Zero Point Energy Does Not Imply the Radiation Law

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Abstract. Zero point energy was introduced by Planck in 1912 in his second radiation law. The minimal average energy of a Planck radiator is $\frac{1}{2}\hbar\omega$, even at absolute temperature zero.

In 1913, Einstein-Stern proposed that Zero Point Energy implies the radiation law with no need for the quantum hypothesis.

In 1916 Einstein gave up quietly on that claim. But in 1969 it was revived in a paper by Boyer.

Boyer never supplied any derivation of the radiation law, with or without quantum assumptions. We show that his proposed equation leads to a Ricatti type differential equation, with infinitely many solutions, that are inconsistent with the radiation law.

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Contents

Zero Point Energy, and the Quantum Hypothesis

In 1912, Planck added $\frac{1}{2}\hbar\omega$ photon of Zero Point Energy to his Radiation Law of 1901, and proved that the average internal radiation energy per mode in the frequency interval $[\omega, \omega + d\omega]$ at a given absolute temperature T is

$$u(\omega) = \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1} + \tfrac{1}{2}\hbar\omega\,.$$

There are $\dfrac{\omega^2}{\pi^2 c^3}$ modes in $[\omega, \omega + d\omega]$, and the average radiation energy density in it is

$$\rho(\omega) = \frac{\omega^2}{\pi^2 c^3}\left(\frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1} + \tfrac{1}{2}\hbar\omega\right)$$

In 1913, Einstein-Stern proposed that Zero Point Energy implies the radiation law with no need for the quantum hypothesis.

Using the differential operator

$$kT\left(3\rho(\omega) - \omega\frac{d\rho}{d\omega}\right) - \frac{\pi^2 c^3}{\omega^2}\rho^2(\omega)\,,$$

Einstein-Stern failed to show that Zero Point Energy alone, without quantum assumptions, implies Planck's Radiation Law.

In 1916 Einstein gave up quietly on that claim. But in 1969 it was revived in a paper by Boyer.

Boyer never supplied any derivation of the radiation law, with or without quantum assumptions.

In [Boyer1], Boyer plugged Planck's $\rho(\omega)$ into the Einstein differential operator, and found that

$$kT\left(3\rho(\omega) - \omega\frac{d\rho}{d\omega}\right) - \frac{\pi^2 c^3}{\omega^2}\rho^2(\omega) =$$

$$= 3kT\frac{\omega^2}{\pi^2 c^3}\left(\frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1} + \tfrac{1}{2}\hbar\omega\right)$$

$$-k\omega T\frac{d}{d\omega}\left\{\frac{\omega^2}{\pi^2 c^3}\left(\frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1} + \tfrac{1}{2}\hbar\omega\right)\right\}$$

$$-\frac{\pi^2 c^3}{\omega^2}\left\{\frac{\omega^2}{\pi^2 c^3}\left(\frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1} + \tfrac{1}{2}\hbar\omega\right)\right\}^2$$

$$= -\frac{\omega^2}{\pi^2 c^3}\left(\tfrac{1}{2}\hbar\omega\right)^2.$$

That confirms that $\rho(\omega)$ satisfies

$$kT\left(3\rho(\omega) - \omega\frac{d\rho}{d\omega}\right) - \frac{\pi^2 c^3}{\omega^2}\rho^2(\omega) = -\frac{\omega^2}{\pi^2 c^3}\left(\tfrac{1}{2}\hbar\omega\right)^2.$$

If this is a derivation without quantum assumptions,

where does the quantization constant $\hbar$ come from?

Furthermore, without solving the above equation, Boyer claimed that this is the only solution to it. In [Boyer1,p.1379], he called $\rho(\omega)$ "The solution…", and in [Boyer2, p.1310], "a unique solution…”.

He concluded that this is a derivation of Planck's Radiation Law without assuming that radiation energy is quantized.

Indeed, the equation has infinitely many solutions which are inconsistent with the radiation law.

And nowhere did Boyer derive Planck's Radiation Law, with or without assuming, that radiation energy is quantized.

1.

Boyer's Model has Infinitely Many Solutions

Boyer's equation for $u(\omega)$,

$$kT\left(u(\omega) - \omega u'(\omega)\right) - u^2(\omega) = -\left(\tfrac{1}{2}\hbar\omega\right)^2$$

is a Ricatti differential equation.

Thus, the Lie group transformation

$$u(\omega) = \frac{y'(\omega)}{y(\omega)}$$

will yield a second order differential equation with no $y^2(\omega)$ term.

To eliminate also the $y'(\omega)$ term, we use the Lie transformation

$$u(\omega) = \alpha(\omega)\frac{y'(\omega)}{y(\omega)}.$$

Then,

$$u' = \alpha'\frac{y'}{y} + \alpha\frac{y''}{y} - \alpha\frac{y'}{y^2}y'$$

Substituting into the Ricatti equation,

$$kT\alpha\frac{y'}{y} - kT\omega\left(\alpha'\frac{y'}{y} + \alpha\frac{y''}{y} - \alpha\frac{y'}{y^2}y'\right) - \left(\alpha\frac{y'}{y}\right)^2 = -\left(\tfrac{1}{2}\hbar\omega\right)^2$$

To eliminate the $\frac{(y')^2}{y^2}$ terms, we equate

$$kT\omega\alpha = \alpha^2$$

Then,

$$\alpha(\omega) = kT\omega\,,$$

$$\alpha' = kT$$

Then, both the $\dfrac{(y')^2}{y^2}$, and the $\dfrac{y'}{y}$ terms vanish.

And the equation reduces to

$$-kT\omega\left(\alpha\frac{y''}{y}\right) = -\left(\tfrac{1}{2}\hbar\omega\right)^2,$$

$$(kT\omega)^2\frac{y''}{y} = \left(\tfrac{1}{2}\hbar\omega\right)^2$$

$$\frac{y''}{y} = \left(\frac{1}{2kT}\hbar\right)^2$$

$$y'' - \left(\frac{\hbar}{2kT}\right)^2 y = 0$$

Therefore,

$$y(\omega) = C_1 e^{\frac{\hbar}{2kT}\omega} + C_2 e^{-\frac{\hbar}{2kT}\omega}$$

$$y'(\omega) = \frac{\hbar}{2kT}\left(C_1 e^{\frac{\hbar}{2kT}\omega} - C_2 e^{-\frac{\hbar}{2kT}\omega}\right)$$

$$u(\omega) = \alpha(\omega)\frac{y'}{y}$$

$$= kT\omega\frac{\dfrac{\hbar}{2kT}\left(C_1 e^{\frac{\hbar}{2kT}\omega} - C_2 e^{-\frac{\hbar}{2kT}\omega}\right)}{C_1 e^{\frac{\hbar}{2kT}\omega} + C_2 e^{-\frac{\hbar}{2kT}\omega}}$$

$$= \tfrac{1}{2}\hbar\omega \frac{C_1 e^{\frac{\hbar}{kT}\omega} - C_2}{C_1 e^{\frac{\hbar}{kT}\omega} + C_2}$$

$$= \tfrac{1}{2}\hbar\omega \frac{-\frac{C_1}{C_2} e^{\frac{\hbar}{kT}\omega} + 1}{-\frac{C_1}{C_2} e^{\frac{\hbar}{kT}\omega} - 1}$$

Denoting $-\dfrac{C_1}{C_2} = C$

$$= \tfrac{1}{2}\hbar\omega \frac{C e^{\frac{\hbar}{kT}\omega} + 1}{C e^{\frac{\hbar}{kT}\omega} - 1}$$

$$= \tfrac{1}{2}\hbar\omega \frac{C e^{\frac{\hbar}{kT}\omega} - 1 + 2}{C e^{\frac{\hbar}{kT}\omega} - 1}$$

$$= \tfrac{1}{2}\hbar\omega \underbrace{\frac{C e^{\frac{\hbar}{kT}\omega} - 1}{C e^{\frac{\hbar}{kT}\omega} - 1}}_{1} + \tfrac{1}{2}\hbar\omega \frac{2}{C e^{\frac{\hbar}{kT}\omega} - 1}$$

$$= \tfrac{1}{2}\hbar\omega + \hbar\omega \frac{1}{C e^{\frac{\hbar}{kT}\omega} - 1}$$

$$= \hbar\omega \left(\frac{1}{C e^{\frac{\hbar}{kT}\omega} - 1} + \tfrac{1}{2} \right)$$

This is a variation on the radiation Law, with what will turn out to be a pesky constant C that we cannot get rid of. Hence, infinitely many solutions.

We have,

$$\rho(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3}\left(\frac{1}{Ce^{\frac{\hbar}{kT}\omega} - 1} + \tfrac{1}{2}\right).$$

Can we determine C from the boundary conditions

$\rho(\omega) \approx 0$, at $\omega \approx 0$, and at $\omega \approx \infty$?

For $\omega \approx 0$,

$$e^{\frac{\hbar}{kT}\omega} \approx e^{\frac{\hbar}{kT}0} = 1,$$

and

$$\rho(\omega) \approx \frac{\hbar 0^3}{\pi^2 c^3}\left(\frac{1}{C - 1} + \tfrac{1}{2}\right).$$

For any $C \neq 1$, we get $\rho(\omega) \approx 0$.

If $C = 1$, then $\frac{\hbar 0^3}{\pi^2 c^3}\frac{1}{e^{\frac{\hbar}{kT}0} - 1} \approx \frac{0}{0}$. But by L'hospital,

$$\frac{\frac{d}{d\omega}\left(\frac{\hbar\omega^3}{\pi^2 c^3}\right)}{\frac{d}{d\omega}(e^{\frac{\hbar}{kT}\omega} - 1)} = \frac{\frac{\hbar}{\pi^2 c^3}3\omega^2}{\frac{\hbar}{kT}e^{\frac{\hbar}{kT}\omega}}$$

$$= 3\frac{kT}{\pi^2 c^3}\frac{\omega^2}{e^{\frac{\hbar}{kT}\omega}}$$

$$\approx 3\frac{kT}{\pi^2 c^3}\frac{0^2}{e^{\frac{\hbar}{kT}0}} = 0$$

Thus, for any arbitrary constant C,

$$\omega \approx 0 \Rightarrow \rho(\omega) \approx 0.$$

For $\omega \approx \infty$,

$$e^{\frac{\hbar}{kT}\omega} \approx e^{\frac{\hbar}{kT}\infty} = \infty\,,$$

and

$$\rho(\omega) \approx \frac{\hbar\infty^3}{\pi^2 c^3}\left(\frac{1}{C\infty - 1} + \tfrac{1}{2}\right).$$

For any $C \neq 0$, we get

$$\frac{\hbar\infty^3}{\pi^2 c^3}\frac{1}{C\infty - 1} \approx \frac{\infty}{\infty}.$$

By L'hospital,

$$\frac{\left(\dfrac{d}{d\omega}\right)^3\left(\dfrac{\hbar\omega^3}{\pi^2 c^3}\right)}{\left(\dfrac{d}{d\omega}\right)^3\left(Ce^{\frac{\hbar}{kT}\omega} - 1\right)} = \frac{\dfrac{\hbar}{\pi^2 c^3}6}{C\left(\dfrac{\hbar}{kT}\right)^3 e^{\frac{\hbar}{kT}\omega}}$$

$$\approx \frac{6}{C\hbar^2\pi^2}\left(\frac{kT}{c}\right)^3\frac{1}{e^{\frac{\hbar}{kT}\infty}} = 0$$

Although $\dfrac{\hbar\omega^3}{\pi^2 c^3}\left(\dfrac{1}{Ce^{\frac{\hbar}{kT}\omega} - 1} + \tfrac{1}{2}\right) \approx \infty$, because of the zero point energy part.

In any event, the problem has infinitely many solutions.

2.

Removing the Zero Point Energy from Boyer's Model is Inconsistent with the Radiation Law.

To be consistent with the radiation law, the removal of the zero point energy from the dipole model should reduce the solution to Planck’s radiation law without ZPE.

Removing the zero point energy from Ricatti's equation, gives

$$kT\left(u(\omega)-\omega u'(\omega)\right)-u^2(\omega)=0$$

The Lie Transformation

$$u(\omega)=\alpha(\omega)\frac{y'(\omega)}{y(\omega)}.$$

with

$$\alpha(\omega)=kT\omega$$

reduces the equation to

$$-kT\omega\left(\alpha\frac{y''}{y}\right)=0$$

$$y''=0$$

$$y=A_1\omega+A_2$$

$$u(\omega)=kT\omega\frac{A_1}{A_1\omega+A_2}$$

$$= kT\omega \frac{1}{\omega + \frac{A_2}{A_1}}$$

Denoting $C = \frac{A_2}{A_1}$,

$$= kT\omega \frac{1}{\omega + C}$$

Therefore,

$$\rho(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{kT\omega}{\omega + C}.$$

For $\omega \approx 0$,

$$\rho(\omega) \approx \frac{\hbar}{\pi^2 c^3} \frac{kT}{C} 0^4 = 0$$

and C remains undetermined.

For $\omega \approx \infty$,

$$\frac{\omega}{\omega + C} = \frac{1}{1 + \frac{C}{\omega}}$$

$$\approx \frac{1}{1 + \frac{C}{\infty}} = 1$$

$$\rho(\omega) \approx \frac{\hbar kT}{\pi^2 c^3} \omega^3$$

$$\approx \frac{\hbar kT}{\pi^2 c^3} \infty^3 = \infty$$

Hence, for any C, we obtain the ultra violet catastrophe.

Thus, the removal of the zero point energy destroys the model, and reveals its inconsistency with the radiation law.

3.

Adding Zero Point Energy to Boyer's Model is Inconsistent with the Radiation Law

Had Planck assumed his zero point energy to be $\hbar\omega$ rather than $\frac{1}{2}\hbar\omega$, his radiation law would be

$$u(\omega) = \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1} + \hbar\omega.$$

This does not solve Boyer's model with $\hbar\omega$ zero point energy.

The Ricatti equation

$$kT\left(u(\omega) - \omega u'(\omega)\right) - u^2(\omega) = (\hbar\omega)^2$$

leads to

$$y'' - \left(\frac{\hbar}{kT}\right)^2 y = 0,$$

and to the infinitely many solutions

$$u(\omega) = \frac{2\hbar\omega}{Ce^{2\frac{\hbar}{kT}\omega} - 1} + \hbar\omega$$

The factor

$$e^{2\frac{\hbar}{kT}\omega}$$

further distinguishes the solutions from Planck's radiation law.

The solutions are non-physical, and inconsistent with the radiation law.

Boyer's claim that zero point energy implies the radiation law fails because the dipole's infinitely many solutions are different from Planck's radiation law, and inconsistent with it.

Nowhere does Boyer derive Planck's Radiation Law without quantum assumptions. The quantum hypothesis has no substitute, and no replacement.

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