quantization of the radiation field, and its description as a collection of oscillators.

Furthermore, their model fails to produce Planck's radiation law.

Even worse, their model that seemed to include Zero Point Energy, produced a Radiation Law with NO Zero Point Energy.

In his 1916 final treatment of the radiation problem, Einstein gave up on the dipole model as a tool to derive Planck's radiation law, and on zero point energy as a replacement for the quantum hypothesis.

In fact, Zero Point Energy has to be assumed separately from the fact that energy is quantized.

Zero Point energy appeared in Planck's Radiation Law, when being unaware of it, he assumed it,

# 1.

# Planck's Zero Point Energy

Planck derived his second radiation law that includes Zero Point Energy by assuming that Zero Point Energy.

Unaware of his assumption, he believed that his quantum hypothesis implied the zero point energy.

Had he utilized his 1901 derivation to obtain his second radiation law, as we do here, he might have noticed that assumption. His 1912 derivation masked the assumption of his Zero Pint Energy.

Thus, he was unable to refute Einstein's claim that zero point energy alone without quantum hypothesis can imply the first radiation law.

Planck was not aware of the necessity of both the ZPE and the quantum hypothesis in his models for the derivation of his second radiation law that includes Zero Point Energy.

## 1.1 Planck's Assumption of Zero Point Energy

Planck's assumptions in his 1912 paper define a harmonic oscillator with evenly spaced energy levels differing by $\hbar\omega$.

The probability that the oscillator will radiate its energy is

$$p$$

The probability that it will retain it is

$$q = 1 - p$$

The probability that the oscillator's energy is between $0$ and $\hbar\omega$ is

$$p_1 .$$

The probability that the energy is between $\hbar\omega$ and $2\hbar\omega$ is

$$p_2 = p_1 q \dots$$

The probability that the energy is between $2\hbar\omega$ and $3\hbar\omega$ is

$$p_3 = p_1 q^2$$

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We have

$$1 = p_1 + p_2 + p_3 + .... + p_n + ....$$

$$= p_1(1 + q + q^2 + .... + q^{n-1} + ....)$$

$$= p_1 \frac{1}{1-q}.$$

Hence,

$$p_1 = 1 - q = p$$

and

$$p_n = pq^{n-1}$$

The oscillator's entropy is

$$s = -kp_1 \log p_1 - kp_2 \log p_2 - kp_3 \log p_3 - kp_4 \log p_4 - ....$$

$$= -kp \log p - kpq \underbrace{\log pq}_{\log p + \log q} - kpq^2 \underbrace{\log pq^2}_{\log p + 2\log q} - kpq^3 \underbrace{\log pq^3}_{\log p + 3\log q} - ....$$

$$= -k \underbrace{(1 + q + q^2 + ...)}_{\frac{1}{1-q} = p} p \log p - k \underbrace{(1 + 2q + 3q^2 + ...)}_{\frac{d}{dq}\underbrace{(q + q^2 + q^3 + ...)}_{\frac{1}{1-q} - 1}} pq \log q - ...$$

$$= -k \log p - k \frac{1}{(1-q)^2} pq \log q$$

$$= -k\log p - k\frac{1-p}{p}\log(1-p)$$

$$= k\left\{-\log p - \frac{1}{p}\log(1-p) + \log(1-p)\right\}$$

$$= k\left\{-\frac{1}{p}\log(1-p) + \log\left(\frac{1}{p}-1\right)\right\}$$

$$= k\left\{-\frac{1}{p}\log p - \frac{1}{p}\underbrace{\left[\log(1-p) - \log p\right]} + \log\left(\frac{1}{p}-1\right)\right\}$$

$$= k\left\{\frac{1}{p}\log\frac{1}{p} - \left(\frac{1}{p}-1\right)\log\left(\frac{1}{p}-1\right)\right\}.$$

To obtain the Radiation Law with Zero Point Energy, the oscillator must have

at the first energy level, between $0$, and $\hbar\omega$,

an average energy of $\frac{1}{2}\hbar\omega$ with probability $p_1$, that is, $\frac{1}{2}\hbar\omega p_1$

at the second energy level, between $\hbar\omega$, and $2\hbar\omega$,

an average energy of $\frac{3}{2}\hbar\omega$ with probability $p_2$, that is, $\frac{3}{2}\hbar\omega p_2$

at the third energy level, between $2\hbar\omega$, and $3\hbar\omega$,

an average energy of $\frac{5}{2}\hbar\omega$ with probability $p_3$, that is, $\frac{5}{2}\hbar\omega p_3$

.........................................

the average energy of the oscillator must be

$$u = \tfrac{1}{2}\hbar\omega p_1 + \tfrac{3}{2}\hbar\omega p_2 + \tfrac{5}{2}\hbar\omega p_3 + ...$$

$$= \tfrac{1}{2}\hbar\omega p(1 + 3q + 5q^2 + ...)$$

$$= \tfrac{1}{2}\hbar\omega p(\underbrace{2 + 4q + 6q^2 + ...}_{2(1+2q+3q^2+...)}) - \tfrac{1}{2}\hbar\omega p\underbrace{(1 + q + q^2 + ...)}_{\frac{1}{1-q}=\frac{1}{p}}$$

$$= \hbar\omega p(\underbrace{1 + 2q + 3q^2 + ...}_{\frac{d}{dq}\left\{q+q^2+q^3+...\right\}}) - \tfrac{1}{2}\hbar\omega$$

$$= \hbar\omega p\frac{d}{dq}\underbrace{\left\{q + q^2 + q^3 + ...\right\}}_{\frac{1}{1-q}-1} - \tfrac{1}{2}\hbar\omega$$

$$= \hbar\omega p\frac{1}{\underbrace{(1-q)^2}_{\frac{1}{p^2}}} - \tfrac{1}{2}\hbar\omega$$

$$= \hbar\omega\frac{1}{p} - \tfrac{1}{2}\hbar\omega$$

$$= \hbar\omega\left(\frac{1}{p} - 1\right) + \tfrac{1}{2}\hbar\omega$$

Here we note the assumption of $\tfrac{1}{2}\hbar\omega$ Zero Point Energy, which was assigned the nonzero probability $p$. Thus,

$$\frac{1}{p} - 1 = \frac{u}{\hbar\omega} - \frac{1}{2},$$

$$\frac{1}{p} = \frac{u}{\hbar\omega} + \frac{1}{2}$$

$$s = k\left\{\frac{1}{p}\log\frac{1}{p} - \left(\frac{1}{p} - 1\right)\log\left(\frac{1}{p} - 1\right)\right\}$$

$$= k\left\{\left(\frac{u}{\hbar\omega} + \frac{1}{2}\right)\log\left(\frac{u}{\hbar\omega} + \frac{1}{2}\right) - \left(\frac{u}{\hbar\omega} - \frac{1}{2}\right)\log\left(\frac{u}{\hbar\omega} - \frac{1}{2}\right)\right\}.$$

$$\frac{\partial s}{\partial u} = k\left\{\frac{1}{\hbar\omega}\log\left(\frac{u}{\hbar\omega}+\frac{1}{2}\right)+\frac{1}{\hbar\omega}\right\} - k\left\{\frac{1}{\hbar\omega}\log\left(\frac{u}{\hbar\omega}-\frac{1}{2}\right)+\frac{1}{\hbar\omega}\right\}$$

$$= \frac{k}{\hbar\omega}\log\frac{\frac{u}{\hbar\omega}+\frac{1}{2}}{\frac{u}{\hbar\omega}-\frac{1}{2}}$$

Since $\frac{\partial s}{\partial u} = \frac{1}{T}$,

$$\frac{k}{\hbar\omega}\log\frac{\frac{u}{\hbar\omega}+\frac{1}{2}}{\frac{u}{\hbar\omega}-\frac{1}{2}} = \frac{1}{T}$$

$$\frac{\frac{u}{\hbar\omega}+\frac{1}{2}}{\frac{u}{\hbar\omega}-\frac{1}{2}} = e^{\frac{\hbar\omega}{kT}}$$

$$u+\frac{1}{2}\hbar\omega = e^{\frac{\hbar\omega}{kT}}\left(u-\frac{1}{2}\hbar\omega\right)$$

$$\frac{1}{2}\hbar\omega\left(e^{\frac{\hbar\omega}{kT}}+1\right) = \left(e^{\frac{\hbar\omega}{kT}}-1\right)u$$

$$u = \frac{1}{2}\hbar\omega\frac{e^{\frac{\hbar\omega}{kT}}+1}{e^{\frac{\hbar\omega}{kT}}-1}$$

$$= \frac{1}{2}\hbar\omega\frac{e^{\frac{\hbar\omega}{kT}}-1+2}{e^{\frac{\hbar\omega}{kT}}-1}$$

$$= \frac{1}{2}\hbar\omega + \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}}-1}.$$

This derivation's assumptions are equivalent to the harmonic oscillator given by the Schrödinger equation.

The assumptions that guarantee the radiation law with ZPE for the harmonic oscillator model include the evenly spaced energy levels, the ZPE, and the quantum hypothesis.

We see that the quantum hypothesis in Planck's derivation of his radiation law, does not imply the zero point energy.

It is the hidden assumption of ZPE in the oscillator's average energy that leads to ZPE in the radiation law.

Thus, Planck's derivation does not establish the existence of ZPE. It only recovers what was assumed at the start.

### 1.2 The Likelihood of Zero Point Energy

From

$$\frac{1}{p} = \frac{u}{\hbar\omega} + \frac{1}{2}$$

$$= \frac{1}{2} + \frac{1}{e^{\frac{\hbar\omega}{kT}} - 1} + \frac{1}{2}$$

$$= \frac{e^{\frac{\hbar\omega}{kT}}}{e^{\frac{\hbar\omega}{kT}} - 1},$$

we have

$$p = \frac{e^{\frac{\hbar\omega}{kT}} - 1}{e^{\frac{\hbar\omega}{kT}}}$$

$$= 1 - e^{-\frac{\hbar\omega}{kT}}$$

Zero Point Energy is assumed with probability.

The ZPE appears in the radiation law for $u(\omega)$ with probability

$$p = 1 - e^{-\frac{\hbar\omega}{kT}}.$$

Computing with

$$h = (6.6260755)10^{-34}\text{Joul/Kelvin}$$

$$\nu \sim (5) \cdot 10^{14}\text{Cycles/sec, for visible light}$$

$$k = (1.380658) \cdot 10^{-23}\text{Joul/Kelvin}$$

$$T = 300^{\circ}\text{Kelvin, for room Temperature}$$

$$-\frac{h\nu}{kT} \sim -79.98692773$$

$$p = 1 - e^{-\frac{\hbar\omega}{kT}}$$

$$\sim 1 - e^{-80}$$

$$\approx 1.$$

That is, Planck assumed Zero Point Energy in his derivation with almost 100% certainty.

Planck's "discovery" of zero point energy was instrumental in prompting Einstein-Stern to look for an experimental confirmation in Eucken's specific heat data. Their analysis precedes Mulliken's observation of zero point energy in vibrations of isotopes. [Mulliken].

# 2.

# Einstein's Zero Point Energy

Einstein and Stern utilized an oscillating dipole model that was used by Planck to obtain

$$\rho(\omega),$$

the radiation energy volume density per frequency in the frequency interval

$$[\omega, \omega + d\omega],$$

in terms of $u(\omega)$, the average energy per mode in $[\omega, \omega + d\omega]$.

$$\rho(\omega) = \frac{\omega^2}{\pi^2 c^3} u(\omega),$$

where

$$\frac{\omega^2}{\pi^2 c^3},$$

is the mode volume density per frequency.

$$\omega = 2\pi\nu,$$

$$\rho(\nu) = 2\pi\rho(\omega) = \frac{8\pi\nu^2}{c^3} u(\nu),$$

$$\frac{d\rho(\nu)}{d\nu} = 4\pi^2 \frac{d\rho(\omega)}{d\omega},$$

The dipole is at thermal equilibrium with an electromagnetic radiation field.

The dipole has mass

$$m,$$

and charge

$$q_e,$$

and it oscillates at frequency

$$\omega$$

in the

$$z$$

direction of the electromagnetic field.

In 1910, Einstein-Hopf showed [EinsteinHopf, equation 9] that the dipole is retarded at random speeds

$$\upsilon(t) \ll c,$$

by a random force

$$R(\nu)\upsilon(t) = \frac{3c\sigma}{10\pi\nu}\left(\rho(\nu) - \frac{1}{3}\nu\frac{d\rho}{d\nu}\right)\upsilon(t),$$

where

$$\sigma,$$

is the radiation-reaction constant.

That force decreases the dipole's kinetic energy.

Over the time period

$$\tau = \Delta t,$$

a random recoil impulse

$$\Delta \equiv P(\Delta t),$$

due to emission or absorption from the radiation field, increases the dipole's kinetic energy.

The thermal equilibrium is attained at the momentum balance between the recoil and the retardation in the mean.

$$m\upsilon(t+\Delta t) = m\upsilon(t) + P(\Delta t) - R(\nu)\upsilon(t)\Delta t \quad (9)$$

Squaring both sides, and keeping terms to first order of $\Delta t$ leads to the stochastic (energy times mass) equation

$$m^2 v^2(t+\Delta t) = m^2 v^2(t) + P^2(\Delta t)$$

$$+2mv(t)P(\Delta t) - 2mv^2(t)R(\nu)\Delta t - 2P(\Delta t)v(t)R(\nu)\Delta t$$

Assuming that

$$\left\langle P(\Delta t)v(t) \right\rangle = 0,$$

and using the equipartition theorem

$$\left\langle mv^2(t+\Delta t) \right\rangle = \left\langle mv^2(t) \right\rangle = kT,$$

the average of the squared momentum equation is

$$\left\langle P^2(\Delta t) \right\rangle = 2kTR(\nu)\Delta t$$

$$= 2kT\frac{3c\sigma}{10\pi\nu}\left(\rho(\nu) - \frac{1}{3}\nu\frac{d\rho}{d\nu}\right)\Delta t.$$

Einstein-Hopf showed [ref. 3, equation 15] that

$$\left\langle P^2(\Delta t) \right\rangle = \frac{c^4\sigma\Delta t}{40\pi^2\nu^3}\rho^2(\nu).$$

Hence,

$$2kT\frac{3c\sigma}{10\pi\nu}\left(\rho(\nu) - \frac{1}{3}\nu\frac{d\rho}{d\nu}\right)\Delta t = \frac{c^4\sigma\Delta t}{40\pi^2\nu^3}\rho^2(\nu),$$

$$3kT\left(\rho(\nu) - \frac{1}{3}\nu\frac{d\rho}{d\nu}\right) = \frac{c^3}{8\pi\nu^2}\rho^2(\nu).$$

They confirmed by substitution that

$$\rho(\nu) = \frac{8\pi\nu^2}{c^3}kT.$$

is a solution. Hence,

$$u(\nu) = kT.$$

is the average energy per mode.

This is the classical Rayleigh-Jeans spectrum that blows at $\omega \approx \infty$, and led to the ultra-violet catastrophe, and the black body radiation problem.

So Einstein-Hopf had to let it go.

But in 1913 Einstein-Stern tried to revive that model by adding to it a zero point energy term.

They derived [EinsteinStern]

$$\left\langle P^2(\Delta t)\right\rangle = \frac{hc\sigma\Delta t}{5\pi}\rho(\nu),$$

and noticed that equating it with

$$2kT\frac{3c\sigma}{10\pi\nu}\left(\rho(\nu) - \frac{1}{3}\nu\frac{d\rho}{d\nu}\right)\Delta t$$

leads to Wien's law.

Adding it to the Right Hand Side, they had

$$2kT\frac{3c\sigma}{10\pi\nu}\left(\rho(\nu) - \frac{1}{3}\nu\frac{d\rho}{d\nu}\right)\Delta t = \frac{c^4\sigma\Delta t}{40\pi^2\nu^3}\rho^2(\nu) + \frac{hc\sigma\Delta t}{5\pi}\rho(\nu),$$

$$3kT\left(\rho(\nu) - \frac{1}{3}\nu\frac{d\rho}{d\nu}\right) = \frac{c^3}{8\pi\nu^2}\rho^2(\nu) + h\nu\rho(\nu).$$

Einstein-Stern substituted into their modified equation

$$\rho(\nu) = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{\frac{h\nu}{kT}} - 1},$$

and confirmed that it satisfies the equation.

That is,

$$u(\nu) = \frac{h\nu}{e^{\frac{h\nu}{kT}} - 1}.$$

This is Planck's Radiation Law WITHOUT Zero Point Energy. Planck's Radiation Law with Zero Point Energy is

$$u(\nu) = \frac{h\nu}{e^{\frac{h\nu}{kT}} - 1} + \frac{1}{2}h\nu\,.$$

The assumed zero point energy $\frac{1}{2}\hbar\omega$ that modifies the Einstein-Stern model, disappears from the final $u(\omega)$

Paradoxically,

*The Einstein-Stern model that included Zero Point Energy, produced a Radiation Law with NO Zero Point Energy*

They speculated that the $h\nu\rho(\nu)$ in their equation meant Zero Point Energy of $h\nu$, while Planck had $\frac{1}{2}h\nu$, and Eucken experiments on specific heats supported $\frac{1}{2}h\nu$.

They wrote [EinsteinStern,p.143]

*"…with the method of calculation sketched here, the zero point energy must be set equal to $h\nu$ in order to arrive at the Planck radiation formula. Future investigation must show whether the discrepancy between this assumption and the assumption underlying the investigation on hydrogen disappears if the calculation is more rigorous…"*

In fact, their equation assumes $\frac{1}{2}h\nu$ zero point energy.

# 3.

# Zero Point Energy of the Solution to Einstein-Stern Model

The solution to the Einstein-Stern Model has indeed $\frac{1}{2}h\nu$ Zero Point Energy.

To add Zero Point Energy to the Einstein-Hopf model,

$$2kT\,\frac{3c\sigma}{10\pi\nu}\left(\rho(\nu)-\frac{1}{3}\nu\frac{d\rho}{d\nu}\right)\Delta t=\frac{c^4\sigma\Delta t}{40\pi^2\nu^3}\rho^2(\nu),$$

we assume that

$$u(\nu)\to u(\nu)+\tfrac{1}{2}h\nu\,.$$

Then,

$$\rho(\nu)=\frac{8\pi\nu^2}{c^3}u(\nu)\to\frac{8\pi\nu^2}{c^3}\left(u(\nu)+\tfrac{1}{2}h\nu\right)=\rho(\nu)+\frac{4\pi h\nu^3}{c^3}$$

$$\frac{d\rho(\nu)}{d\nu}=\frac{16\pi\nu}{c^3}u(\nu)+\frac{8\pi\nu^2}{c^3}\frac{du}{d\nu}$$

$$\to\frac{16\pi\nu}{c^3}\left(u(\nu)+\tfrac{1}{2}h\nu\right)+\frac{8\pi\nu^2}{c^3}\left(\frac{du}{d\nu}+\tfrac{1}{2}h\right)$$

$$=\frac{d\rho(\nu)}{d\nu}+\frac{16\pi\nu}{c^3}\tfrac{1}{2}h\nu+\frac{8\pi\nu^2}{c^3}\tfrac{1}{2}h$$

$$=\frac{d\rho(\nu)}{d\nu}+\frac{12\pi h\nu^2}{c^3}$$

Thus, for the left hand side of the Einstein-Hopf model

$$2kT\frac{3c\sigma}{10\pi\nu}\left(\rho(\nu)-\frac{1}{3}\nu\frac{d\rho}{d\nu}\right)\Delta t\xrightarrow[u(\nu)\to u(\nu)+\frac{1}{2}h\nu]{}$$

$$2kT\frac{3c\sigma}{10\pi\nu}\left(\rho(\nu)+\frac{4\pi h\nu^3}{c^3}-\frac{1}{3}\nu\left[\frac{d\rho(\nu)}{d\nu}+\frac{12\pi h\nu^2}{c^3}\right]\right)\Delta t=$$

$$=2kT\frac{3c\sigma}{10\pi\nu}\left(\rho(\nu)-\frac{1}{3}\nu\frac{d\rho(\nu)}{d\nu}\right)\Delta t.$$

That is, the Left hand Side of the model does not changed when

$$u(\nu)\to u(\nu)+\tfrac{1}{2}h\nu$$

For the Right Hand Side of the Einstein-Hopf Model,

$$\frac{c^4\sigma\Delta t}{40\pi^2\nu^3}\rho^2(\nu)\xrightarrow[u(\nu)\to u(\nu)+\frac{1}{2}h\nu]{}\frac{c^4\sigma\Delta t}{40\pi^2\nu^3}\left(\frac{8\pi\nu^2}{c^3}\right)^2\left(u(\nu)+\tfrac{1}{2}h\nu\right)^2$$

To first order of $h$,

$$=\frac{c^4\sigma\Delta t}{40\pi^2\nu^3}\underbrace{\left(\frac{8\pi\nu^2}{c^3}\right)^2u^2(\nu)}_{\rho^2(\nu)}+\frac{c^4\sigma\Delta t}{40\pi^2\nu^3}\left(\frac{8\pi\nu^2}{c^3}\right)\underbrace{\left(\frac{8\pi\nu^2}{c^3}\right)u(\nu)}_{\rho(\nu)}h\nu$$

$$=\frac{c^4\sigma\Delta t}{40\pi^2\nu^3}\rho^2(\nu)+\frac{c\sigma\Delta t}{5\pi\nu}\rho(\nu)h\nu$$

which is the Right Hand Side of the Einstein-Stern modified model.

Thus, the Einstein-Stern Model indicates $\frac{1}{2}h\nu$ Zero Point Energy.

However, the Einstein-Stern Model fails because it has infinitely many solutions.

# 4.

# Einstein-Stern Dipole Model Has Infinitely Many Solutions

In $\omega$, the Einstein-Stern equation is

$$kT\left(3\rho(\omega)-\omega\frac{d\rho}{d\omega}\right)=\frac{\pi^2c^3}{\omega^2}\rho^2(\nu)+\hbar\omega\rho(\omega).$$

Substituting

$$\rho(\omega)=\frac{\omega^2}{\pi^2c^3}u(\omega)$$

$$\frac{d\rho}{d\omega}=\frac{2\omega}{\pi^2c^3}u(\omega)+\frac{\omega^2}{\pi^2c^3}\frac{du(\omega)}{d\omega}$$

we have

$$kT\underbrace{\left(3\frac{\omega^2}{\pi^2c^3}u(\omega)-\omega(\frac{2\omega}{\pi^2c^3}u(\omega)+\frac{\omega^2}{\pi^2c^3}\frac{du(\omega)}{d\omega})\right)}_{\frac{\omega^2}{\pi^2c^3}\left[u(\omega)-\frac{du(\omega)}{d\omega}\right]}=$$

$$=\underbrace{\frac{\pi^2c^3}{\omega^2}\left(\frac{\omega^2}{\pi^2c^3}\right)^2u^2(\omega)+\hbar\omega\frac{\omega^2}{\pi^2c^3}u(\omega)}_{\frac{\omega^2}{\pi^2c^3}\left[u^2(\omega)+\hbar\omega u(\omega)\right]}$$

$$kT\left(u(\omega)-\frac{du(\omega)}{d\omega}\right)=u^2(\omega)+\hbar\omega u(\omega).$$

The equation is satisfied by

$$u(\omega) = \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1},$$

as well as infinitely many other solutions

It is a Bernoulli differential equation that is reduced to a first order equation by the Lie group transformation

$$u(\omega) = \frac{1}{y(\omega)}.$$

Then,

$$\frac{1}{y} + \omega\frac{y'}{y^2} = \frac{1}{kT}\frac{1}{y^2} + \frac{\hbar\omega}{kT}\frac{1}{y}$$

$$y + \omega y' = \frac{1}{kT} + \frac{\hbar\omega}{kT}y$$

$$\omega y' + y(1 - \frac{\hbar\omega}{kT}) = \frac{1}{kT}$$

For the homogeneous equation,

$$\omega y' + y(1 - \frac{\hbar\omega}{kT}) = 0,$$

$$\frac{y'}{y} = \frac{\hbar}{kT} - \frac{1}{\omega}$$

$$\log y = \frac{\hbar\omega}{kT} - \log\omega + \log C_1$$

$$\log\frac{\omega y}{C_1} = \frac{\hbar\omega}{kT}$$

$$\frac{\omega y}{C_1} = e^{\frac{\hbar\omega}{kT}}$$

$$y(\omega) = \frac{C_1}{\omega} e^{\frac{\hbar\omega}{kT}}$$

By the variation of parameter method, we substitute

$$y(\omega) = \frac{C_1(\omega)}{\omega} e^{\frac{\hbar\omega}{kT}}.$$

into

$$y + \omega y' = \frac{1}{kT} + \frac{\hbar\omega}{kT} y$$

We obtain

$$\frac{C_1(\omega)}{\omega} e^{\frac{\hbar\omega}{kT}} + \omega\left( -\frac{C_1(\omega)}{\omega^2} e^{\frac{\hbar\omega}{kT}} + \frac{C_1'(\omega)}{\omega} e^{\frac{\hbar\omega}{kT}} + \frac{C_1(\omega)}{\omega} e^{\frac{\hbar\omega}{kT}} \frac{\hbar}{kT} \right) =$$

$$= \frac{1}{kT} + \frac{\hbar\omega}{kT} \frac{C_1(\omega)}{\omega} e^{\frac{\hbar\omega}{kT}},$$

$$C_1'(\omega) e^{\frac{\hbar\omega}{kT}} = \frac{1}{kT},$$

$$C_1'(\omega) = \frac{1}{kT} e^{-\frac{\hbar\omega}{kT}}$$

$$C_1(\omega) = -\frac{1}{\hbar} e^{-\frac{\hbar\omega}{kT}} + C_0$$

$$y(\omega) = \frac{1}{\omega}\left( -\frac{1}{\hbar} e^{-\frac{\hbar\omega}{kT}} + C_0 \right) e^{\frac{\hbar\omega}{kT}}$$

$$= -\frac{1}{\hbar\omega} + \frac{1}{\omega} C_0 e^{\frac{\hbar\omega}{kT}}$$

$$u(\omega) = \frac{1}{-\frac{1}{\hbar\omega} + \frac{1}{\omega} C_0 e^{\frac{\hbar\omega}{kT}}},$$

$$= \frac{\hbar\omega}{-1 + \hbar C_0 e^{\frac{\hbar\omega}{kT}}},$$

Denoting $C = \hbar C_0$,

$$= \frac{\hbar\omega}{Ce^{\frac{\hbar\omega}{kT}} - 1}.$$

$$\rho(\omega) = \frac{\omega^2}{\pi^2 c^3} \frac{\hbar\omega}{Ce^{\frac{\hbar\omega}{kT}} - 1}.$$

Can we determine $C$ by imposing the boundary conditions

$\rho(\omega) \approx 0$, at $\omega \approx 0$, and at $\omega \approx \infty$?

For $\omega \approx 0$,

$$e^{\frac{\hbar}{kT}\omega} \approx e^{\frac{\hbar}{kT}0} = 1,$$

and

$$\rho(\omega) \approx \frac{\hbar 0^3}{\pi^2 c^3} \frac{1}{C - 1}.$$

For any $C \neq 1$, we get $\rho(\omega) \approx 0$.

If $C = 1$, then $\frac{\hbar 0^3}{\pi^2 c^3} \frac{1}{e^{\frac{\hbar}{kT}0} - 1} \approx \frac{0}{0}$. But by L'hospital,

$$\frac{\frac{d}{d\omega}\left(\frac{\hbar\omega^3}{\pi^2 c^3}\right)}{\frac{d}{d\omega}(e^{\frac{\hbar}{kT}\omega} - 1)} = \frac{\frac{\hbar}{\pi^2 c^3} 3\omega^2}{\frac{\hbar}{kT} e^{\frac{\hbar}{kT}\omega}}$$

$$= 3\frac{kT}{\pi^2 c^3} \frac{\omega^2}{e^{\frac{\hbar}{kT}\omega}}$$

$$\approx 3\frac{kT}{\pi^2 c^3}\frac{0^2}{e^{\frac{\hbar}{kT}0}} = 0$$

Thus, for any arbitrary constant $C$,

$$\omega \approx 0 \Rightarrow \rho(\omega) \approx 0\,.$$

For $\omega \approx \infty$,

$$e^{\frac{\hbar}{kT}\omega} \approx e^{\frac{\hbar}{kT}\infty} = \infty\,,$$

and

$$\rho(\omega) \approx \frac{\hbar\infty^3}{\pi^2 c^3}\frac{1}{C\infty - 1}\,.$$

For any $C \neq 0$, we get

$$\frac{\hbar\infty^3}{\pi^2 c^3}\frac{1}{C\infty - 1} \approx \frac{\infty}{\infty}\,.$$

By L'hospital,

$$\frac{\left(\frac{d}{d\omega}\right)^3\left(\frac{\hbar\omega^3}{\pi^2 c^3}\right)}{\left(\frac{d}{d\omega}\right)^3(Ce^{\frac{\hbar}{kT}\omega} - 1)} = \frac{\frac{\hbar}{\pi^2 c^3}6}{C\left(\frac{\hbar}{kT}\right)^3 e^{\frac{\hbar}{kT}\omega}}$$

$$\approx \frac{6}{C\hbar^2\pi^2}\left(\frac{kT}{c}\right)^3\frac{1}{e^{\frac{\hbar}{kT}\infty}} = 0\,.$$

Thus, Einstein-Stern dipole model has infinitely many solutions.

# 5.

# Removing Zero Point Energy From Einstein's Model is Inconsistent With The Radiation Law

To be consistent with the radiation law, the removal of the $\frac{1}{2}h\nu$ Zero Point Energy from Einstein's dipole model should reduce the solution to a radiation law without ZPE.

Einstein Dipole model solution is already a Radiation Law without ZPE, and the removal of the ZPE from the model contradicts the Radiation Law.

Removing the zero point energy from Bernoulli's equation, gives

$$u(\omega) - \omega u'(\omega) = \frac{1}{kT} u^2(\omega)$$

The Lie Transformation

$$u = \frac{1}{y}$$

reduces the equation to

$$\frac{1}{y} + \omega \frac{y'}{y^2} = \frac{1}{kT}\frac{1}{y^2},$$

$$y + \omega y' = \frac{1}{kT}.$$

$$y(\omega) = \frac{C_1}{\omega}$$

solves the homogeneous equation, and we substitute

$$y(\omega) = \frac{C_1(\omega)}{\omega}$$

into the equation. Then,

$$\frac{C_1(\omega)}{\omega} + \omega\frac{C_1'(\omega)}{\omega} - \omega\frac{C_1(\omega)}{\omega^2} = \frac{1}{kT},$$

$$C_1'(\omega) = \frac{1}{kT},$$

$$C_1(\omega) = \frac{1}{kT}\omega + C_0.$$

$$y = \frac{C_1(\omega)}{\omega} = \frac{1}{kT} + \frac{1}{\omega}C_0$$

$$= \frac{\omega + kTC_0}{kT\omega}$$

$$u(\omega) = \frac{kT\omega}{\omega + kTC_0}.$$

$$\rho(\omega) = \frac{\omega^2}{\pi^2 c^3}\frac{kT\omega}{\omega + kTC_0}$$

For $\omega \approx 0$,

For any $C_0 \neq 0$, we get

$$\rho(\omega) \approx \frac{\omega^2}{\pi^2 c^3}\frac{\cancel{kT}\omega}{0 + \cancel{kT}C_0}$$

$$\approx \frac{0^2}{\pi^2 c^3} \frac{0}{C_0} = 0$$

If $C_0 = 0$, then

$$\rho(\omega) \approx \frac{0^2}{\pi^2 c^3} \frac{kT\cancel{\omega}}{\cancel{\omega}} = 0 \ .$$

Thus, for any $C_0$,

$$\omega \approx 0 \Rightarrow \rho(\omega) \approx 0 \, .$$

For $\omega \approx \infty$,

$$\rho(\omega) = \frac{\omega^2}{\pi^2 c^3} \frac{kT\omega}{\omega + kTC_0} \, .$$

$$\approx \frac{\omega^2}{\pi^2 c^3} \frac{kT}{1 + \dfrac{kTC_0}{\omega}}$$

$$\approx \frac{\infty^2}{\pi^2 c^3} \frac{kT}{1 + \dfrac{kTC_0}{\infty}} \approx \infty$$

That is, for any $C_0$,

$$\omega \approx \infty \Rightarrow \rho(\omega) \approx \infty \, ,$$

indicating the ultra violet catastrophe, and the classical Rayleigh-Jeans spectrum.

Thus, the removal of the zero point energy destroys the model, and reveals its inconsistency with the radiation law.

# 6.

# Adding Zero Point Energy to Einstein's Model is Inconsistent with the Radiation Law.

Had Planck assumed his zero point energy to be $\hbar\omega$ rather than $\frac{1}{2}\hbar\omega$, his radiation law would have been

$$u(\omega) = \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1} + \hbar\omega.$$

showing the added zero point energy.

The addition of $\hbar\omega$ zero point energy in the Einstein-Stern model gives the Bernoulli equation

$$kT\left(u(\omega) - \frac{du(\omega)}{d\omega}\right) = u^2(\omega) + 2\hbar\omega u(\omega)$$

Then,

$$u(\omega) - \omega u'(\omega) = \frac{1}{kT}u^2(\omega) + 2\frac{\hbar\omega}{kT}u(\omega),$$

with infinitely many solutions

$$u(\omega) = \frac{2\hbar\omega}{Ce^{2\frac{\hbar}{kT}\omega} - 1}$$

The factor

$$e^{2\frac{\hbar}{kT}\omega}$$

further corrupts Planck's radiation law.

The solutions have no zero point energy because

$$T \approx 0 \Rightarrow e^{2\frac{\hbar\omega}{kT}} \approx \infty \Rightarrow u(\omega) \approx 0$$

The additional zero point energy does not show up in $u(\omega)$, but generates a nonphysical radiation law.

# 7.

# Einstein's Giving Up on Zero Point Energy

In his last treatment of the quantum theory of radiation [Einstein], Einstein assumes the quantum hypothesis, Wien's law, and spontaneous emission with Bohr rules, to derive Planck's radiation law without zero point energy, and to confirm that

$$\left\langle P^2(\Delta t)\right\rangle = 2kTR(\nu)\Delta t\,.$$

He claims that the radiation-reaction is given by

$$R(\nu) = \frac{h\nu}{c^2 S} p_n B_n^m e^{-\frac{\varepsilon_n}{kT}}\left(\rho(\nu) - \frac{1}{3}\nu\rho'(\nu)\right)\left(1 - e^{-\frac{h\nu}{kT}}\right)$$

He substitutes

$$\rho(\nu) = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{\frac{h\nu}{kT}} - 1},$$

and obtains

$$R(\nu) = \frac{h\nu}{c^2 S} p_n B_n^m e^{-\frac{\varepsilon_n}{kT}}\frac{\rho(\nu)}{3}\frac{h\nu}{kT}$$

Thus,

$$2kTR(\nu)\Delta t = \left(\frac{2}{S} p_n B_n^m e^{-\frac{\varepsilon_n}{kT}}\rho(\nu)\Delta t\right)\left(\frac{1}{3}\left[\frac{h\nu}{c}\right]^2\right).$$

The first factor on the right is the number of induced transitions from the molecular state $z_m$, into the molecular state $z_n$, The second factor is the average momentum transfer per transition.

Therefore, the product equals

$$\left\langle P^2(\Delta t)\right\rangle.$$

And this confirms that radiation is a directional collision process.

While these claims need to be checked out, Zero Point Energy is not amongst them.

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