4-space Curl is a 4-Vector

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Abstract It is commonly believed that the curl in 4-space vector Calculus is a six dimensional vector. We show that contrary to that belief, the 4-space curl is a 4-vector, and supply the correct formula for it. This allows better understanding of the 4-Space Curl Theorem.

Keywords: Infinitesimal, Infinite-Hyper-real, Hyper-real, Cardinal, Infinity. Non-Archimedean, Calculus, Limit, Continuity, Derivative, Integral, Gradient, Divergence, Curl,

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Contents

Introduction

The definition of the Curl of a Vector Function requires the concept of circulation, and the use of infinitesimals.

Since infinitesimals were avoided, and limits are vague, the Curl in many texts is defined by the result that follows from its definition based on the area density of circulation.

Namely, in Cartesian Coordinates, the Curl is defined by

$$\nabla \times \begin{bmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z \end{bmatrix} = \begin{bmatrix} \partial_y R - \partial_z Q \\ \partial_z P - \partial_x R \\ \partial_x Q - \partial_y P \end{bmatrix}$$

$$= \begin{vmatrix} \partial_y & \partial_z \\ Q & R \end{vmatrix} \vec{1}_x + \begin{vmatrix} \partial_z & \partial_x \\ R & Q \end{vmatrix} \vec{1}_y + \begin{vmatrix} \partial_x & \partial_y \\ P & Q \end{vmatrix} \vec{1}_z$$

$$= \begin{vmatrix} \vec{1}_x & \vec{1}_y & \vec{1}_z \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix}.$$

Clearly, the 3-Space Curl is a 3-dimensional vector function, with components in the direction of the unit vectors $\vec{1}_x$, $\vec{1}_y$, and $\vec{1}_z$.

It is not self evident how this result, that became definition, may be generalized to 4-Space with its base of four unit vectors $\vec{1}_x$, $\vec{1}_y$, $\vec{1}_z$, and $\vec{1}_t$.

For instance, we can add a column to obtain

$$\begin{vmatrix} \vec{1}_x & \vec{1}_y & \vec{1}_z & \vec{1}_t \\ \partial_x & \partial_y & \partial_z & \partial_t \\ P & Q & R & S \\ ? & ? & ? & ? \end{vmatrix},$$

but what will be a fourth raw of that 4×4 determinant?

It is less evident how the fact that there are six terms that determine the 4-space curl, lead to the belief that the 4-Space curl is six dimensional.

To obtain the formula for the 4-Space curl, we need to define it through the circulations in 4-space.

To that end, we need to use Infinitesimals, Infinitesimal Calculus, and Infinitesimal Vector Calculus.

We have constructed infinitesimals in [Dan1], established the Infinitesimal Calculus in [Dan2], and derived the Infinitesimal Vector Calculus in [Dan3].

In particular, the Hyper-real Plane, Hyper-real Vector Functions, Hyper-real Continuity, Derivatives, and Hyper-real Integration are presented in [Dan3].

1.

Hyper-real Plane

The Hyper-real 4-Space is a cross product of four Hyper-real lines.

We present the Hyper-real 2-Space, which is a cross product of two Hyper-real lines.

Each 2-vector of real numbers (α, β) can be represented by a Cauchy sequence of rational numbers, $(r_1, q_1), (r_2, q_2), (r_3, q_3)...$ so that $(r_n, q_n) \to (\alpha, \beta)$.

The constant sequence $(\alpha, \beta), (\alpha, \beta), (\alpha, \beta)...$ is a constant hyper-real 2-vector.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to $(0, 0)$ sequences of 2-vectors $(\iota_1, o_1)(\iota_2, o_2), (\iota_3, o_3)...$ constitutes a family of infinitesimal hyper-real 2-vectors.
2. The infinitesimal 2-vectors are smaller than any real 2-vector, yet strictly greater than the zero 2-vector.

3. Their reciprocals $(\frac{1}{\iota_1}, \frac{1}{o_1}), (\frac{1}{\iota_2}, \frac{1}{o_2}), (\frac{1}{\iota_3}, \frac{1}{o_3}), ...$ are the infinite hyper-real 2-vectors.
4. The infinite hyper-real 2-vectors are greater than any real 2-vector, yet strictly smaller than the infinity 2-vector.
5. The infinite hyper-real 2-vectors with negative signs are smaller than any real 2-vector, yet strictly greater than $(-\infty, -\infty)$.
6. The sum of a real 2-vector with an infinitesimal 2-vector is a non-constant hyper-real 2-vector.
7. The Hyper-real 2-vectors are the totality of

   constant hyper-real 2-vectors,

   a family of infinitesimal 2-vectors, with signs that may be $(+,+)$, $(+,-)$, $(-,+)$, or $(-,-)$,

   a family of infinite hyper-real 2-vectors with signs that may be $(+,+)$, $(+,-)$, $(-,+)$, or $(-,-)$, and

   non-constant hyper-real 2-vectors.
8. The hyper-real 2-vectors constitute the <u>Hyper-real Plane</u>.
9. That plane includes the real 2-vectors separated by the non-constant hyper-real 2-vectors. Each real 2-vector is

the center of a disk of infinitesimal radius of hyper-real 2-vectors, that includes no other real 2-vector.

10. In particular, the zero 2-vector is separated from any real 2-vector by infinitesimal 2-vectors that lie in a disk of infinitesimal radius around the zero.

11. The Zero 2-vector is not an infinitesimal 2-vector, because zero is not strictly greater than zero.

12. We do not add the infinity 2-vector to the hyper-real Plane.

13. The infinitesimal 2-vectors, and the infinite hyper-real 2-vectors, are semi-groups with respect to addition. Neither set includes zero.

14. The hyper-real Plane is embedded in $\mathbb{R}^{\infty} \times \mathbb{R}^{\infty}$, and is not homeomorphic to the real Plane. There is no bi-continuous one-one mapping from the hyper-real Plane onto the real plane.

15. In particular, there are no points in the real Plane that can be assigned uniquely to the infinitesimal hyper-real 2-vectors, or to the infinite hyper-real 2-vectorss, or to the non-constant hyper-real 2-vectors.

16. No neighbourhood of a hyper-real 2-vector is homeomorphic to an $\mathbb{R}^n \times \mathbb{R}^n$ ball. Therefore, the hyper-real plane is not a manifold.

17. The hyper-real plane is not spanned by two elements, and it is not two-dimensional.

2.

Hyper-real Vector Function

2.1 Definition of a hyper-real function

$f(x,y)$ *is a hyper-real function, iff it is from the hyper-real 2-vectors into the hyper-reals.*

This means that any number in the domain, or in the range of a hyper-real $f(x,y)$ is either one of the following

real vector

real vector + infinitesimal vector

infinitesimal vector

infinite hyper-real vector

Clearly,

2.2 *Every function from the real plane into the reals is a hyper-real function.*

3.

4-Space Path Integral

The definition of Path Integration extends to a vector of four Hyper-real functions

3.1 4-space Path Integral Definition

the Hyper-real Path Integral of $\begin{bmatrix} P(x,y,z,t) \\ Q(x,y,z,t) \\ R(x,y,z,t) \\ (S(x,y,z,t) \end{bmatrix}$ over a path

$(x(\tau), y(\tau), z(\tau), t(\tau))$, $\tau \in [\alpha, \beta]$, is the sum of the areas

$$\sum_{t\in[\alpha,\beta]} \left\{ P(x(\tau),y(\tau),z(\tau),t(\tau))dx(\tau) + Q(x(\tau),y(\tau),z(\tau),t(\tau))dy(\tau) + \right.$$

$$\left. + R(x(\tau),y(\tau),z(\tau),t(\tau))dz(\tau) + S(x(\tau),y(\tau),z(\tau),t(\tau))dt(\tau) \right\}.$$

If for any infinitesimal dt, the Integration Sum equals the same hyper-real number, then $\begin{bmatrix} P(x,y,z,t) \\ Q(x,y,z,t) \\ R(x,y,z,t) \\ (S(x,y,z,t) \end{bmatrix}$ is Hyper-Real Integrable over the path $\gamma(a,b)$.

Then, we call the Integration Sum the Hyper-Real Path Integral of $\begin{bmatrix} P(x,y,z,t) \\ Q(x,y,z,t) \\ R(x,y,z,t) \\ (S(x,y,z,t) \end{bmatrix}$ over $\gamma(a,b)$, and denote it by

$$\int_{\gamma(a,b)} P(x,y,z,t)dx + Q(x,y,z,t)dy + R(x,y,z,t)dz + S(x,y,z,t)dt .$$

Since there are countably many real numbers in $[\alpha, \beta]$,

5.2 The Integration Sum has countably many terms.

5.3 Continuous $\begin{bmatrix} P(x,y,z,t) \\ Q(x,y,z,t) \\ R(x,y,z,t) \\ (S(x,y,z,t) \end{bmatrix}$ **is Path-Integrable**

If $P(x,y,z,t)$, $Q(x,y,z,t)$, $R(x,y,z,t)$, *and* $S(x,y,z,t)$ *are Continuous on a domain* D

Then $\begin{bmatrix} P(x,y,z,t) \\ Q(x,y,z,t) \\ R(x,y,z,t) \\ (S(x,y,z,t) \end{bmatrix}$ *is Path-Integrable in* D

4.

Hyper-real Area Integral

4.1 Area Integral of $P(x,y)$ Definition

Let $P(x,y)$ be a hyper-real function, defined on a bounded domain in the Hyper-Real Plane.

$P(x,y)$ may take infinite hyper-real values.

An area element is

$$dA = dxdy,$$

For each (x,y), there is an infinitesimal rectangular box with base area $dA = dxdy$, height $P(x,y)$, and volume $P(x,y)dxdy$.

We form the **Double Sum** of all the volumes that are enclosed between the surface of $P(x,y)$, and the $P(x,y)$ domain in the plane

$$\sum_{y=b_1}^{y=b_2} \sum_{x=a_1}^{x=a_2} P(x,y)dxdy.$$

If for any infinitesimals dx, and dy, the Double Sum equals the same hyper-real number, then $P(x,y)$ is Hyper-Real Integrable over the plain domain.

Then, we call the Double Integration Sum the Hyper-Real Area Integral of $P(x,y)$ over the Domain, and denote it by

$$\int_{y=b_1}^{y=b_2}\int_{x=a_1}^{x=a_2} P(x,y)dxdy\,.$$

If the number is an infinite hyper-real, then it equals

$$\int_{y=b_1}^{y=b_2}\int_{x=a_1}^{x=a_2} P(x,y)dxdy\,.$$

If the number is a finite hyper-real, then its constant part equals $\int_{y=b_1}^{y=b_2}\int_{x=a_1}^{x=a_2} P(x,y)dxdy\,.\square$

The Integration Sum may take infinite hyper-real values, such as $\frac{1}{(dx)(dy)}$, but may not equal to ∞.

Since there are countably many real numbers in the plane,

4.2 The Integration Sum is countable.

4.3 Continuous $P(x,y)$ is Area-Integrable

5.

3-Space Surface Integral

5.1 The Surface Area Element

A point on a surface is determined by two parameters u, and v, so that in the x, y, z coordinate system,

$$\vec{r}(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}.$$

At the point $\vec{r}(u,v)$,

the tangent to the surface in the direction of u is

$$\frac{\partial \vec{r}}{\partial u},$$

the tangent to the surface in the direction of v is

$$\frac{\partial \vec{r}}{\partial v},$$

and the normal to the tangent plane is

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{bmatrix} \vec{1}_x & \vec{1}_y & \vec{1}_z \\ \partial_u x & \partial_u y & \partial_u z \\ \partial_v x & \partial_v y & \partial_v z \end{bmatrix}$$

$$= \left(\frac{\partial(y,z)}{\partial(u,v)}, \quad \frac{\partial(z,x)}{\partial(u,v)} \quad \frac{\partial(x,y)}{\partial(u,v)} \right).$$

The unit normal is

$$\vec{1}_n = \frac{\dfrac{\partial \vec{r}}{\partial u} \times \dfrac{\partial \vec{r}}{\partial v}}{\left| \dfrac{\partial \vec{r}}{\partial u} \times \dfrac{\partial \vec{r}}{\partial v} \right|},$$

$$= \begin{bmatrix} (\vec{1}_x, \vec{1}_n) \\ (\vec{1}_y, \vec{1}_n) \\ (\vec{1}_z, \vec{1}_n) \end{bmatrix},$$

$$= \begin{bmatrix} \cos(\vec{1}_x, \vec{1}_n) \\ \cos(\vec{1}_y, \vec{1}_n) \\ \cos(\vec{1}_z, \vec{1}_n) \end{bmatrix}.$$

The surface area element is

$$d\vec{S} = (\frac{\partial \vec{r}}{\partial u} du) \times (\frac{\partial \vec{r}}{\partial v} dv)$$

$$= \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} dudv = \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| \vec{1}_n dudv$$

$$= \begin{bmatrix} \dfrac{\partial(y,z)}{\partial(u,v)} dudv \\ \dfrac{\partial(x,z)}{\partial(u,v)} dudv \\ \dfrac{\partial(x,y)}{\partial(u,v)} dudv \end{bmatrix} = \begin{bmatrix} dydz \\ dxdz \\ dxdy \end{bmatrix} = \begin{bmatrix} dS_x \\ dS_y \\ dS_z \end{bmatrix} = \begin{bmatrix} \vec{1}_x \cdot d\vec{S} \\ \vec{1}_y \cdot d\vec{S} \\ \vec{1}_z \cdot d\vec{S} \end{bmatrix}$$

5.2 Surface Integral of $\varphi(x,y,z)$ Definition

Let

$$\varphi(x,y,z) = \varphi(x(u,v), y(u,v), z(u,v))$$
$$= \varphi(u,v),$$

be hyper-real function, defined on a bounded surface $\vec{r}(u,v)$ in the Hyper-Real 3-space.

$\varphi(u,v)$ may take infinite hyper-real values.

For each (u,v), there is an infinitesimal volume

$$\varphi(u,v)\vec{1}_x \cdot d\vec{S} = \varphi(x,y,z)(\vec{1}_x \cdot \vec{1}_n)dydz\,.$$

We form the **Double Sum**

$$\sum_{z=c_1}^{z=c_2} \sum_{y=b_1}^{y=b_2} \varphi(x,y,z)(\vec{1}_x \cdot \vec{1}_n)dydz\,.$$

If for any infinitesimals dy, and dz, the Double Sum equals the same hyper-real number, then $\varphi(x,y,z)$ is Hyper-Real Integrable over the surface.

Then, we call the Double Integration Sum the Hyper-Real Surface Integral of $\varphi(u,v)$ over the surface, and denote it by

$$\int_{z=c_1}^{z=c_2} \int_{y=b_1}^{y=b_2} \varphi(x,y,z)(\vec{1}_x \cdot \vec{1}_n)dydz\,.$$

If the number is an infinite hyper-real, then it equals $\int\limits_{z=c_1}^{z=c_2}\int\limits_{y=b_1}^{y=b_2}\varphi(x,y,z)(\vec{1}_x\cdot\vec{1}_n)dydz$.

If the number is a finite hyper-real, then its constant part equals $\int\limits_{z=c_1}^{z=c_2}\int\limits_{y=b_1}^{y=b_2}\varphi(x,y,z)(\vec{1}_x\cdot\vec{1}_n)dydz$.□

The Integration Sum may take infinite hyper-real values, such as $\frac{1}{(dy)(dz)}$, but may not equal to ∞.

Since there are countably many real numbers in the plane,

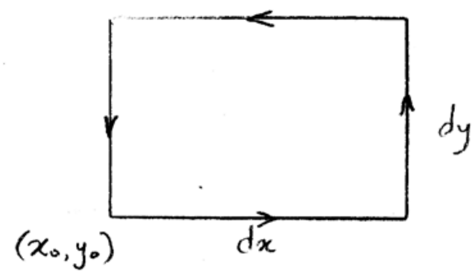
5.3 The Integration Sum is countable.

5.4 Continuous $\varphi(x,y,z)$ is Surface-Integrable.

6.

Plane Curl

Let $P(x,y)$, and $Q(x,y)$, be hyper-real differentiable functions, defined on a rectangle with vertex at (x_0,y_0) and sides dx, dy.



6.1 The Area Curl

The **circulation** of $\begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$ along the rectangle is

$$\oint_{\text{rectangle}} P(x,y)\vec{1}_x \cdot \vec{1}_l dl + Q(x,y)\vec{1}_y \cdot \vec{1}_l dl \,.$$

We define the **area density** of that circulation multiplied by $\vec{1}_z$ as the Curl of $\begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$

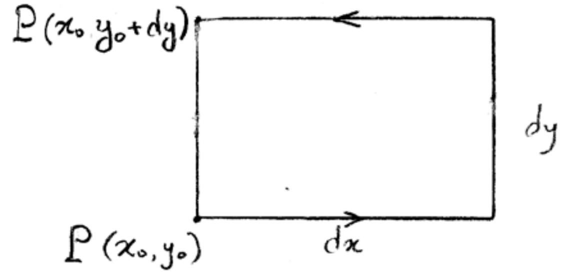
$$\underbrace{\text{Curl}}_{\nabla\times}\begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix} = \vec{1}_z \frac{1}{dxdy} \oint_{\text{rectangle}} P(x,y)\vec{1}_x \cdot \vec{1}_l dl + Q(x,y)\vec{1}_y \cdot \vec{1}_l dl$$

6.2 $$\underbrace{\text{Curl}}_{\nabla\times}\begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}_{x_0,y_0} = \begin{vmatrix} \partial_x & \partial_y \\ P(x,y) & Q(x,y) \end{vmatrix}_{x_0,y_0} \vec{1}_z$$

Proof:

Since $P(x,y)\vec{1}_x \cdot \vec{1}_l dl$ is nonzero when $\vec{1}_l$ is along the x axis,

the circulation path starts at (x_0, y_0), and ends at $(x_0, y_0 + dy)$



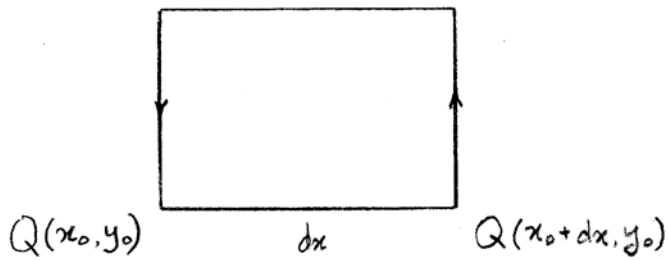
$$\oint_{\text{rectangle}} P(x,y)\vec{1}_x \cdot \vec{1}_l dl = \int_{(x_0,y_0)}^{(x_0,y_0+dy)} P(x,y)\vec{1}_x \cdot \vec{1}_l dl$$

$$= P(x_0,y_0)\vec{1}_x \cdot \vec{1}_x dx + P(x_0,y_0+y)\vec{1}_x \cdot (-\vec{1}_x)dx$$

$$= \left\{P(x_0,y_0) - P(x_0,y_0+y)\right\}dx\,.$$

$$= \left\{-\frac{\partial P}{\partial y}\bigg|_{x_0,y_0} dy\right\}dx\,.$$

Since $Q(x,y)\vec{1}_y \cdot \vec{1}_l dl$ is nonzero when $\vec{1}_l$ is along the y axis, the circulation path starts at $(x_0 + dx, y_0)$, and ends at (x_0, y_0)



$$\oint_{\text{rectangle}} Q(x,y)\vec{1}_y \cdot \vec{1}_l dl = \int_{(x_0+dx,y_0)}^{(x_0,y_0)} Q(x,y)\vec{1}_x \cdot \vec{1}_l dl$$

$$= Q(x_0 + dx, y_0)\vec{1}_y \cdot \vec{1}_y dy + P(x_0,y_0)\vec{1}_y \cdot (-\vec{1}_y) dy$$

$$= \left\{Q(x_0 + dx, y_0) - Q(x_0,y_0)\right\} dy \,.$$

$$= \left\{\frac{\partial Q}{\partial x}\bigg|_{x_0,y_0} dx\right\} dy \,.$$

Therefore,

$$\underbrace{\text{Curl}}_{\nabla\times}\begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix} = \vec{1}_z \frac{1}{dxdy} \oint_{\text{rectangle}} P(x,y)\vec{1}_x \cdot \vec{1}_l dl + Q(x,y)\vec{1}_y \cdot \vec{1}_l dl$$

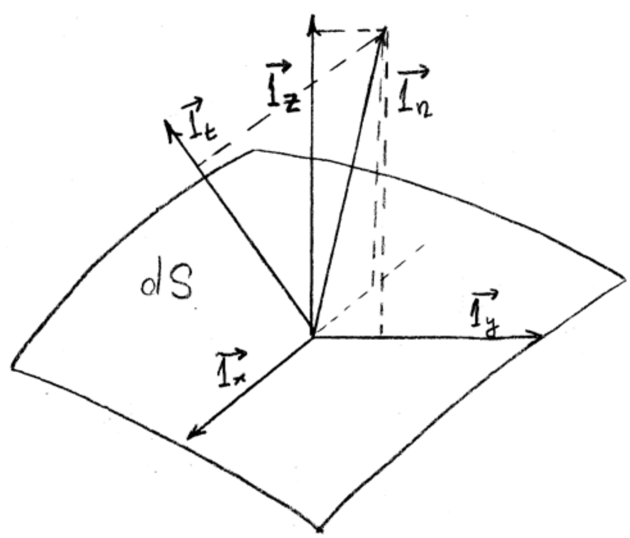
$$= \vec{1}_z \frac{1}{dxdy}\left\{-\frac{\partial P}{\partial y}\bigg|_{x_0,y_0} + \frac{\partial Q}{\partial x}\bigg|_{x_0,y_0}\right\} dxdy$$

$$= (\partial_x Q - \partial_y P)\vec{1}_z . \square$$

7.

4-Space Curl is a 4-Vector

Let $P(x,y,z,t)$, $Q(x,y,z,t)$, $R(x,y,z,t)$, and $S(x,y,z,t)$ be hyper-real differentiable functions, defined on an infinitesimal area element dS. dS projects onto six 2-planes generated by the unit vectors $\vec{1}_x, \vec{1}_y, \vec{1}_z$, and $\vec{1}_t$.



<u>x-y projection</u> with area $dxdy$ and normal $\vec{1}_x \times \vec{1}_y = \vec{1}_z$

<u>y-z projection</u> with area $dydz$ and normal $\vec{1}_y \times \vec{1}_z = \vec{1}_t$

<u>z-t projection</u> with area $dzdt$ and normal $\vec{1}_z \times \vec{1}_t = \vec{1}_x$

<u>t-x projection</u> with area $dtdx$ and normal $\vec{1}_t \times \vec{1}_x = \vec{1}_y$

<u>t-y projection</u> with area $dtdy$ and normal

$$\vec{1}_t \times \vec{1}_y = (\vec{1}_y \times \vec{1}_z) \times \vec{1}_y = \vec{1}_z \underbrace{(\vec{1}_y \cdot \vec{1}_y)}_{1} - \vec{1}_y \underbrace{(\vec{1}_z \cdot \vec{1}_y)}_{0} = \vec{1}_z$$

<u>z-x projection</u> with area $dzdx$ and normal

$$\vec{1}_z \times \vec{1}_x = (\vec{1}_x \times \vec{1}_y) \times \vec{1}_x = \vec{1}_y \underbrace{(\vec{1}_x \cdot \vec{1}_x)}_{1} - \vec{1}_x \underbrace{(\vec{1}_y \cdot \vec{1}_x)}_{0} = \vec{1}_y$$

The projected areas are

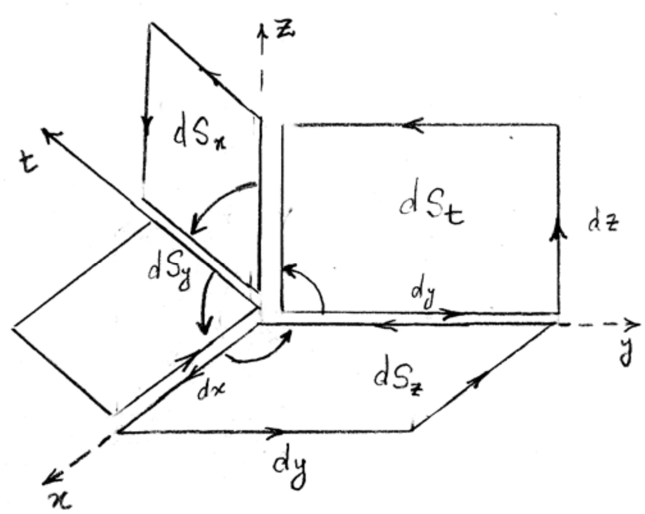
$$dS_x = \vec{1}_x \cdot \vec{1}_n dS = dzdt,$$

$$dS_y = \vec{1}_y \cdot \vec{1}_n dS = dtdx + dzdx,$$

$$dS_z = \vec{1}_z \cdot \vec{1}_n dS = dxdy + dtdy,$$

$$dS_t = \vec{1}_t \cdot \vec{1}_n dS = dydz$$

The projections areas are walls of a box with vertex at (x_0, y_0, z_0, t_0) and sides dx, dy, dz, and dt.



Given positive orientation of a right hand system,

$$\nabla \times \begin{bmatrix} P(x,y,z,t) \\ Q(x,y,z,t) \end{bmatrix} = \vec{1}_z \frac{1}{dxdy} \oint_{\partial(dS_z)} P(x,y,z,t)\vec{1}_x \cdot \vec{1}_l dl + Q(x,y,z,t)\vec{1}_y \cdot \vec{1}_l dl \,,$$

$$= \begin{vmatrix} \partial_x & \partial_y \\ P(x,y,z,t) & Q(x,y,z,t) \end{vmatrix}_{x_0,y_0,z_0,t_0} \vec{1}_z$$

$$\nabla \times \begin{bmatrix} Q(x,y,z,t) \\ R(x,y,z,t) \end{bmatrix} = \vec{1}_t \frac{1}{dydz} \oint_{\partial(dS_t)} Q(x,y,z,t)\vec{1}_y \cdot \vec{1}_l dl + R(x,y,z,t)\vec{1}_z \cdot \vec{1}_l dl \,,$$

$$= \begin{vmatrix} \partial_y & \partial_z \\ Q(x,y,z,t) & R(x,y,z,t) \end{vmatrix}_{x_0,y_0,z_0,t_0} \vec{1}_t$$

$$\nabla \times \begin{bmatrix} R(x,y,z,t) \\ S(x,y,z,t) \end{bmatrix} = \vec{1}_x \frac{1}{dzdt} \oint_{\partial(dS_x)} R(x,y,z,t)\vec{1}_z \cdot \vec{1}_l dl + S(x,y,z,t)\vec{1}_t \cdot \vec{1}_l dl \,,$$

$$= \begin{vmatrix} \partial_z & \partial_t \\ R(x,y,z,t) & S(x,y,z,t) \end{vmatrix}_{x_0,y_0,z_0,t_0} \vec{1}_x \,.$$

$$\nabla \times \begin{bmatrix} S(x,y,z,t) \\ P(x,y,z,t) \end{bmatrix} = \vec{1}_y \frac{1}{dtdx} \oint_{\partial(dS_y)} S(x,y,z,t)\vec{1}_t \cdot \vec{1}_l dl + P(x,y,z,t)\vec{1}_x \cdot \vec{1}_l dl \,,$$

$$= \begin{vmatrix} \partial_t & \partial_x \\ S(x,y,z,t) & P(x,y,z,t) \end{vmatrix}_{x_0,y_0,z_0,t_0} \vec{1}_y$$

$$\nabla \times \begin{bmatrix} S(x,y,z,t) \\ Q(x,y,z,t) \end{bmatrix} = \vec{1}_z \frac{1}{dtdy} \oint_{\partial(dS_z)} S(x,y,z,t)\vec{1}_t \cdot \vec{1}_l dl + Q(x,y,z,t)\vec{1}_y \cdot \vec{1}_l dl \,,$$

$$= \begin{vmatrix} \partial_t & \partial_y \\ S(x,y,z,t) & Q(x,y,z,t) \end{vmatrix}_{x_0,y_0,z_0,t_0} \vec{1}_z$$

$$\nabla \times \begin{bmatrix} P(x,y,z,t) \\ R(x,y,z,t) \end{bmatrix} = \vec{1}_y \frac{1}{dzdx} \oint_{\partial(dS_y)} P(x,y,z,t)\vec{1}_x \cdot \vec{1}_l dl + R(x,y,z,t)\vec{1}_z \cdot \vec{1}_l dl\,,$$

$$= \begin{vmatrix} \partial_x & \partial_z \\ P(x,y,z,t) & R(x,y,z,t) \end{vmatrix}_{x_0,y_0,z_0,t_0} \vec{1}_y$$

7.1 The 4-space Curl is a 4-vector

The 4-space Curl is the sum of the six area curls. That is,

$$\nabla \times \begin{bmatrix} P(x,y,z,t) \\ Q(x,y,z,t) \\ R(x,y,z,t) \\ S(x,y,z,t) \end{bmatrix} =$$

$$= \underbrace{\nabla \times \begin{bmatrix} R \\ S \end{bmatrix}}_{\begin{vmatrix} \partial_z & \partial_t \\ R & S \end{vmatrix} \vec{1}_x} + \underbrace{\nabla \times \begin{bmatrix} S \\ P \end{bmatrix}}_{\begin{vmatrix} \partial_t & \partial_x \\ S & P \end{vmatrix} \vec{1}_y} + \underbrace{\nabla \times \begin{bmatrix} P \\ R \end{bmatrix}}_{\begin{vmatrix} \partial_x & \partial_z \\ P & R \end{vmatrix} \vec{1}_y} + \underbrace{\nabla \times \begin{bmatrix} P \\ Q \end{bmatrix}}_{\begin{vmatrix} \partial_x & \partial_y \\ P & Q \end{vmatrix} \vec{1}_z} + \underbrace{\nabla \times \begin{bmatrix} S \\ Q \end{bmatrix}}_{\begin{vmatrix} \partial_t & \partial_y \\ S & Q \end{vmatrix} \vec{1}_z} + \underbrace{\nabla \times \begin{bmatrix} Q \\ R \end{bmatrix}}_{\begin{vmatrix} \partial_y & \partial_z \\ Q & R \end{vmatrix} \vec{1}_t}$$

$$= \begin{pmatrix} \begin{vmatrix} \partial_z & \partial_t \\ R & S \end{vmatrix} \\ \begin{vmatrix} \partial_t & \partial_x \\ S & P \end{vmatrix} + \begin{vmatrix} \partial_x & \partial_z \\ P & R \end{vmatrix} \\ \begin{vmatrix} \partial_x & \partial_y \\ P & Q \end{vmatrix} + \begin{vmatrix} \partial_t & \partial_y \\ S & Q \end{vmatrix} \\ \begin{vmatrix} \partial_y & \partial_z \\ Q & R \end{vmatrix} \end{pmatrix} = \begin{pmatrix} S_z - R_t \\ P_t - S_x + R_x - P_z \\ Q_x - P_y + Q_t - S_y \\ R_y - Q_z \end{pmatrix}.$$

8.

4-Space Curl Theorem

Let $P(x,y,z,t)$, $Q(x,y,z,t)$, $R(x,y,z,t)$, and $S(x,y,z,t)$ be hyper-real differentiable functions, defined on a surface Σ, with closed boundary $\partial\Sigma$.

8.1 4-Space Curl Theorem

$$\oint_{\partial\Sigma} \Big\{ P(x,y,z,t)\vec{1}_x \cdot \vec{1}_l dl + Q(x,y,z,t)\vec{1}_y \cdot \vec{1}_l dl +$$

$$+ R(x,y,z,t)\vec{1}_z \cdot \vec{1}_l dl + S(x,y,z,t)\vec{1}_t \cdot \vec{1}_l dl \Big\} =$$

$$= \oiint_{\Sigma} \Big\{ (S_z - R_t)dzdt + (P_t - S_x)dtdx + (R_x - P_z)dxdz +$$

$$+(Q_x - P_y)dxdy + (Q_t - S_y)dtdy + (R_y - Q_z)dydz \Big\}$$

Proof: The sum of the $\left\{\partial_z S - \partial_t R\right\} dzdt$ over the infinitesimal rectangles enclosed in a plane area, Σ_x projected by Σ, perpendicular to $\vec{1}_x$,

equals the sum of the circulations

$$\int_{\text{rectangle}} R(x,y,z,t)\vec{1}_z \cdot \vec{1}_l dl + S(x,y,z,t)\vec{1}_t \cdot \vec{1}_l dl = \int_{\text{rectangle}} Rdz + Sdt$$

along the sides of the infinitesimal rectangles enclosed in the area Σ_x,

$$\sum_{z=c_1}^{z=c_2} \sum_{t=d_1}^{t=d_2} \left\{\partial_z S - \partial_t R\right\} dydz =$$

$$= \sum_{z=c_1}^{yz=c_2} \oint_{\text{rectangle}} R(x,y,z,t)dz + \sum_{t=d_1}^{t=d_2} \oint_{\text{rectangle}} S(x,y,z,t)dt$$

The sum over the areas is

$$\oiint_{\Sigma_x} \left\{\partial_z S(x,y,z,t) - \partial_t R(x,y,z,t)\right\} dzdt.$$

The Interior path integrals of

$$\int_{\text{rectangle}} R(x,y,z,t)\vec{1}_z \cdot \vec{1}_l dl, \quad \text{and} \quad \int_{\text{rectangle}} S(x,y,z,t)\vec{1}_t \cdot \vec{1}_l dl,$$

appear in pairs of opposite signs and cancel, leaving the Integral over the boundary line,

$$\oint_{\partial\Sigma_x} R(x,y,z,t)dz + S(x,y,z,t)dt = \oiint_{\Sigma_x} \left\{ \partial_z S - \partial_t R \right\} dzdt\,.$$

Similarly,

$$\oint_{\partial\Sigma_y} \left\{ P(x,y,z,t)dx + S(x,y,z,t)dt \right\} = \oiint_{\Sigma_y} \left\{ \partial_t P - \partial_x S \right\} dtdx\,,$$

$$\oint_{\partial\Sigma_y} \left\{ P(x,y,z,t)dx + R(x,y,z,t)dz \right\} = \oiint_{\Sigma_y} \left\{ \partial_x R - \partial_z P \right\} dxdz\,,$$

$$\oint_{\partial\Sigma_z} \left\{ P(x,y,z,t)dx + Q(x,y,z,t)dy \right\} = \oiint_{\Sigma_z} \left\{ \partial_x Q - \partial_y P \right\} dxdy\,,$$

$$\oint_{\partial\Sigma_z} \left\{ Q(x,y,z,t)dy + S(x,y,z,t)dt \right\} = \oiint_{\Sigma_z} \left\{ \partial_y S - \partial_t Q \right\} dydt$$

$$\oint_{\partial\Sigma_t} \left\{ Q(x,y,z,t)dy + R(x,y,z,t)dz \right\} = \oiint_{\Sigma_t} \left\{ \partial_y R - \partial_z Q \right\} dydz\,.$$

The sum of the left hand sides of the equalities is written as

$$\oint_{\partial\Sigma} P(x,y,z,t)dx + Q(x,y,z,t)dy + R(x,y,z,t)dz + S(x,y,z,t)dt\,.$$

The sum of the right hand sides is written as

$$\oiint_{\Sigma} \left\{ (\partial_z S - \partial_t R)dzdt + (\partial_t P - \partial_x S)dtdx + (\partial_x R - \partial_z P)dxdz + \right.$$

$$\left. +(\partial_x Q - \partial_y P)dxdy + (\partial_y S - \partial_t Q)dydt + (\partial_y R - \partial_z Q)dxdz \right\}.\square$$

References

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