Space-Time Electrodynamics, and Magnetic Monopoles

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Abstract Maxwell's Electrodynamics Equations for the 3-Space Vector Fields,

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix}, \quad \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}, \quad \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}, \quad \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix},$$

disallow magnetic monopoles.

Those equations could not be written for the Space-Time Vector Fields

$$\begin{bmatrix} D_x \\ D_y \\ D_z \\ D_t \end{bmatrix}, \quad \begin{bmatrix} B_x \\ B_y \\ B_z \\ B_t \end{bmatrix}, \quad \begin{bmatrix} E_x \\ E_y \\ E_z \\ E_t \end{bmatrix}, \quad \begin{bmatrix} H_x \\ H_y \\ H_z \\ H_t \end{bmatrix},$$

because it is believed that the curl of a <u>four</u> dimensional Vector Field is a <u>six</u> dimensional Vector Field.

Recently, we showed that the 4-space curl is a 4-vector, and supplied the correct formula for it. Thus, we obtain the Curls of the Space-Time $\vec{E}$, and of the Space-time $\vec{H}$.

We show that the Curl of the space-time $\vec{E}$ with $E_t = 0$ differs from the Curl of the Spatial $\vec{E}$. Similarly, the Curl of the space-time $\vec{H}$ with $H_t = 0$ differs from the Curl of the Spatial $\vec{H}$.

On the other hand, the divergence of the space-time $\vec{D}$ with $E_t = 0$ equals the divergence of the spatial $\vec{D}$, and the divergence of the space-time $\vec{B}$ with $H_t = 0$ equals the divergence of the spatial $\vec{B}$.

Also, the Power Density of the Space-time $\vec{E}$ with $E_t = 0$, and $\vec{H}$ with $H_t = 0$, equals the Power density of the Spatial $\vec{E}$, and $\vec{H}$.

The Divergence of the space-time $\vec{B}$ allows Magnetic Monopoles.

The space-time $\vec{E}$, and $\vec{B}$, can be derived from space-time potentials φ, and $\vec{\mathrm{A}}$, and remain unchanged under gauge transformations of these potentials.

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Contents

Introduction

0.1 Electro-Magnetic Fields

In Electrodynamics we denote

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} D_x(x,y,z,t) \\ D_y(x,y,z,t) \\ D_z(x,y,z,t) \end{bmatrix} = \text{Displacement, (Coulomb/(meter)}^2\text{)}$$

$$\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \begin{bmatrix} E_x(x,y,z,t) \\ E_y(x,y,z,t) \\ E_z(x,y,z,t) \end{bmatrix} = \text{Electric. (Volts/meter)}$$

The Displacement, and the Electric Fields are related by

$$D = \varepsilon E,$$

where, in an isotropic medium,

$$\varepsilon = \varepsilon_0 n^2 = \varepsilon_0(1+\chi) = \text{Permittivity},$$

$$n = \text{Refractive Index},$$

$$\chi = \text{Susceptibility}.$$

In nonlinear isotropic medium, χE may depend on powers of the Electric field E. In Optical Fibers, we assume

$$\chi E = \chi^{(1)} E + \chi^{(3)} E^3.$$

Then, assuming a harmonic plane electric field oscillating at angular speed ω, propagating along the Fiber, the refractive index depends on ω too, and $n = n(\omega, E)$

In a non-isotropic medium,

$$\varepsilon = \varepsilon_{ij}$$

is a 3×3 matrix. If non-linear, each ε_{ij} depends on ω, and the electric field.

Similarly, we denote

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} B_x(x,y,z,t) \\ B_y(x,y,z,t) \\ B_z(x,y,z,t) \end{bmatrix} = \text{Induction (Weber/(meter)}^2\text{)}$$

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} H_x(x,y,z,t) \\ H_y(x,y,z,t) \\ H_z(x,y,z,t) \end{bmatrix} = \text{Magnetic (Ampere/meter)}$$

The Induction, and the Magnetic Fields are related by

$$B = \mu H,$$

where, in an isotropic medium,

$$\mu = \mu_0 \mu_r = \text{Permeability},$$

$$\mu_r = \text{Relative Permeability}.$$

0.2 Electrodynamics Equations in 3-space

Maxwell's Electrodynamics Equations for the 3-Space Vector Fields,

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix}, \quad \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}, \quad \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}, \quad \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix},$$

are

(I) $$\underbrace{\mathrm{Div}\, D}_{\partial_x D_x + \partial_y D_y + \partial_z D_z} = \rho, \quad \text{(Coulomb/(meter)}^3\text{)},$$

where ρ = electric charge density.

(II) $$\underbrace{\mathrm{Div}\, B}_{\partial_x B_x + \partial_y B_y + \partial_z B_z} = 0, \quad \text{(Weber/(meter)}^3\text{)},$$

where the magnetic charge density is zero.

Thus, Equation (II) means no magnetic monopoles.

(III) $$\underbrace{\mathrm{Curl}\, E}_{\begin{bmatrix} \partial_y E_z - \partial_z E_y \\ \partial_z E_x - \partial_x E_z \\ \partial_x E_y - \partial_y E_x \end{bmatrix}} = - \underbrace{\partial_t B}_{\begin{bmatrix} \partial_t B_x \\ \partial_t B_y \\ \partial_t B_z \end{bmatrix}}, \quad \text{(Volt/(meter)}^2\text{)}$$

(IV) $$\underbrace{\mathrm{Curl}\, H}_{\begin{bmatrix} \partial_y H_z - \partial_z H_y \\ \partial_z H_x - \partial_x H_z \\ \partial_x H_y - \partial_y H_x \end{bmatrix}} = \underbrace{J}_{\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix}} + \underbrace{\partial_t D}_{\begin{bmatrix} \partial_t D_x \\ \partial_t D_y \\ \partial_t D_z \end{bmatrix}}, \quad \text{(Ampere/(meter)}^2\text{)}$$

where J = conduction Current Density.

0.3 Erroneous Curl in Four Dimensions

Maxwell's equations could not be written for the Space-Time Vector Fields

$$\begin{bmatrix} D_x \\ D_y \\ D_z \\ D_t \end{bmatrix}, \quad \begin{bmatrix} B_x \\ B_y \\ B_z \\ B_t \end{bmatrix}, \quad \begin{bmatrix} E_x \\ E_y \\ E_z \\ E_t \end{bmatrix}, \quad \begin{bmatrix} H_x \\ H_y \\ H_z \\ H_t \end{bmatrix},$$

because it is believed that the curl of a <u>four</u> dimensional Vector Field is a <u>six</u> dimensional Vector Field.

The definition of the Curl of a Vector Function is based on the area density of circulation, and requires the concept of circulation, and the use of infinitesimals.

Since infinitesimals were avoided, and limits are vague, the Curl in many texts is defined by the result that follows from its definition.

Namely, in Cartesian Coordinates, the Curl is defined by

$$\nabla \times \begin{bmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z \end{bmatrix} = \begin{bmatrix} \partial_y R - \partial_z Q \\ \partial_z P - \partial_x R \\ \partial_x Q - \partial_y P \end{bmatrix}$$

$$= \begin{vmatrix} \partial_y & \partial_z \\ Q & R \end{vmatrix} \vec{1}_x + \begin{vmatrix} \partial_z & \partial_x \\ R & Q \end{vmatrix} \vec{1}_y + \begin{vmatrix} \partial_x & \partial_y \\ P & Q \end{vmatrix} \vec{1}_z$$

$$= \begin{vmatrix} \vec{1}_x & \vec{1}_y & \vec{1}_z \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix}.$$

Clearly, the 3-Space Curl is a 3-dimensional vector function, with components in the direction of the unit vectors $\vec{1}_x$, $\vec{1}_y$, and $\vec{1}_z$.

It is not self evident how this result, that became definition, may be generalized to 4-Space with its base of four unit vectors $\vec{1}_x$, $\vec{1}_y$, $\vec{1}_z$, and $\vec{1}_t$.

For instance, we can add a column to obtain

$$\begin{vmatrix} \vec{1}_x & \vec{1}_y & \vec{1}_z & \vec{1}_t \\ \partial_x & \partial_y & \partial_z & \partial_t \\ P & Q & R & S \\ ? & ? & ? & ? \end{vmatrix},$$

but what will be a fourth raw of that 4×4 determinant?

It is less evident how the fact that there are six terms of the form

$$\partial_i E_j - \partial_j E_i$$

that determine the 4-space curl, lead to the belief that the 4-Space curl is six dimensional.

To obtain the formula for the 4-Space curl, we need to define it through the circulations in 4-space.

0.4 Correct 4-space Curl

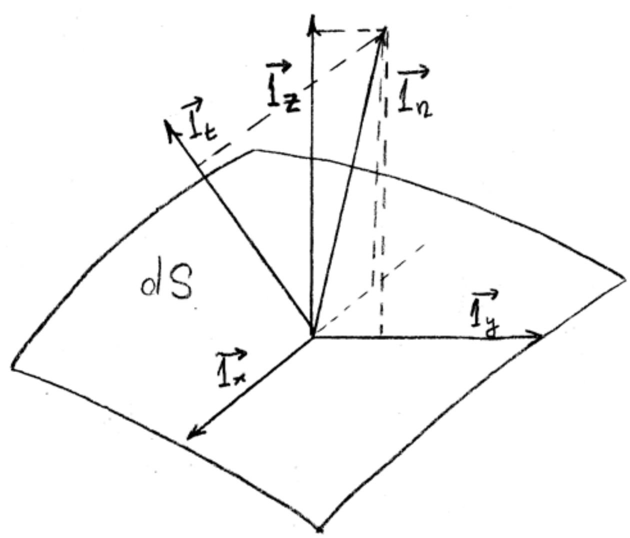
In [Dan4], we showed that the 4-space curl is a 4-vector, and supplied the correct formula for it.

Applying this formula to the Space-Time $\vec{E}$, and $\vec{H}$, we obtain the Electrodynamics Equations for the Space-Time Electro-Magnetic Fields. Those equations allow Magnetic Monopoles.

1.

Curl of 4-Vector

Let $P(x,y,z,t)$, $Q(x,y,z,t)$, $R(x,y,z,t)$, and $S(x,y,z,t)$ be hyper-real differentiable functions, defined on an infinitesimal area element dS. dS projects onto six 2-planes generated by the unit vectors $\vec{1}_x, \vec{1}_y, \vec{1}_z$, and $\vec{1}_t$.



x-y projection with area $dxdy$ and normal $\vec{1}_x \times \vec{1}_y = \vec{1}_z$

y-z projection with area $dydz$ and normal $\vec{1}_y \times \vec{1}_z = \vec{1}_t$

z-t projection with area $dzdt$ and normal $\vec{1}_z \times \vec{1}_t = \vec{1}_x$

t-x projection with area $dtdx$ and normal $\vec{1}_t \times \vec{1}_x = \vec{1}_y$

<u>t-y projection</u> with area $dtdy$ and normal

$$\vec{1}_t \times \vec{1}_y = (\vec{1}_y \times \vec{1}_z) \times \vec{1}_y = \vec{1}_z \underbrace{(\vec{1}_y \cdot \vec{1}_y)}_{1} - \vec{1}_y \underbrace{(\vec{1}_z \cdot \vec{1}_y)}_{0} = \vec{1}_z$$

<u>z-x projection</u> with area $dzdx$ and normal

$$\vec{1}_z \times \vec{1}_x = (\vec{1}_x \times \vec{1}_y) \times \vec{1}_x = \vec{1}_y \underbrace{(\vec{1}_x \cdot \vec{1}_x)}_{1} - \vec{1}_x \underbrace{(\vec{1}_y \cdot \vec{1}_x)}_{0} = \vec{1}_y$$

The projected areas are

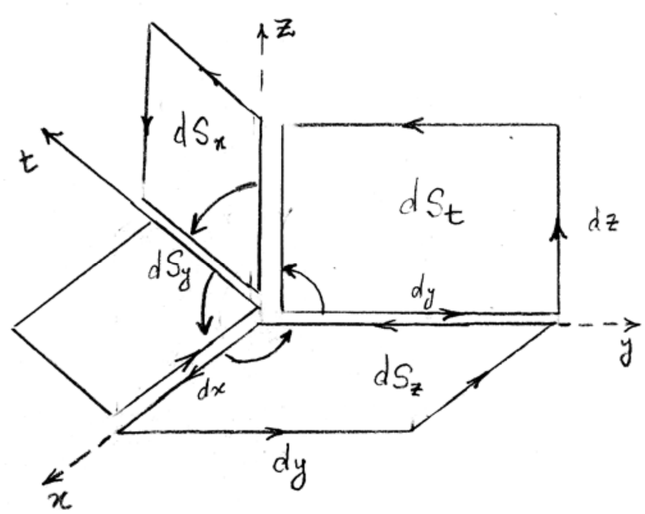
$$dS_x = \vec{1}_x \cdot \vec{1}_n dS = dzdt,$$

$$dS_y = \vec{1}_y \cdot \vec{1}_n dS = dtdx + dzdx,$$

$$dS_z = \vec{1}_z \cdot \vec{1}_n dS = dxdy + dtdy,$$

$$dS_t = \vec{1}_t \cdot \vec{1}_n dS = dydz$$

The projections areas are walls of a box with vertex at (x_0, y_0, z_0, t_0) and sides dx, dy, dz, and dt.



Given positive orientation of a right hand system,

$$\nabla \times \begin{bmatrix} P(x,y,z,t) \\ Q(x,y,z,t) \end{bmatrix} = \vec{1}_z \frac{1}{dxdy} \oint_{\partial(dS_z)} P(x,y,z,t)\vec{1}_x \cdot \vec{1}_l dl + Q(x,y,z,t)\vec{1}_y \cdot \vec{1}_l dl \,,$$

$$= \begin{vmatrix} \partial_x & \partial_y \\ P(x,y,z,t) & Q(x,y,z,t) \end{vmatrix}_{x_0,y_0,z_0,t_0} \vec{1}_z$$

$$\nabla \times \begin{bmatrix} Q(x,y,z,t) \\ R(x,y,z,t) \end{bmatrix} = \vec{1}_t \frac{1}{dydz} \oint_{\partial(dS_t)} Q(x,y,z,t)\vec{1}_y \cdot \vec{1}_l dl + R(x,y,z,t)\vec{1}_z \cdot \vec{1}_l dl \,,$$

$$= \begin{vmatrix} \partial_y & \partial_z \\ Q(x,y,z,t) & R(x,y,z,t) \end{vmatrix}_{x_0,y_0,z_0,t_0} \vec{1}_t$$

$$\nabla \times \begin{bmatrix} R(x,y,z,t) \\ S(x,y,z,t) \end{bmatrix} = \vec{1}_x \frac{1}{dzdt} \oint_{\partial(dS_x)} R(x,y,z,t)\vec{1}_z \cdot \vec{1}_l dl + S(x,y,z,t)\vec{1}_t \cdot \vec{1}_l dl \,,$$

$$= \begin{vmatrix} \partial_z & \partial_t \\ R(x,y,z,t) & S(x,y,z,t) \end{vmatrix}_{x_0,y_0,z_0,t_0} \vec{1}_x \,.$$

$$\nabla \times \begin{bmatrix} S(x,y,z,t) \\ P(x,y,z,t) \end{bmatrix} = \vec{1}_y \frac{1}{dtdx} \oint_{\partial(dS_y)} S(x,y,z,t)\vec{1}_t \cdot \vec{1}_l dl + P(x,y,z,t)\vec{1}_x \cdot \vec{1}_l dl \,,$$

$$= \begin{vmatrix} \partial_t & \partial_x \\ S(x,y,z,t) & P(x,y,z,t) \end{vmatrix}_{x_0,y_0,z_0,t_0} \vec{1}_y$$

$$\nabla \times \begin{bmatrix} S(x,y,z,t) \\ Q(x,y,z,t) \end{bmatrix} = \vec{1}_z \frac{1}{dtdy} \oint_{\partial(dS_z)} S(x,y,z,t)\vec{1}_t \cdot \vec{1}_l dl + Q(x,y,z,t)\vec{1}_y \cdot \vec{1}_l dl \,,$$

$$= \begin{vmatrix} \partial_t & \partial_y \\ S(x,y,z,t) & Q(x,y,z,t) \end{vmatrix}_{x_0,y_0,z_0,t_0} \vec{1}_z$$

$$\nabla \times \begin{bmatrix} P(x,y,z,t) \\ R(x,y,z,t) \end{bmatrix} = \vec{1}_y \frac{1}{dzdx} \oint_{\partial(dS_y)} P(x,y,z,t)\vec{1}_x \cdot \vec{1}_l dl + R(x,y,z,t)\vec{1}_z \cdot \vec{1}_l dl\,,$$

$$= \begin{vmatrix} \partial_x & \partial_z \\ P(x,y,z,t) & R(x,y,z,t) \end{vmatrix}_{x_0,y_0,z_0,t_0} \vec{1}_y$$

The 4-space Curl is the sum of the six area curls. That is,

$$\nabla \times \begin{bmatrix} P(x,y,z,t) \\ Q(x,y,z,t) \\ R(x,y,z,t) \\ S(x,y,z,t) \end{bmatrix} =$$

$$= \underbrace{\nabla \times \begin{bmatrix} R \\ S \end{bmatrix}}_{\begin{vmatrix} \partial_z & \partial_t \\ R & S \end{vmatrix}\vec{1}_x} + \underbrace{\nabla \times \begin{bmatrix} S \\ P \end{bmatrix}}_{\begin{vmatrix} \partial_t & \partial_x \\ S & P \end{vmatrix}\vec{1}_y} + \underbrace{\nabla \times \begin{bmatrix} P \\ R \end{bmatrix}}_{\begin{vmatrix} \partial_x & \partial_z \\ P & R \end{vmatrix}\vec{1}_y} + \underbrace{\nabla \times \begin{bmatrix} P \\ Q \end{bmatrix}}_{\begin{vmatrix} \partial_x & \partial_y \\ P & Q \end{vmatrix}\vec{1}_z} + \underbrace{\nabla \times \begin{bmatrix} S \\ Q \end{bmatrix}}_{\begin{vmatrix} \partial_t & \partial_y \\ S & Q \end{vmatrix}\vec{1}_z} + \underbrace{\nabla \times \begin{bmatrix} Q \\ R \end{bmatrix}}_{\begin{vmatrix} \partial_y & \partial_z \\ Q & R \end{vmatrix}\vec{1}_t}$$

$$\boxed{= \begin{pmatrix} S_z - R_t \\ P_t - S_x + R_x - P_z \\ Q_x - P_y + Q_t - S_y \\ R_y - Q_z \end{pmatrix}}.$$

2.

Cross-Product of 4-Vectors

The Cross-product of 4-vectors is the sum of six cross-products. That is,

$$\begin{bmatrix} P_1 \\ Q_1 \\ R_1 \\ S_1 \end{bmatrix} \times \begin{bmatrix} P_2 \\ Q_2 \\ R_2 \\ S_2 \end{bmatrix} = \underbrace{\begin{bmatrix} R_1 \\ S_1 \end{bmatrix} \times \begin{bmatrix} R_2 \\ S_2 \end{bmatrix}}_{\begin{vmatrix} R_1 & S_1 \\ R_2 & S_2 \end{vmatrix} \vec{1}_x} + \underbrace{\begin{bmatrix} S_1 \\ P_1 \end{bmatrix} \times \begin{bmatrix} S_2 \\ P_2 \end{bmatrix}}_{\begin{vmatrix} S_1 & P_1 \\ S_2 & P_2 \end{vmatrix} \vec{1}_y} + \underbrace{\begin{bmatrix} P_1 \\ R_1 \end{bmatrix} \times \begin{bmatrix} P_2 \\ R_2 \end{bmatrix}}_{\begin{vmatrix} P_1 & R_1 \\ P_2 & R_2 \end{vmatrix} \vec{1}_y} +$$

$$+ \underbrace{\begin{bmatrix} P_1 \\ Q_1 \end{bmatrix} \times \begin{bmatrix} P_2 \\ Q_2 \end{bmatrix}}_{\begin{vmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{vmatrix} \vec{1}_z} + \underbrace{\begin{bmatrix} S_1 \\ Q_1 \end{bmatrix} \times \begin{bmatrix} S_2 \\ Q_2 \end{bmatrix}}_{\begin{vmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{vmatrix} \vec{1}_z} + \underbrace{\begin{bmatrix} Q_1 \\ R_1 \end{bmatrix} \times \begin{bmatrix} Q_2 \\ R_2 \end{bmatrix}}_{\begin{vmatrix} Q_1 & R_1 \\ Q_2 & R_2 \end{vmatrix} \vec{1}_t}$$

$$= \left[\begin{array}{c} \begin{vmatrix} R_1 & S_1 \\ R_2 & S_2 \end{vmatrix} \\ \begin{vmatrix} S_1 & P_1 \\ S_2 & P_2 \end{vmatrix} + \begin{vmatrix} P_1 & R_1 \\ P_2 & R_2 \end{vmatrix} \\ \begin{vmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{vmatrix} + \begin{vmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{vmatrix} \\ \begin{vmatrix} Q_1 & R_1 \\ Q_2 & R_2 \end{vmatrix} \end{array}\right].$$

3.

Space-time Electric and Magnetic Fields

We shall assume that Space-time electric and magnetic fields have four components along the axes x, y, z, t.

3.1 Space-Time <u>Electric-Flux</u> (= Displacement) Field

$$\begin{bmatrix} D_x \\ D_y \\ D_z \\ D_t \end{bmatrix} = \begin{bmatrix} D_x(x,y,z,t) \\ D_y(x,y,z,t) \\ D_z(x,y,z,t) \\ D_t(x,y,z,t) \end{bmatrix} \text{ (Coulomb/(meter)}^2\text{)}.$$

3.2 Space-Time <u>Potential-Gradient</u> (=Electric) Field

$$\begin{bmatrix} E_x \\ E_y \\ E_z \\ E_t \end{bmatrix} = \begin{bmatrix} E_x(x,y,z,t) \\ E_y(x,y,z,t) \\ E_z(x,y,z,t) \\ E_t(x,y,z,t) \end{bmatrix} \text{ (Volts/meter)}.$$

3.3 Space-time <u>Magnetic-Flux</u> (=Induction) Field

$$\begin{bmatrix} B_x \\ B_y \\ B_z \\ B_t \end{bmatrix} = \begin{bmatrix} B_x(x,y,z,t) \\ B_y(x,y,z,t) \\ B_z(x,y,z,t) \\ B_t(x,y,z,t) \end{bmatrix} \text{ (Weber/(meter)}^2\text{)}$$

3.4 Space-time <u>Current-Gradient</u> (=Magnetic) Field

$$\begin{bmatrix} H_x \\ H_y \\ H_z \\ H_t \end{bmatrix} = \begin{bmatrix} H_x(x,y,z,t) \\ H_y(x,y,z,t) \\ H_z(x,y,z,t) \\ H_t(x,y,z,t) \end{bmatrix} \text{ (Ampere/meter)}$$

We shall assume that the Space-time Magnetic and the Electric Fields are perpendicular.

3.5 Assuming $\boxed{E \perp H}$

$$\boxed{E_x H_x + E_y H_y + E_z H_z + E_t H_t = 0}$$

3.6 $E_t = H_t = 0 \Rightarrow$ Spatial E $\perp$ Spatial H

<u>Proof</u>: $E_t = H_t = 0 \Rightarrow E_x H_x + E_y H_y + E_z H_z = 0$

We shall assume that the Space-time Magnetic and the Electric Fields propagate in direction perpendicular to both

3.7 Assuming Propagation in direction Perpendicular to Space-time E, and to Space-time H.

3.8 $E_t = H_t = 0 \Rightarrow$ **Propagation is in direction perpendicular to the spatial E, and to the spatial H**

3.9 Space-time Power Density

$$\boxed{\begin{bmatrix} E_x \\ E_y \\ E_z \\ E_t \end{bmatrix} \times \begin{bmatrix} H_x \\ H_y \\ H_z \\ H_t \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} E_z & E_t \\ H_z & H_t \end{vmatrix} \\ \begin{vmatrix} E_t & E_x \\ H_t & H_x \end{vmatrix} + \begin{vmatrix} E_x & E_z \\ H_x & H_z \end{vmatrix} \\ \begin{vmatrix} E_x & E_y \\ H_x & H_y \end{vmatrix} + \begin{vmatrix} E_t & E_y \\ H_t & H_y \end{vmatrix} \\ \begin{vmatrix} E_y & E_z \\ H_y & H_z \end{vmatrix} \end{bmatrix}}$$

<u>*Proof*</u>: By 2.□

3.10
$$\begin{bmatrix} E_x \\ E_y \\ E_z \\ 0 \end{bmatrix} \times \begin{bmatrix} H_x \\ H_y \\ H_z \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \begin{vmatrix} E_x & E_z \\ H_x & H_z \end{vmatrix} \\ \begin{vmatrix} E_x & E_y \\ H_x & H_y \end{vmatrix} \\ \begin{vmatrix} E_y & E_z \\ H_y & H_z \end{vmatrix} \end{bmatrix}$$

3.11 $E_t = H_t = 0 \Rightarrow$ **Power Density of Space-time E×H=**

Power Density of Spatial E×H

4.

Curls of Space-Time Fields

4.1 Curl of Space-time E

$$\boxed{\nabla \times \begin{bmatrix} E_x(x,y,z,t) \\ E_y(x,y,z,t) \\ E_z(x,y,z,t) \\ E_t(x,y,z,t) \end{bmatrix} = \begin{pmatrix} \partial_z \\ -\partial_x \\ -\partial_y \\ 0 \end{pmatrix} E_t - \partial_t \begin{bmatrix} E_z \\ -E_x - B_y \\ -E_y + B_z \\ B_x \end{bmatrix}}$$

<u>*Proof*</u>: Applying the Formula for the 4-dimensional Curl,

$$\nabla \times \begin{bmatrix} E_x(x,y,z,t) \\ E_y(x,y,z,t) \\ E_z(x,y,z,t) \\ E_t(x,y,z,t) \end{bmatrix} = \begin{pmatrix} \partial_z E_t - \partial_t E_z \\ \partial_t E_x - \partial_x E_t + \underbrace{\partial_x E_z - \partial_z E_x}_{\partial_t B_y} \\ \underbrace{\partial_x E_y - \partial_y E_x}_{-\partial_t B_z} + \partial_t E_y - \partial_y E_t \\ \underbrace{\partial_y E_z - \partial_z E_y}_{-\partial_t B_x} \end{pmatrix}$$

$$= \begin{pmatrix} \partial_z \\ -\partial_x \\ -\partial_y \\ 0 \end{pmatrix} E_t - \partial_t \begin{bmatrix} E_z \\ -E_x - B_y \\ -E_y + B_z \\ B_x \end{bmatrix} . \square$$

4.2 $$\nabla \times \begin{bmatrix} E_x(x,y,z,t) \\ E_y(x,y,z,t) \\ E_z(x,y,z,t) \\ 0 \end{bmatrix} = -\partial_t \begin{bmatrix} E_z \\ -E_x - B_y \\ -E_y + B_z \\ B_x \end{bmatrix}$$

4.3 $E_t = 0 \Rightarrow$ Curl of Space-time E $\neq$ Curl of Spatial E

4.4 Curl of Space-time H

$$\boxed{\nabla \times \begin{bmatrix} H_x(x,y,z,t) \\ H_y(x,y,z,t) \\ H_z(x,y,z,t) \\ H_t(x,y,z,t) \end{bmatrix} = \begin{pmatrix} \partial_z \\ -\partial_x \\ -\partial_y \\ 0 \end{pmatrix} H_t - \begin{bmatrix} 0 \\ J_y \\ -J_z \\ J_x \end{bmatrix} - \partial_t \begin{bmatrix} H_z \\ -H_x - D_y \\ -H_y + D_z \\ D_x \end{bmatrix}}$$

<u>*Proof*</u>: Applying the Formula for the 4-dimensional Curl,

$$\nabla \times \begin{bmatrix} H_x(x,y,z,t) \\ H_y(x,y,z,t) \\ H_z(x,y,z,t) \\ H_t(x,y,z,t) \end{bmatrix} = \begin{pmatrix} \partial_z H_t - \partial_t H_z \\ \partial_t H_x - \partial_x H_t + \underbrace{\partial_x H_z - \partial_z H_x}_{-J_y - \partial_t D_y} \\ \underbrace{\partial_x H_y - \partial_y H_x}_{J_z + \partial_t D_z} + \partial_t H_y - \partial_y H_t \\ \underbrace{\partial_y H_z - \partial_z H_y}_{J_x + \partial_t D_x} \end{pmatrix}$$

$$= \begin{pmatrix} \partial_z \\ -\partial_x \\ -\partial_y \\ 0 \end{pmatrix} H_t - \begin{bmatrix} 0 \\ J_y \\ -J_z \\ J_x \end{bmatrix} - \partial_t \begin{bmatrix} H_z \\ -H_x - D_y \\ -H_y + D_z \\ D_x \end{bmatrix} . \square$$

4.5 $$\nabla \times \begin{bmatrix} H_x(x,y,z,t) \\ H_y(x,y,z,t) \\ H_z(x,y,z,t) \\ 0 \end{bmatrix} = -\begin{bmatrix} 0 \\ J_y \\ -J_z \\ J_x \end{bmatrix} - \partial_t \begin{bmatrix} H_z \\ -H_x - D_y \\ -H_y + D_z \\ D_x \end{bmatrix}$$

4.6 $H_t = 0 \Rightarrow$ **Curl of Space-time H $\neq$ Curl of Spatial H**

5.

Divergences of Space-time Fields, and Magnetic Monopoles

5.1 Divergence of Space-time D

$$\boxed{\nabla \cdot \begin{bmatrix} D_x(x,y,z,t) \\ D_y(x,y,z,t) \\ D_z(x,y,z,t) \\ D_t(x,y,z,t) \end{bmatrix} = \rho + \partial_t D_t}$$

Proof:

$$\nabla \cdot \begin{bmatrix} D_x(x,y,z,t) \\ D_y(x,y,z,t) \\ D_z(x,y,z,t) \\ D_t(x,y,z,t) \end{bmatrix} = \underbrace{\partial_x D_x + \partial_y D_y + \partial_z D_z}_{\rho} + \partial_t D_t . \square$$

5.2 $\boxed{\partial_t D_t \neq 0 \Rightarrow \textbf{Electric Monopoles even with } \rho = 0}$

5.3
$$\nabla \cdot \begin{bmatrix} D_x(x,y,z,t) \\ D_y(x,y,z,t) \\ D_z(x,y,z,t) \\ 0 \end{bmatrix} = \rho$$

5.4 $E_t = 0 \Rightarrow$ **Divergence of Space-time D = Divergence of Spatial D**

5.5
$$\boxed{\nabla \cdot \begin{bmatrix} B_x(x,y,z,t) \\ B_y(x,y,z,t) \\ B_z(x,y,z,t) \\ B_t(x,y,z,t) \end{bmatrix} = \partial_t B_t}$$

Proof:
$$\nabla \cdot \begin{bmatrix} B_x(x,y,z,t) \\ B_y(x,y,z,t) \\ B_z(x,y,z,t) \\ B_t(x,y,z,t) \end{bmatrix} = \underbrace{\partial_x B_x + \partial_y B_y + \partial_z B_z}_{0} + \partial_t B_t . \square$$

5.6 $\boxed{\partial_t B_t \neq 0 \Rightarrow \textbf{Magnetic Monopoles}}$

5.7
$$\nabla \cdot \begin{bmatrix} B_x(x,y,z,t) \\ B_y(x,y,z,t) \\ B_z(x,y,z,t) \\ 0 \end{bmatrix} = 0$$

5.8 $H_t = 0 \Rightarrow$ **Divergence of Space-time B =**

Divergence of Spatial B

6.

Space-time Potentials, and their Gauge Transformations

Denote a 4-vector by $\vec{V}^{(4)}$, and a 3-vector by $\vec{V}^{(3)}$

We define a space-time scalar electric potential

$$\varphi(x,y,z,t),$$

and a space-time 4-vector Magnetic Potential

$$\vec{A}^{(4)}(x,y,z,t)=\begin{bmatrix} A_x(x,y,z,t) \\ A_y(x,y,z,t) \\ A_z(x,y,z,t) \\ A_t(x,y,z,t) \end{bmatrix}.$$

We define

6.1 Space-time Electric Field

$$\vec{E}^{(4)}=-\vec{\nabla}^{(4)}\varphi-\partial_t\vec{A}^{(4)}.$$

6.2
$$\begin{bmatrix} E_x^{(4)} \\ E_y^{(4)} \\ E_z^{(4)} \\ E_t^{(4)} \end{bmatrix}=\begin{bmatrix} -\partial_x\varphi-\partial_t A_x \\ -\partial_y\varphi-\partial_t A_y \\ -\partial_z\varphi-\partial_t A_z \\ -\partial_t\varphi-\partial_t A_t \end{bmatrix}=\begin{bmatrix} E_x^{(3)} \\ E_y^{(3)} \\ E_z^{(3)} \\ -\partial_t\varphi-\partial_t A_t \end{bmatrix}.$$

We define

6.3 Space-time Magnetic Induction

$$\vec{B}^{(4)} = \vec{\nabla}^{(4)} \times \vec{A}^{(4)}.$$

6.4
$$\begin{bmatrix} B_x^{(4)} \\ B_y^{(4)} \\ B_z^{(4)} \\ B_t^{(4)} \end{bmatrix} = \begin{pmatrix} \partial_z A_t - \partial_t A_z \\ \partial_t A_x - \partial_x A_t + \underbrace{\partial_x A_z - \partial_z A_x}_{-B_y^{(3)}} \\ \underbrace{\partial_x A_y - \partial_y A_x}_{B_z^{(3)}} + \partial_t A_y - \partial_y A_t \\ \underbrace{\partial_y A_z - \partial_z A_y}_{B_x^{(3)}} \end{pmatrix}$$

$$= \begin{pmatrix} \partial_z A_t - \partial_t A_z \\ \partial_t A_x - \partial_x A_t - B_y^{(3)} \\ B_z^{(3)} + \partial_t A_y - \partial_y A_t \\ B_x^{(3)} \end{pmatrix}.$$

6.5 $\vec{\nabla}^{(4)} \times \vec{\nabla}^{(4)} \psi = 0^{(4)}$

Proof:

$$\vec{\nabla}^{(4)} \times \begin{bmatrix} \partial_x \psi \\ \partial_y \psi \\ \partial_z \psi \\ \partial_t \psi \end{bmatrix} = \begin{pmatrix} \partial_z \partial_t \psi - \partial_t \partial_z \psi \\ \partial_t \partial_x \psi - \partial_x \partial_t \psi + \partial_x \partial_z \psi - \partial_z \partial_x \psi \\ \partial_x \partial_y \psi - \partial_y \partial_x \psi + \partial_t \partial_y \psi - \partial_y \partial_t \psi \\ \partial_y \partial_z \psi - \partial_z \partial_y \psi \end{pmatrix} = 0^{(4)}. \square$$

6.6 Gauge Transformation of φ, and $\vec{A}^{(4)}$

For any scalar Potential $\Lambda(x,y,z,t)$,

the Potentials

$$\Phi = \varphi - \partial_t \Lambda,$$

$$\vec{\mathcal{A}}^{(4)} = \vec{A}^{(4)} + \vec{\nabla}^{(4)}\Lambda,$$

are called ***Gauge Transformation of*** φ, **and** $\vec{A}^{(4)}$

6.7 *For any scalar Potential* $\Lambda(x,y,z,t)$,

$$\vec{E}^{(4)} = -\vec{\nabla}^{(4)}\Phi - \partial_t \vec{\mathcal{A}}^{(4)},$$

$$\vec{B}^{(4)} = \vec{\nabla}^{(4)} \times \vec{\mathcal{A}}^{(4)}.$$

Proof:

$$\vec{E}^{(4)} = -\vec{\nabla}^{(4)}\varphi - \partial_t \vec{A}^{(4)}$$

$$= -\vec{\nabla}^{(4)}(\underbrace{\varphi - \partial_t \Lambda}_{\Phi}) - \partial_t(\underbrace{\vec{A}^{(4)} + \vec{\nabla}^{(4)}\Lambda}_{\vec{\mathcal{A}}^{(4)}}).\square$$

$$\vec{B}^{(4)} = \vec{\nabla}^{(4)} \times \vec{A}^{(4)}$$

$$= \vec{\nabla}^{(4)} \times (\underbrace{\vec{A}^{(4)} + \vec{\nabla}^{(4)}\Lambda}_{\vec{\mathcal{A}}^{(4)}}).\square$$

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