

Infinitesimal Vector Calculus

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Abstract We extend the Infinitesimal Calculus of one variable, to the Calculus of Vector-valued, Hyper-real Functions of several variables.

We define the Gradient, Divergence, and Curl, with the precision of Infinitesimal Calculus, and derive their presentations in the main Coordinate systems.

We establish the Integral Theorems of Green, Gauss, and Stokes, in Infinitesimal Vector Calculus.

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Introduction

The controversy surrounding the infinitesimals, obstructed the development of the Infinitesimal Calculus.

Postulating infinitesimals, only revealed Logic's inapplicability to Mathematical Analysis, where claims must be proved, and unproven claims are ignored.

Recently we have shown that when the Real Line is represented as the infinite dimensional space of all the Cauchy sequences of rational numbers, the hyper-reals are spanned by the constant hyper-reals, a family of infinitesimal hyper-reals, and the associated family of infinite hyper-reals.

The infinitesimal hyper-reals are smaller than any real number, yet bigger than zero.

The reciprocals of the infinitesimal hyper-reals are the infinite hyper-reals. They are greater than any real number, yet strictly smaller than infinity.

A neighborhood of infinitesimals separates the zero hyper-real from the reals, and each real number is the center of an interval of hyper-reals, that includes no other real number.

The Hyper-reals are totally ordered, and are lined up on a line, the hyper-real line.

A hyper-real function is a mapping from the hyper-real line into the hyper-real line.

Infinitesimal Calculus is the Calculus of hyper-real functions.

Infinitesimal Calculus is far more precise, and effective than the ε, δ Calculus.

In particular, Infinitesimal Calculus enables us to differentiate over a discontinuity jump, as large as an infinite hyper-real, and integrate over such discontinuities.

We proceed to present the Infinitesimal Calculus of Functions that map vectors of Hyper-real into vectors of Hyper-reals.

To check continuity, we focus on the oscillation of a Hyper-real function over an infinitesimal interval.

Differentiation in the ε, δ Calculus, is undefined over a jump discontinuity.

In infinitesimal Calculus, we can differentiate over jump discontinuities, that may be as large as infinite hyper-reals.

The derivative is sharply defined in terms of infinitesimals, compared with its definition in terms of limits.

Integration in the ε, δ Calculus is undefined over jump discontinuities, and for unbounded functions.

In infinitesimal Calculus, using infinite Riemann Sums, we can integrate over jump discontinuities, that may be as large as infinite hyper-reals.

Infinitesimal Calculus is far more effective than the ε, δ Calculus, because being based on almost zero numbers, it allows us to deal with their reciprocals, the almost infinite numbers. We have no use for infinity by itself, but to comprehend the effects of singularities, we have use for the almost infinite.

Infinitesimals are a precise tool compared to the vague limit concept, and the awkward ε, δ statements.

Infinitesimal Calculus is the Calculus of hyper-real functions.

The minimal domain and range, needed for the definition and analysis of a hyper-real function, is the hyper-real line.

1.

Hyper-real Plane

The Hyper-real Plane is a cross product of two Hyper-real lines.

Each 2-vector of real numbers (α, β) can be represented by a Cauchy sequence of rational numbers, $(r_1, q_1), (r_2, q_2), (r_3, q_3) \dots$ so that $(r_n, q_n) \rightarrow (\alpha, \beta)$.

The constant sequence $(\alpha, \beta), (\alpha, \beta), (\alpha, \beta) \dots$ is a constant hyper-real 2-vector.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to $(0, 0)$ sequences of 2-vectors $(t_1, o_1), (t_2, o_2), (t_3, o_3) \dots$ constitutes a family of infinitesimal hyper-real 2-vectors.
2. The infinitesimal 2-vectors are smaller than any real 2-vector, yet strictly greater than the zero 2-vector.
3. Their reciprocals $(\frac{1}{t_1}, \frac{1}{o_1}), (\frac{1}{t_2}, \frac{1}{o_2}), (\frac{1}{t_3}, \frac{1}{o_3}), \dots$ are the infinite hyper-real 2-vectors.

4. The infinite hyper-real 2-vectors are greater than any real 2-vector, yet strictly smaller than the infinity 2-vector.
5. The infinite hyper-real 2-vectors with negative signs are smaller than any real 2-vector, yet strictly greater than $(-\infty, -\infty)$.
6. The sum of a real 2-vector with an infinitesimal 2-vector is a non-constant hyper-real 2-vector.
7. The Hyper-real 2-vectors are the totality of
 - constant hyper-real 2-vectors,
 - a family of infinitesimal 2-vectors, with signs that may be $(+, +)$, $(+, -)$, $(-, +)$, or $(-, -)$,
 - a family of infinite hyper-real 2-vectors with signs that may be $(+, +)$, $(+, -)$, $(-, +)$, or $(-, -)$, and
 - non-constant hyper-real 2-vectors.
8. The hyper-real 2-vectors constitute the Hyper-real Plane.
9. That plane includes the real 2-vectors separated by the non-constant hyper-real 2-vectors. Each real 2-vector is the center of a disk of infinitesimal radius of hyper-real 2-vectors, that includes no other real 2-vector.

10. In particular, the zero 2-vector is separated from any real 2-vector by infinitesimal 2-vectors that lie in a disk of infinitesimal radius around the zero.
11. The Zero 2-vector is not an infinitesimal 2-vector, because zero is not strictly greater than zero.
12. We do not add the infinity 2-vector to the hyper-real Plane.
13. The infinitesimal 2-vectors, and the infinite hyper-real 2-vectors, are semi-groups with respect to addition. Neither set includes zero.
14. The hyper-real Plane is embedded in $\mathbb{R}^\infty \times \mathbb{R}^\infty$, and is not homeomorphic to the real Plane. There is no bi-continuous one-one mapping from the hyper-real Plane onto the real plane.
15. In particular, there are no points in the real Plane that can be assigned uniquely to the infinitesimal hyper-real 2-vectors, or to the infinite hyper-real 2-vectors, or to the non-constant hyper-real 2-vectors.

16. No neighbourhood of a hyper-real 2-vector is homeomorphic to an $\mathbb{R}^n \times \mathbb{R}^n$ ball. Therefore, the hyper-real plane is not a manifold.
17. The hyper-real plane is not spanned by two elements, and it is not two-dimensional.

2.

Hyper-real Vector Function

2.1 Definition of a hyper-real function

$f(x, y)$ is a hyper-real function, iff it is from the hyper-real 2-vectors into the hyper-reals.

This means that any number in the domain, or in the range of a hyper-real $f(x, y)$ is either one of the following

real vector

real vector + infinitesimal vector

infinitesimal vector

infinite hyper-real vector

Clearly,

2.2 *Every function from the real plane into the reals is a hyper-real function.*

3.

Continuity

3.1 Oscillation of $f(x, y)$ over $|(x, y) - (x_0, y_0)| < d\rho$

The oscillation of a hyper-real function $f(x, y)$, over the infinitesimal disk $|(x, y) - (x_0, y_0)| < d\rho$ is

$$\sup_{|(x,y)-(x_0,y_0)|<d\rho} f(x, y) - \inf_{|(x,y)-(x_0,y_0)|<d\rho} f(x, y)$$

where the sup, and the inf are taken over the disk.

3.2 Continuity Definition

We require that the oscillation of the hyper-real $f(x, y)$, over the infinitesimal disk $|(x, y) - (x_0, y_0)| < d\rho$ will be infinitesimal. That is,

$f(x, y)$ is continuous at (x_0, y_0) iff for any infinitesimal dx ,

$$\sup_{|(x,y)-(x_0,y_0)|<d\rho} f(x, y) - \inf_{|(x,y)-(x_0,y_0)|<d\rho} f(x, y) = \text{infinitesimal}.$$

3.3 Continuity Example

$f(x, y) = xy$ is continuous at $(0, 0)$, because for any $d\rho$,

$$\sup_{|(x,y)| < d\rho} xy - \inf_{|(x,y)| < d\rho} xy = (d\rho)^2. \square$$

3.3 Discontinuity Example

$f(x, y) = \frac{x^2y}{x^4 + y^2}$ is discontinuous at $(0, 0)$, because for any $d\rho$,

$$\sup_{|(x,y)| < d\rho} \frac{x^2y}{x^4 + y^2} \geq \frac{x^2y}{x^4 + y^2} \Big|_{x^2=y; |(x,y)| < d\rho} = \frac{x^4}{2x^4} = \frac{1}{2},$$

$$\inf_{|(x,y)| < d\rho} \frac{x^2y}{x^4 + y^2} \leq \frac{x^2y}{x^4 + y^2} \Big|_{x=0; |(x,y)| < d\rho} = 0,$$

$$\sup_{|(x,y)| < d\rho} \frac{x^2y}{x^4 + y^2} - \inf_{|(x,y)| < d\rho} \frac{x^2y}{x^4 + y^2} \geq \frac{1}{2}. \square$$

3.4 Discontinuity Example

$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$ is discontinuous at $(0, 0)$, because for any $d\rho$,

$$\sup_{|(x,y)| < d\rho} \frac{x^2 - y^2}{x^2 + y^2} \geq \frac{x^2 - y^2}{x^2 + y^2} \Big|_{y=0; |(x,y)| < d\rho} = 1,$$

$$\inf_{|(x,y)| < d\rho} \frac{x^2 - y^2}{x^2 + y^2} \leq \frac{x^2 - y^2}{x^2 + y^2} \Big|_{x=0; |(x,y)| < d\rho} = -1,$$

$$\sup_{|(x,y)| < d\rho} \frac{x^2 - y^2}{x^2 + y^2} - \inf_{|(x,y)| < d\rho} \frac{x^2 - y^2}{x^2 + y^2} \geq 2. \square$$

4.

$$f(x, y) = \frac{1}{xy}, \quad x \neq 0, \quad y \neq 0$$

In the Calculus of Limits, the function $f(x, y) = \frac{1}{xy}$ is

defined for all $x \neq 0, y \neq 0$

We avoid $x = 0$, or $y = 0$, because the oscillation of

$f(x, y) = \frac{1}{xy}$ over a disk that includes $(0, 0)$ is infinite.

Avoiding $x = 0$, still allows the statement

$$x \downarrow 0 \Rightarrow \frac{1}{x} \uparrow \infty,$$

which is imprecise due to the appearance of ∞ on the right.

It follows that the domain of $f(x, y) = \frac{1}{xy}$ has to avoid a disk

that includes $(0, 0)$.

Consequently, our conception of the function close to its singularity is imprecise.

4.1 *In the Calculus of limits, $f(x, y) = \frac{1}{xy}$, is precisely*

defined only out of a disk of real radius centered at $(0,0)$

In Infinitesimal Calculus, infinite hyper-reals give a precise value.

For instance, the oscillation of $f(x,y) = \frac{1}{xy}$ over $|(x,y)| < d\rho$

is an infinite hyper-real which is .

Therefore,

4.2 *In Infinitesimal Calculus, The Hyper-real function*

$f(x,y) = \frac{1}{xy}$, *is precisely defined for any $x \neq 0$.*

5.

$$\log(x + y), x \neq 0$$

In the Calculus of Limits, the function $f(x, y) = \log(x + y)$ is defined for all $x \neq 0$, and $y \neq 0$.

We avoid $(0, 0)$, because the oscillation of $f(x, y) = \log(x + y)$ over an interval that includes $(0, 0)$, is infinite.

5.1 *In the Calculus of limits, $f(x, y) = \log(x + y)$, is precisely defined only out of a disk of real radius about $(0, 0)$*

In Infinitesimal Calculus, infinite hyper-reals give a precise value.

Consequently,

5.2 *In Infinitesimal Calculus, The Hyper-real function*

$$f(x, y) = \log(x + y) \text{ is defined for any } |(x, y)| > 0.$$

6.

Derivative

6.1 Partial Derivative

Hyper-real $f(x, y)$ defined at (x_0, y_0) , has a Partial Derivative in the direction of x , at (x_0, y_0) , $\partial_x f|_{(x_0, y_0)}$, iff for any infinitesimal dx ,

$$\frac{f(x_0 + dx, y_0) - f(x_0, y_0)}{dx},$$

equals a unique hyper-real number.

If the hyper-real is infinite, it is denoted $\partial_x f|_{(x_0, y_0)}$.

If the hyper-real is finite, its constant part is $\partial_x f|_{(x_0, y_0)}$.

6.2 Derivative

Hyper-real $f(x, y)$ defined at (x_0, y_0) , has a Derivative at (x_0, y_0) , $Df|_{(x_0, y_0)}$, iff for any infinitesimals dx , and dy

$$\frac{f(x_0 + dx, y_0 + dy) - f(x_0, y_0)}{\sqrt{dx^2 + dy^2}},$$

equals a unique hyper-real number.

If the hyper-real is infinite, it is denoted $Df|_{(x_0, y_0)}$.

If the hyper-real is finite, its constant part is $Df|_{(x_0, y_0)}$.

7.

Plane Path Integral

Following the definition of the Hyper-real Integral in [Dan3],

the Hyper-real Plane Path Integral of $\begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix}$ over a path

$(x(t), y(t))$, $t \in [\alpha, \beta]$, is the sum of the areas

$$P(x(t), y(t))dx(t)$$

of the rectangles with base $dx(t)$ and height $P(x(t), y(t))$,

Plus the sum of the areas

$$Q(x(t), y(t))dy(t)$$

of the rectangles with base $dy(t)$ and height $Q(x(t), y(t))$

7.1 Hyper-Real Plane Path Integral Definition

Let $P(x, y)$, and $Q(x, y)$ be hyper-real functions, defined on a bounded domain in the Hyper-Real Plane.

$P(x, y)$, and $Q(x, y)$ may take infinite hyper-real values.

$x(t), y(t)$, $t \in [\alpha, \beta]$, are hyper-real differentiable functions that constitute a path, $\gamma(a, b)$,

An arc length element is

$$dl = \sqrt{[dx(t)]^2 + [dy(t)]^2},$$

A tangent unit vector is

$$\vec{1}_l = \begin{bmatrix} \frac{dx(t)}{\sqrt{[dx(t)]^2 + [dy(t)]^2}} \\ \frac{dy}{\sqrt{[dx(t)]^2 + [dy(t)]^2}} \end{bmatrix}$$

Hence,

$$\vec{1}_x \cdot \vec{1}_l dl = dx$$

$$\vec{1}_y \cdot \vec{1}_l dl = dy.$$

For each t , there is an infinitesimal rectangle with base $dx(t)$, height $P(x(t), y(t))$, and area $P(x(t), y(t))dx(t)$, and there is an infinitesimal rectangle with base $dy(t)$, height $Q(x(t), y(t))$, and area $Q(x(t), y(t))dy(t)$,

We form the **Integration Sum** of all the areas that start at $(x(\alpha), y(\alpha))$, and end at $(x(\beta), y(\beta))$,

$$\sum_{t \in [\alpha, \beta]} \{ P(x(t), y(t))dx(t) + Q(x(t), y(t))dy(t) \}.$$

If for any infinitesimals $dx(t) = \dot{x}(t)dt$, and $dy(t) = \dot{y}(t)dt$ the Integration Sum equals the same hyper-real number, then

$\begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix}$ is Hyper-Real Integrable over the path $\gamma(a, b)$.

Then, we call the Integration Sum the Hyper-Real Path

Integral of $\begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix}$ over $\gamma(a, b)$, and denote it by

$$\int_{\gamma(a,b)} P(x, y)dx + Q(x, y)dy \equiv \int_{\gamma(a,b)} P(x, y)\vec{1}_x \cdot \vec{1}_l dl + Q(x, y)\vec{1}_y \cdot \vec{1}_l dl.$$

If the hyper-real number is an infinite hyper-real, then it

equals $\int_{\gamma(a,b)} P(x, y)dx + Q(x, y)dy$.

If the hyper-real number is finite, then its constant part

equals $\int_{\gamma(a,b)} P(x, y)dx + Q(x, y)dy$. \square

The Integration Sum may take infinite hyper-real values, such as $\frac{1}{dt}$, but may not equal to ∞ .

The Hyper-real Integral of the function $P(x, y) = \frac{1}{|(x, y)|}$ over a

path that goes through $(0, 0)$ diverges.

7.2 The Countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers,
 $Card\mathbb{N}$, equals the number of Real Numbers,
 $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = \dots = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = \dots \equiv \infty.$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[\alpha, \beta]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(z)dz$.

7.3 Continuous $[P(x, y), Q(x, y)]$ is Path-Integrable

If $P(x, y)$, and $Q(x, y)$ are Continuous on a domain D

Then $[P(x, y), Q(x, y)]$ is Path-Integrable in D

Proof: Let $(x(t), y(t))$, $t \in [\alpha, \beta]$, be a path. Then,

$$P(x(t), y(t))\dot{x}(t) + Q(x(t), y(t))\dot{y}(t) \text{ is continuous on } [\alpha, \beta].$$

Therefore, by [Dan3, 12.4],

$$P(x(t), y(t))\dot{x}(t) + Q(x(t), y(t))\dot{y}(t) \text{ is integrable on } [\alpha, \beta].$$

That is, $[P(x, y), Q(x, y)]$ is Path-Integrable on $\gamma(a, b)$. \square

8.

3-Space Path Integral

The definition of Path Integration extends to a vector of three Hyper-real functions

8.1 Path Integral of $\begin{bmatrix} P(x, y, z) \\ Q(x, y, z) \\ R(x, y, z) \end{bmatrix}$ **Definition**

the Hyper-real Path Integral of $\begin{bmatrix} P(x, y, z) \\ Q(x, y, z) \\ R(x, y, z) \end{bmatrix}$ over a path

$(x(t), y(t), z(t)), t \in [\alpha, \beta]$, is the sum of the areas

$$\sum_{t \in [\alpha, \beta]} \{P(x(t), y(t), z(t))\dot{x}(t) + Q(x(t), y(t), z(t))\dot{y}(t) + R(x(t), y(t), z(t))\dot{z}(t)\}dt.$$

If for any infinitesimal dt , the Integration Sum equals the

same hyper-real number, then $\begin{bmatrix} P(x, y, z) \\ Q(x, y, z) \\ R(x, y, z) \end{bmatrix}$ is Hyper-Real

Integrable over the path $\gamma(a, b)$.

Then, we call the Integration Sum the Hyper-Real Path

$$\text{Integral of } \begin{bmatrix} P(x, y, z) \\ Q(x, y, z) \\ R(x, y, z) \end{bmatrix} \text{ over } \gamma(a, b), \text{ and denote it by}$$

$$\int_{\gamma(a, b)} P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz.$$

Since there are countably many real numbers in $[\alpha, \beta]$,

8.2 The Integration Sum has countably many terms.

8.3 Continuous $\begin{bmatrix} P(x, y, z) \\ Q(x, y, z) \\ R(x, y, z) \end{bmatrix}$ is Path-Integrable

If $P(x, y, z)$, $Q(x, y, z)$, and $R(x, y, z)$ are Continuous on a domain D

Then $\begin{bmatrix} P(x, y, z) \\ Q(x, y, z) \\ R(x, y, z) \end{bmatrix}$ is Path-Integrable in D

9.

Hyper-real Area Integral

9.1 Area Integral of $P(x, y)$ Definition

Let $P(x, y)$ be a hyper-real function, defined on a bounded domain in the Hyper-Real Plane.

$P(x, y)$ may take infinite hyper-real values.

An area element is

$$dA = dx dy,$$

For each (x, y) , there is an infinitesimal rectangular box with base area $dA = dx dy$, height $P(x, y)$, and volume $P(x, y) dx dy$.

We form the **Double Sum** of all the volumes that are enclosed between the surface of $P(x, y)$, and the $P(x, y)$ domain in the plane

$$\sum_{y=b_1}^{y=b_2} \sum_{x=a_1}^{x=a_2} P(x, y) dx dy.$$

If for any infinitesimals dx , and dy , the Double Sum equals the same hyper-real number, then $P(x, y)$ is Hyper-Real Integrable over the plain domain.

Then, we call the Double Integration Sum the Hyper-Real Area Integral of $P(x, y)$ over the Domain, and denote it by

$$\int_{y=b_1}^{y=b_2} \int_{x=a_1}^{x=a_2} P(x, y) dx dy .$$

If the number is an infinite hyper-real, then it equals

$$\int_{y=b_1}^{y=b_2} \int_{x=a_1}^{x=a_2} P(x, y) dx dy .$$

If the number is a finite hyper-real, then its constant part

$$\text{equals } \int_{y=b_1}^{y=b_2} \int_{x=a_1}^{x=a_2} P(x, y) dx dy . \square$$

The Integration Sum may take infinite hyper-real values, such as $\frac{1}{(dx)(dy)}$, but may not equal to ∞ .

Since there are countably many real numbers in the plane,

9.2 The Integration Sum is countable.

9.3 Continuous $P(x, y)$ is Area-Integrable

10.

Hyper-Real Volume Integral

10.1 Volume Integral of $P(x, y, z)$ Definition

Let $P(x, y, z)$ be a hyper-real function, defined on a bounded domain in the Hyper-Real 3-Space.

$P(x, y, z)$ may take infinite hyper-real values.

A volume element is

$$dV = dx dy dz,$$

For each (x, y, z) , there is an infinitesimal 4 dimensional box with base volume $dV = dx dy dz$, height $P(x, y, z)$, and 4-volume $P(x, y) dx dy dz$.

We form the **Triple Sum** of all the 4-volumes that are based on the 3-manifold on which $P(x, y, z)$ is defined

$$\sum_{z=c_1}^{z=c_2} \sum_{y=b_1}^{y=b_2} \sum_{x=a_1}^{x=a_2} P(x, y) dx dy dz.$$

If for any infinitesimals dx , dy , and dz , the triple Sum equals the same hyper-real number, then $P(x, y, z)$ is Hyper-Real Integrable over its domain.

Then, we call the Triple Integration Sum the Hyper-Real Volume Integral of $P(x, y, z)$ over the Domain, and denote it by

$$\int_{z=c_1}^{z=c_2} \int_{y=b_1}^{y=b_2} \int_{x=a_1}^{x=a_2} P(x, y, z) dx dy dz .$$

If the number is an infinite hyper-real, then it equals

$$\int_{z=c_1}^{z=c_2} \int_{y=b_1}^{y=b_2} \int_{x=a_1}^{x=a_2} P(x, y, z) dx dy dz .$$

If the number is a finite hyper-real, then its constant part

$$\text{equals } \int_{z=c_1}^{z=c_2} \int_{y=b_1}^{y=b_2} \int_{x=a_1}^{x=a_2} P(x, y, z) dx dy dz . \square$$

The Integration Sum may take infinite hyper-real values, such as $\frac{1}{(dx)(dy)(dz)}$, but may not equal to ∞ .

Since there are countably many real numbers in the space,

10.2 The Integration Sum is countable.

10.3 Continuous $P(x, y, z)$ is Volume-Integrable

11.

Hyper-Real Surface Integral

11.1 The Surface Area Element

A point on a surface is determined by two parameters u , and v , so that in the x, y, z coordinate system,

$$\vec{r}(u, v) = \begin{bmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{bmatrix}.$$

At the point $\vec{r}(u, v)$,

the tangent to the surface in the direction of u is

$$\frac{\partial \vec{r}}{\partial u},$$

the tangent to the surface in the direction of v is

$$\frac{\partial \vec{r}}{\partial v},$$

and the normal to the tangent plane is

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{bmatrix} \vec{1}_x & \vec{1}_y & \vec{1}_z \\ \partial_u x & \partial_u y & \partial_u z \\ \partial_v x & \partial_v y & \partial_v z \end{bmatrix}$$

$$= \left(\frac{\partial(y, z)}{\partial(u, v)}, \frac{\partial(z, x)}{\partial(u, v)}, \frac{\partial(x, y)}{\partial(u, v)} \right).$$

The unit normal is

$$\begin{aligned} \vec{\mathbf{1}}_n &= \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right|}, \\ &= \begin{bmatrix} (\vec{\mathbf{1}}_x, \vec{\mathbf{1}}_n) \\ (\vec{\mathbf{1}}_y, \vec{\mathbf{1}}_n) \\ (\vec{\mathbf{1}}_z, \vec{\mathbf{1}}_n) \end{bmatrix}, \\ &= \begin{bmatrix} \cos(\vec{\mathbf{1}}_x, \vec{\mathbf{1}}_n) \\ \cos(\vec{\mathbf{1}}_y, \vec{\mathbf{1}}_n) \\ \cos(\vec{\mathbf{1}}_z, \vec{\mathbf{1}}_n) \end{bmatrix}. \end{aligned}$$

The surface area element is

$$\begin{aligned} d\vec{S} &= \left(\frac{\partial \vec{r}}{\partial u} du \right) \times \left(\frac{\partial \vec{r}}{\partial v} dv \right) \\ &= \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} dudv = \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| \vec{\mathbf{1}}_n dudv \\ &= \begin{bmatrix} \frac{\partial(y, z)}{\partial(u, v)} dudv \\ \frac{\partial(x, z)}{\partial(u, v)} dudv \\ \frac{\partial(x, y)}{\partial(u, v)} dudv \end{bmatrix} = \begin{bmatrix} dydz \\ dx dz \\ dx dy \end{bmatrix} = \begin{bmatrix} dS_x \\ dS_y \\ dS_z \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{1}}_x \cdot d\vec{S} \\ \vec{\mathbf{1}}_y \cdot d\vec{S} \\ \vec{\mathbf{1}}_z \cdot d\vec{S} \end{bmatrix} \end{aligned}$$

11.2 Surface Integral of $\varphi(x, y, z)$ Definition

Let

$$\begin{aligned}\varphi(x, y, z) &= \varphi(x(u, v), y(u, v), z(u, v)) \\ &= \varphi(u, v),\end{aligned}$$

be hyper-real function, defined on a bounded surface $\vec{r}(u, v)$ in the Hyper-Real 3-space.

$\varphi(u, v)$ may take infinite hyper-real values.

For each (u, v) , there is an infinitesimal volume

$$\varphi(u, v)\vec{1}_x \cdot d\vec{S} = \varphi(x, y, z)(\vec{1}_x \cdot \vec{1}_n)dydz.$$

We form the **Double Sum**

$$\sum_{z=c_1}^{z=c_2} \sum_{y=b_1}^{y=b_2} \varphi(x, y, z)(\vec{1}_x \cdot \vec{1}_n)dydz.$$

If for any infinitesimals dy , and dz , the Double Sum equals the same hyper-real number, then $\varphi(x, y, z)$ is Hyper-Real Integrable over the surface.

Then, we call the Double Integration Sum the Hyper-Real Surface Integral of $\varphi(u, v)$ over the surface, and denote it by

$$\int_{z=c_1}^{z=c_2} \int_{y=b_1}^{y=b_2} \varphi(x, y, z)(\vec{1}_x \cdot \vec{1}_n)dydz.$$

If the number is an infinite hyper-real, then it equals

$$\int_{z=c_1}^{z=c_2} \int_{y=b_1}^{y=b_2} \varphi(x, y, z) (\vec{1}_x \cdot \vec{1}_n) dy dz .$$

If the number is a finite hyper-real, then its constant part

equals $\int_{z=c_1}^{z=c_2} \int_{y=b_1}^{y=b_2} \varphi(x, y, z) (\vec{1}_x \cdot \vec{1}_n) dy dz . \square$

The Integration Sum may take infinite hyper-real values, such as $\frac{1}{(dy)(dz)}$, but may not equal to ∞ .

Since there are countably many real numbers in the plane,

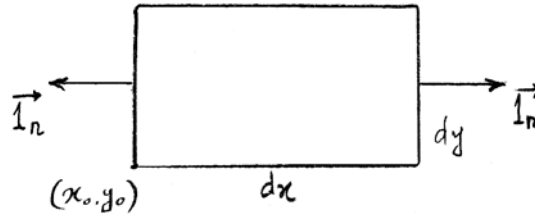
11.3 The Integration Sum is countable.

11.4 Continuous $\varphi(x, y, z)$ is Surface-Integrable.

12.

Plane Gradient

Let $\varphi(x, y)$ be a hyper-real differentiable Potential function, defined on a rectangle with vertex at (x_0, y_0) and sides dx , and dy .



12.1 x -Component of Gradient Definition

The potential drop at $y = y_0$, from $x = x_0$, to $x = x_0 + dx$ is

$$\varphi(x_0 + dx, y_0) - \varphi(x_0, y_0).$$

We define the **linear density** of that potential drop as the x -component of the Gradient of $\varphi(x, y)$ at (x_0, y_0) ,

$$\underbrace{(\text{Grad } \varphi)}_{\nabla} \Big|_{x_0, y_0} \Big|_x = \frac{1}{dx} [\varphi(x_0 + dx, y_0) - \varphi(x_0, y_0)] = \frac{\partial \varphi}{\partial x} \Big|_{x_0, y_0} .$$

12.2 y -Component of Gradient Definition

The potential drop at $x = x_0$, from $y = y_0$, to $y = y_0 + dy$ is

$$\varphi(x_0, y_0 + dy) - \varphi(x_0, y_0).$$

We define the **linear density** of that potential drop as the y -component of the Gradient of $\varphi(x, y)$ at (x_0, y_0) ,

$$\underbrace{(\text{Grad } \varphi)}_{\nabla}_y \Big|_{x_0, y_0} = \frac{1}{dy} [\varphi(x_0, y_0 + dy) - \varphi(x_0, y_0)] = \frac{\partial \varphi}{\partial y} \Big|_{x_0, y_0}$$

12.3

$$\nabla \varphi = \begin{bmatrix} \partial_x \varphi \\ \partial_y \varphi \end{bmatrix}$$

13.

Plane Gradient Theorem

13.1 The Plane Gradient Theorem

$$\left[\begin{array}{c} \oiint (\partial_x \varphi) dx dy \\ \oiint_S (\partial_y \varphi) dx dy \end{array} \right] = \left[\begin{array}{c} \oint \varphi(x, y) \vec{1}_x \cdot \vec{1}_n dy \\ \oint_{\partial S} \varphi(x, y) \vec{1}_y \cdot \vec{1}_n dx \end{array} \right]$$

Proof: The sum of the $\left[\begin{array}{c} \partial_x \varphi dx dy \\ \partial_y \varphi dx dy \end{array} \right]$ over the infinitesimal

rectangles enclosed in a area S ,

equals the sum of $\left[\begin{array}{c} \int_{\text{rectangle}} \varphi(x, y) \vec{1}_x \cdot \vec{1}_n dy \\ \int_{\text{rectangle}} \varphi(x, y) \vec{1}_y \cdot \vec{1}_n dx \end{array} \right]$, over the sides of the

infinitesimal rectangles enclosed in the area S ,

$$\left[\begin{array}{c} \sum_{y=b_1}^{y=b_2} \sum_{x=a_1}^{x=a_2} (\partial_x \varphi) dx dy \\ \sum_{y=b_1}^{y=b_2} \sum_{x=a_1}^{x=a_2} (\partial_y \varphi) dx dy \end{array} \right] = \left[\begin{array}{c} \sum_{y=b_1}^{y=b_2} \oint_{\text{rectangle}} \varphi(x, y) \vec{1}_x \cdot \vec{1}_n dy \\ \sum_{x=a_1}^{x=a_2} \oint_{\text{rectangle}} \varphi(x, y) \vec{1}_y \cdot \vec{1}_n dx \end{array} \right]$$

The sum over the areas is

$$\left[\begin{array}{c} \oiint \partial_x \varphi dx dy \\ \overset{S}{\iint} \\ \oiint \partial_y \varphi dx dy \\ \underset{S}{} \end{array} \right].$$

The Interior line integrals of

$$\int_{\text{rectangle}} \varphi(x, y) \vec{1}_x \cdot \vec{1}_n dy, \quad \text{and} \quad \int_{\text{rectangle}} \varphi(x, y) \vec{1}_x \cdot \vec{1}_n dy$$

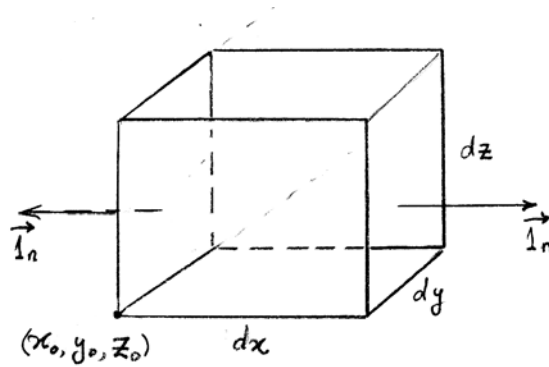
appear in pairs of opposite signs and cancel, leaving the
Integral over the boundary line,

$$\left[\begin{array}{c} \oint \varphi(x, y) \vec{1}_x \cdot \vec{1}_n dy \\ \overset{\partial S}{} \\ \oint \varphi(x, y) \vec{1}_y \cdot \vec{1}_n dx \\ \underset{\partial S}{} \end{array} \right]. \square$$

14.

3-Space Gradient

Let $\varphi(x, y, z)$ be a hyper-real differentiable Potential function, defined on a box with vertex at (x_0, y_0, z_0) and sides dx , dy , and dz .



14.1 x -Component of Gradient Definition

The potential drop at y_0, z_0 , from $x = x_0$, to $x = x_0 + dx$ is

$$\varphi(x_0 + dx, y_0, z_0) - \varphi(x_0, y_0, z_0).$$

We define the **linear density** of that potential drop as the x -component of the Gradient of $\varphi(x, y, z)$ at (x_0, y_0, z_0) ,

$$\underbrace{(\text{Grad } \varphi)}_{\nabla}_x \Big|_{x_0, y_0} = \frac{1}{dx} [\varphi(x_0 + dx, y_0, z_0) - \varphi(x_0, y_0, z_0)] = \frac{\partial \varphi}{\partial x} \Big|_{x_0, y_0, z_0}.$$

14.2 y -Component of Gradient Definition

$$\underbrace{(\text{Grad } \varphi)}_{\nabla}_y \Big|_{x_0, y_0, z_0} = \frac{1}{dy} [\varphi(x_0, y_0 + dy, z_0) - \varphi(x_0, y_0, z_0)] = \frac{\partial \varphi}{\partial y} \Big|_{x_0, y_0, z_0}$$

14.3 z -Component of Gradient Definition

$$\underbrace{(\text{Grad } \varphi)}_{\nabla}_z \Big|_{x_0, y_0, z_0} = \frac{1}{dz} [\varphi(x_0, y_0, z_0 + dz) - \varphi(x_0, y_0, z_0)] = \frac{\partial \varphi}{\partial z} \Big|_{x_0, y_0, z_0}$$

$$\mathbf{14.4} \quad \nabla \varphi = \begin{bmatrix} \partial_x \varphi \\ \partial_y \varphi \\ \partial_z \varphi \end{bmatrix}$$

15.

3-Space Gradient Theorem

15.1 The Gradient Theorem

$$\iiint_V (\nabla \varphi) dx dy dz = \iint_{S=\partial V} \varphi(x, y, z) d\vec{S}$$

Proof: The sum of the $(\nabla \varphi) dx dy dz$ over the infinitesimal box enclosed in a volume V ,

equals the sum of the fluxes $\iint_{\text{parallelepiped}} \varphi(x, y, z) d\vec{S}$ over the

surfaces of the infinitesimal boxes enclosed in the volume V ,

$$\left[\begin{array}{l} \sum_{z=c_1}^{z=c_2} \sum_{y=b_1}^{y=b_2} \sum_{x=a_1}^{x=a_2} (\partial_x \varphi) dx dy dz \\ \sum_{z=c_1}^{z=c_2} \sum_{y=b_1}^{y=b_2} \sum_{x=a_1}^{x=a_2} (\partial_y \varphi) dx dy dz \\ \sum_{z=c_1}^{z=c_2} \sum_{y=b_1}^{y=b_2} \sum_{x=a_1}^{x=a_2} (\partial_z \varphi) dx dy dz \end{array} \right] = \left[\begin{array}{l} \sum_{z=c_1}^{z=c_2} \sum_{y=b_1}^{y=b_2} \iint_{\text{parallelepiped}} \varphi(x, y, z) \vec{1}_x \cdot \vec{1}_n dy dz \\ \sum_{z=c_1}^{z=c_2} \sum_{x=a_1}^{x=a_2} \iint_{\text{parallelepiped}} \varphi(x, y, z) \vec{1}_y \cdot \vec{1}_n dx dz \\ \sum_{y=b_1}^{y=b_2} \sum_{x=a_1}^{x=a_2} \iint_{\text{parallelepiped}} \varphi(x, y, z) \vec{1}_z \cdot \vec{1}_n dx dy \end{array} \right]$$

The sum over the volumes is

$$\left[\begin{array}{c} \iiint_V \partial_x \varphi dx dy dz \\ \iiint_V \partial_y \varphi dx dy dz \\ \iiint_V \partial_z \varphi dx dy dz \end{array} \right] = \iiint_V (\nabla \varphi) dx dy dz .$$

The Interior area integrals of $\iint_{\text{parallelepiped}} \varphi(x, y, z) d\vec{S}$ appear in

pairs of opposite signs and cancel, leaving the surface
Integral over the boundary surface,

$$\left[\begin{array}{c} \iint_{S=\partial V} \varphi(x, y, z) \vec{1}_x \cdot \vec{1}_n dy dz \\ \iint_{S=\partial V} \varphi(x, y, z) \vec{1}_y \cdot \vec{1}_n dx dz \\ \iint_{S=\partial V} \varphi(x, y, z) \vec{1}_z \cdot \vec{1}_n dx dy \end{array} \right] = \iint_{S=\partial V} \varphi(x, y, z) d\vec{S} . \square$$

16.

Plane Directional Derivative

16.1 The derivative of $\varphi(x, y)$ in the direction of \vec{l} is

$$\frac{d\varphi}{dl} = (\nabla\varphi) \cdot \vec{l}$$

Proof:

$$\begin{aligned} \frac{d\varphi}{dl} &= \frac{\partial\varphi}{\partial x} \frac{dx}{dl} + \frac{\partial\varphi}{\partial y} \frac{dy}{dl} \\ &= (\partial_x\varphi)(\vec{l}_x \cdot \vec{l}) + (\partial_y\varphi)(\vec{l}_y \cdot \vec{l}) \\ &= (\nabla\varphi) \cdot \vec{l}. \square \end{aligned}$$

16.2 The maximal $\frac{d\varphi(x, y)}{dl}$ is $\left| \begin{matrix} \partial_x\varphi \\ \partial_y\varphi \end{matrix} \right|$, where $\vec{l} \parallel \nabla\varphi$.

Proof:

$$\frac{d\varphi}{dl} = |\nabla\varphi| |\vec{l}| \cos(\nabla\varphi, \vec{l}) = |\nabla\varphi| \cos(\nabla\varphi, \vec{l}). \square$$

16.3 $\text{Grad } \varphi$ is normal to the surface $\varphi(x, y) = c$

Proof: For any \vec{l} on the surface $\varphi(x, y, z) = c$,

$$(\nabla\varphi) \cdot \vec{l} = \frac{d\varphi}{dl} = 0. \text{ That is, } \text{Grad } \varphi \perp \vec{l}. \square$$

16.4 $\varphi(x, y) = c \Rightarrow \frac{d\varphi}{dn}$ equals $|\nabla\varphi|$, and is maximal

Proof: By 11.11, $\text{Grad } \varphi \parallel \vec{\mathbf{1}}_n$.

$$\text{Hence, } \frac{d\varphi}{dn} = |\text{Grad } \varphi| \|\vec{\mathbf{1}}_n\| \underbrace{\cos(\text{Grad } \varphi, \vec{\mathbf{1}}_n)}_1 = |\text{Grad } \varphi|$$

By 11.10, $|\text{Grad } \varphi|$ is the maximal $\frac{d\varphi}{dl}$.

Therefore, $\frac{d\varphi}{dn}$ is maximal. \square

17.

3-Space Directional Derivative

17.1 *The derivative of $\varphi(x, y, z)$ in the direction of \vec{l} is*

$$\frac{d\varphi}{dl} = (\nabla\varphi) \cdot \vec{l}$$

Proof: $\frac{d\varphi}{dl} = \frac{\partial\varphi}{\partial x_i} \frac{dx_i}{dl} = (\nabla\varphi)_{x_i} (\vec{l}_{x_i} \cdot \vec{l}) = (\nabla\varphi) \cdot \vec{l}. \square$

17.2 *The maximal $\frac{d\varphi}{dl}$ is $|\nabla\varphi|$, where $\vec{l} \parallel \nabla\varphi$.*

Proof: $\frac{d\varphi}{dl} = |\nabla\varphi| |\vec{l}| \cos(\nabla\varphi, \vec{l}) = |\nabla\varphi| \cos(\nabla\varphi, \vec{l}). \square$

17.3 *Grad φ is normal to the surface $\varphi(x, y, z) = c$*

Proof: For any \vec{l} on the surface $\varphi(x, y, z) = c$,

$$(\nabla\varphi) \cdot \vec{l} = \frac{d\varphi}{dl} = 0. \text{ That is, } \text{Grad } \varphi \perp \vec{l}. \square$$

17.4 $\varphi(x, y, z) = c \Rightarrow \frac{d\varphi}{dn}$ equals $|\nabla\varphi|$, and is maximal

Proof: By 11.11, $\text{Grad } \varphi \parallel \vec{\mathbf{1}}_n$.

$$\text{Hence, } \frac{d\varphi}{dn} = |\text{Grad } \varphi| \|\vec{\mathbf{1}}_n\| \underbrace{\cos(\text{Grad } \varphi, \vec{\mathbf{1}}_n)}_1 = |\text{Grad } \varphi|$$

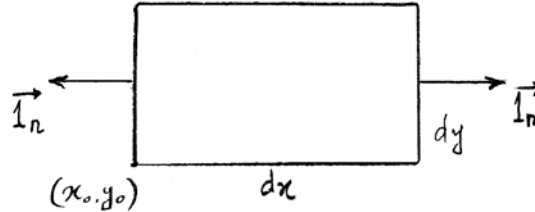
By 11.10, $|\text{Grad } \varphi|$ is the maximal $\frac{d\varphi}{dl}$.

Therefore, $\frac{d\varphi}{dn}$ is maximal. \square

18.

Area Divergence

Let $P(x, y)$, and $Q(x, y)$, be hyper-real differentiable functions, defined on a rectangle with vertex at (x_0, y_0) and sides dx , dy .



18.1 The Area Divergence due to $P(x, y)$

The flux of $P(x, y)$ through $x = x_0$, and $x = x_0 + dx$ is

$$\int_{\text{rectangle}} P(x, y) \vec{i}_x \cdot \vec{i}_n dy.$$

We define the **area density** of that flux as the part of the

Divergence of $\begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix}$ due to $P(x, y)$

$$\left(\underbrace{\text{Div}}_{\nabla \cdot} \begin{bmatrix} P \\ Q \end{bmatrix} \right)_P = \frac{1}{dxdy} \int_{\text{rectangle}} P(x, y) \vec{i}_x \cdot \vec{i}_n dy$$

$$\mathbf{18.2} \quad (\nabla \cdot \begin{bmatrix} P \\ Q \end{bmatrix})_P = \frac{\partial P}{\partial x}$$

Proof:

The flux of $P(x, y)$ through the rectangle side $x = x_0$ is

$$P(x_0, y_0) \vec{\mathbf{1}}_x \cdot (-\vec{\mathbf{1}}_x) dy = -P(x_0, y_0) dy.$$

The flux of $P(x, y)$ through the rectangle side $x = x_0 + dx$ is

$$P(x_0 + dx, y_0) \vec{\mathbf{1}}_x \cdot \vec{\mathbf{1}}_x dy = P(x_0 + dx, y_0) dy.$$

$$\begin{aligned} (\nabla \cdot \begin{bmatrix} P \\ Q \end{bmatrix})_P &= \frac{1}{dxdy} \int_{\text{rectangle}} P(x, y) \vec{\mathbf{1}}_x \cdot \vec{\mathbf{1}}_n dy \\ &= \frac{1}{dxdy} \{ P(x_0 + dx, y_0) dy - P(x_0, y_0) dy \} \\ &= \frac{P(x_0 + dx, y_0) - P(x_0, y_0)}{dx} \\ &= \left. \frac{\partial P}{\partial x} \right|_{x_0, y_0}. \end{aligned}$$

18.3 The Area Divergence due to $Q(x, y)$

The flux of $Q(x, y)$ through $y = y_0$, and $y = y_0 + dy$ is

$$\int_{\text{rectangle}} Q(x, y) \vec{1}_y \cdot \vec{1}_n dx.$$

We define the **area density** of that flux as the part of the

Divergence of $\begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix}$ due to $Q(x, y)$

$$(\nabla \cdot \begin{bmatrix} P \\ Q \end{bmatrix})_Q = \frac{1}{dxdy} \int_{\text{rectangle}} Q(x, y) \vec{1}_y \cdot \vec{1}_n dx$$

$$\mathbf{18.4} \quad (\nabla \cdot \begin{bmatrix} P \\ Q \end{bmatrix})_Q = \frac{\partial Q}{\partial y}$$

Proof:

The flux of $Q(x, y)$ through the rectangle side $y = y_0$ is

$$Q(x_0, y_0) \vec{1}_y \cdot (-\vec{1}_y) dx = -Q(x_0, y_0) dx.$$

The flux of $Q(x, y)$ through the rectangle side $y = y_0 + dy$ is

$$Q(x_0, y_0 + dy) \vec{1}_y \cdot \vec{1}_y dx = Q(x_0, y_0 + dy) dx.$$

$$\begin{aligned} (\nabla \cdot \begin{bmatrix} P \\ Q \end{bmatrix})_y &= \frac{1}{dxdy} \int_{\text{rectangle}} Q(x, y) \vec{1}_y \cdot \vec{1}_n dx \\ &= \frac{1}{dxdy} \{ Q(x_0, y_0 + dy) dx - Q(x_0, y_0) dx \} \end{aligned}$$

$$\begin{aligned}
&= \frac{Q(x_0, y_0 + dy) - Q(x_0, y_0)}{dy} \\
&= \partial_y Q \Big|_{x_0, y_0} . \square
\end{aligned}$$

18.5 The Area Divergence

We define the *Area density of the total flux through the sides of the rectangle* as the **Divergence** of $(P(x, y), Q(x, y))$

$$\begin{aligned}
\text{Div} \begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix} &\equiv \nabla \cdot \begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix} \\
&= \frac{1}{dxdy} \left\{ \int_{\text{rectangle}} P(x, y) \vec{1}_x \cdot \vec{1}_n dy + \int_{\text{rectangle}} Q(x, y) \vec{1}_y \cdot \vec{1}_n dx \right\} \\
&= \frac{\partial P}{\partial x} \Big|_{x_0, y_0} + \frac{\partial Q}{\partial y} \Big|_{x_0, y_0} .
\end{aligned}$$

19.

Area Divergence Theorem

19.1 The Area Divergence Theorem

$$\oiint_S \{ \partial_x P + \partial_y Q \} dx dy = \oint_{\partial S} P(x, y) \vec{1}_x \cdot \vec{1}_n dy + Q(x, y) \vec{1}_y \cdot \vec{1}_n dx$$

Proof: The sum of the $\{ \partial_x P + \partial_y Q \} dx dy$ over the infinitesimal rectangles enclosed in a plane area S , equals the sum of the fluxes

$$\int_{\text{rectangle}} P(x, y) \vec{1}_x \cdot \vec{1}_n dy + Q(x, y) \vec{1}_y \cdot \vec{1}_n dx$$

over the sides of the infinitesimal rectangles enclosed in the area S ,

$$\sum_{y=b_1}^{y=b_2} \sum_{x=a_1}^{x=a_2} \{ \partial_x P + \partial_y Q \} dx dy = \sum_{y=b_1}^{y=b_2} P(x, y) \vec{1}_x \cdot \vec{1}_n dy + \sum_{x=a_1}^{x=a_2} Q(x, y) \vec{1}_y \cdot \vec{1}_n dx$$

The sum over the areas is

$$\oiint_S \{ \partial_x P(x, y) + \partial_y Q(x, y) \} dx dy .$$

The Interior path integrals of

$$\int_{\text{rectangle}} P(x, y) \vec{1}_x \cdot \vec{1}_n dy, \quad \text{and} \quad \int_{\text{rectangle}} Q(x, y) \vec{1}_y \cdot \vec{1}_n dx,$$

appear in pairs of opposite signs and cancel, leaving the
Integral over the boundary line,

$$\oint_{\partial S} P(x, y) \vec{1}_x \cdot \vec{1}_n dy + Q(x, y) \vec{1}_y \cdot \vec{1}_n dx = \iint_S \{ \partial_x P + \partial_y Q \} dx dy . \square$$

$$\mathbf{19.2} \quad \iint_S \nabla \cdot \begin{bmatrix} P \\ Q \end{bmatrix} dS = \oint_{\partial S} P(x, y) \vec{1}_x \cdot \vec{1}_n dy + Q(x, y) \vec{1}_y \cdot \vec{1}_n dx$$

19.3 $V(x, y)$ is *Hyper-real differentiable in a plane domain*

D. Then,

$$\iint_S (\nabla^2 V) dS = \oint_{\partial S} (\partial_x V) \vec{1}_x \cdot \vec{1}_n dy + (\partial_y V) \vec{1}_y \cdot \vec{1}_n dx$$

Proof: In 19.2, put $P = \partial_x V$, $Q = \partial_y V$. Then,

$$\begin{aligned} \iint_S (\nabla^2 V) dS &= \oint_{\partial S} (\partial_x V) \vec{1}_x \cdot \vec{1}_n dy + (\partial_y V) \vec{1}_y \cdot \vec{1}_n dx \\ &= \oint_{\partial S} \begin{bmatrix} \partial_x V \\ \partial_y V \end{bmatrix} \cdot \begin{bmatrix} \vec{1}_x \cdot \vec{1}_n dy \\ \vec{1}_y \cdot \vec{1}_n dx \end{bmatrix} \end{aligned}$$

20.

Plane Green's Identities

20.1 $U(x, y)$, and $V(x, y)$ are Hyper-real differentiable in a plane domain D . Then,

$$\nabla \cdot (U[\nabla V]) = U\nabla^2 V + (\nabla U) \cdot (\nabla V)$$

Proof:

$$\begin{aligned} \nabla \cdot (U[\nabla V]) &= \nabla \cdot \begin{bmatrix} U\partial_x V \\ U\partial_y V \end{bmatrix} \\ &= \partial_x(U\partial_x V) + \partial_y(U\partial_y V) \\ &= U[\partial_x^2 V + \partial_y^2 V] + (\partial_x U)\partial_x V + (\partial_y U)\partial_y V \\ &= U\nabla^2 V + (\nabla U) \cdot (\nabla V). \square \end{aligned}$$

20.2 1st Plane Green Identity

$U(x, y)$, and $V(x, y)$ are Hyper-real differentiable in a plane domain D bounded by a closed curve ∂D . Then,

$$\iint_D [U\nabla^2 V + \nabla U \cdot \nabla V] dx dy = \oint_{\partial D} U \frac{\partial V}{\partial n} dl$$

Proof:

By the Plane Divergence Theorem, 19.1

$$\oint_D \underbrace{\nabla \cdot \begin{bmatrix} U \partial_x V \\ U \partial_y V \end{bmatrix}}_{\nabla \cdot (U[\nabla V])} dS = \oint_{\partial D} U(\partial_x V) \vec{1}_x \cdot \vec{1}_n dy + U(\partial_y V) \vec{1}_y \cdot \vec{1}_n dx$$

By 20.1,

$$\nabla \cdot (U[\nabla V]) = U \nabla^2 V + (\nabla U) \cdot (\nabla V)$$

Therefore,

$$\begin{aligned} \oint_D [U \nabla^2 V + \nabla U \cdot \nabla V] dx dy &= \oint_{\partial D} U \partial_x V \vec{1}_x \cdot \vec{1}_n dy + U \partial_y V \vec{1}_y \cdot \vec{1}_n dx \\ &= \oint_{\partial D} U \frac{\partial V}{\partial n} dl. \square \end{aligned}$$

20.3 2nd Plane Green Identity

$U(x, y)$, and $V(x, y)$ are Hyper-real differentiable in a plane domain D bounded by a closed curve ∂D . Then,

$$\oint_D [U \nabla^2 V - V \nabla^2 U] dx dy = \oint_{\partial D} [U \frac{\partial V}{\partial n} - \frac{\partial U}{\partial n} V] dl$$

Proof: By 20.2,

$$\oint_D [\nabla V \cdot \nabla U] dx dy = - \oint_D [V \nabla^2 U] dx dy + \oint_{\partial D} [V \frac{\partial U}{\partial n}] dl,$$

$$\iint_D [\nabla U \cdot \nabla V] dx dy = -\iint_D [U \nabla^2 V] dx dy + \oint_{\partial D} [U \frac{\partial V}{\partial n}] dl.$$

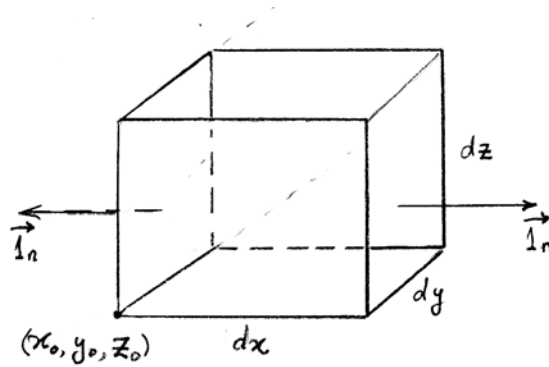
Since $\nabla V \cdot \nabla U = \nabla U \cdot \nabla V$,

$$\iint_D [U \nabla^2 V - V \nabla^2 U] dx dy = \oint_{\partial D} [U \frac{\partial V}{\partial n} - \frac{\partial U}{\partial n} V] dl. \square$$

21.

Volume Divergence

Let $P(x, y, z)$, $Q(x, y, z)$, and $R(x, y, z)$ be hyper-real differentiable functions, defined on a box with vertex at (x_0, y_0, z_0) and sides dx , dy , and dz .



21.1 Volume Divergence due to $P(x, y, z)$

The flux of $P(x, y, z)$ through the box walls $x = x_0$, and $x = x_0 + dx$ is

$$\iint_{box} P(x, y, z) dS_x = \iint_{box} P(x, y, z) \vec{i}_x \cdot \vec{i}_n dy dz.$$

We define the **volume density** of that flux as the part of the Divergence due to $P(x, y, z)$

$$(\nabla \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix})_P = \frac{1}{dxdydz} \iint_{box} P(x, y, z) \vec{1}_x \cdot \vec{1}_n dydz$$

21.2
$$(\nabla \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix})_P = \frac{\partial P}{\partial x}$$

Proof:

The flux of $P(x, y, z)$ through the wall $x = x_0$ is

$$P(x_0, y_0, z_0) \vec{1}_x \cdot (-\vec{1}_x) dydz = -P(x_0, y_0, z_0) dydz.$$

The flux of $P(x, y, z)$ through the wall $x = x_0 + dx$ is

$$P(x_0 + dx, y_0, z_0) \vec{1}_x \cdot \vec{1}_x dydz = P(x_0 + dx, y_0, z_0) dydz.$$

$$\begin{aligned} (\nabla \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix})_P &= \frac{1}{dxdydz} \iint_{box} P(x, y, z) \vec{1}_x \cdot \vec{1}_n dydz \\ &= \frac{1}{dxdydz} \{ P(x_0 + dx, y_0, z_0) dydz - P(x_0, y_0, z_0) dydz \} \\ &= \frac{P(x_0 + dx, y_0, z_0) - P(x_0, y_0, z_0)}{dx} \\ &= \frac{\partial P}{\partial x} \Big|_{x_0, y_0, z_0}. \end{aligned}$$

21.3 The Volume Divergence due to $Q(x, y, z)$

The flux of $Q(x, y, z)$ through the box walls $y = y_0$, and $y = y_0 + dy$ is

$$\iint_{box} Q(x, y, z) dS_y = \iint_{box} Q(x, y, z) \vec{1}_y \cdot \vec{1}_n dx dz.$$

We define the **volume density** of that flux as the part of the Divergence due to $Q(x, y, z)$

$$\left(\nabla \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \right)_Q = \frac{1}{dxdydz} \iint_{box} Q(x, y, z) \vec{1}_y \cdot \vec{1}_n dx dz$$

21.4
$$\left(\nabla \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \right)_Q = \frac{\partial Q}{\partial y}$$

Proof:

The flux of $Q(x, y, z)$ through the wall $y = y_0$ is

$$Q(x_0, y_0, z_0) \vec{1}_y \cdot (-\vec{1}_y) dx dz = -Q(x_0, y_0, z_0) dx dz.$$

The flux of $Q(x, y, z)$ through the wall $y = y_0 + dy$ is

$$Q(x_0, y_0 + dy, z_0) \vec{1}_y \cdot \vec{1}_y dx dz = Q(x_0, y_0 + dy, z_0) dx dz.$$

$$\begin{aligned}
(\nabla \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix})_Q &= \frac{1}{dxdydz} \iint_{box} Q(x, y, z) \vec{1}_x \cdot \vec{1}_n dx dz \\
&= \frac{1}{dxdydz} \{ Q(x_0, y_0 + dy, z_0) dx dz - Q(x_0, y_0, z_0) dx dz \} \\
&= \frac{Q(x_0, y_0 + dy, z_0) - Q(x_0, y_0, z_0)}{dy} \\
&= \left. \frac{\partial Q}{\partial y} \right|_{x_0, y_0, z_0} .
\end{aligned}$$

21.5 The Volume Divergence due to $R(x, y, z)$

The flux of $R(x, y, z)$ through the box walls $z = z_0$, and $z = z_0 + dz$ is

$$\iint_{box} R(x, y, z) dS_z = \iint_{box} R(x, y, z) \vec{1}_z \cdot \vec{1}_n dx dy .$$

We define the **volume density** of that flux as the part of the Divergence due to $R(x, y, z)$

$$(\nabla \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix})_R = \frac{1}{dxdydz} \iint_{box} R(x, y, z) \vec{1}_z \cdot \vec{1}_n dx dy$$

$$\mathbf{21.6} \quad (\nabla \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix})_R = \frac{\partial R}{\partial z}$$

The flux of $R(x, y, z)$ through the wall $z = z_0$ is

$$R(x_0, y_0, z_0) \vec{1}_z \cdot (-\vec{1}_z) dx dy = -R(x_0, y_0, z_0) dx dy.$$

The flux of $R(x, y, z)$ through the wall $z = z_0 + dz$ is

$$R(x_0, y_0, z_0 + dz) \vec{1}_z \cdot \vec{1}_z dx dy = R(x_0, y_0, z_0 + dz) dx dy.$$

$$\begin{aligned} (\nabla \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix})_R &= \frac{1}{dxdydz} \iint_{box} R(x, y, z) \vec{1}_z \cdot \vec{1}_n dx dy \\ &= \frac{1}{dxdydz} \{ R(x_0, y_0, z_0 + dz) dx dz - R(x_0, y_0, z_0) dx dz \} \\ &= \frac{R(x_0, y_0, z_0 + dz) - R(x_0, y_0, z_0)}{dz} \\ &= \left. \frac{\partial R}{\partial z} \right|_{x_0, y_0, z_0}. \end{aligned}$$

21.7 Divergence of $(P(x, y, z), Q(x, y, z), R(x, y, z))$

We define the *volume density of the total flux through the walls of the box* as the **Divergence** of the vector

$$(P(x, y, z), Q(x, y, z), R(x, y, z))$$

$$\nabla \cdot \begin{bmatrix} P(x, y, z) \\ Q(x, y, z) \\ R(x, y, z) \end{bmatrix} = \frac{1}{dxdydz} \left\{ \iint_{box} P(x, y, z) \vec{1}_x \cdot \vec{1}_n dydz \right. \\ \left. + \iint_{box} Q(x, y, z) \vec{1}_y \cdot \vec{1}_n dx dz \right. \\ \left. + \iint_{box} P(x, y, z) \vec{1}_x \cdot \vec{1}_n dydz \right\}$$

$$\mathbf{21.8} \quad \nabla \cdot \begin{bmatrix} P(x, y, z) \\ Q(x, y, z) \\ R(x, y, z) \end{bmatrix} = \frac{\partial P}{\partial x} \Big|_{x_0, y_0, z_0} + \frac{\partial Q}{\partial y} \Big|_{x_0, y_0, z_0} + \frac{\partial R}{\partial z} \Big|_{x_0, y_0, z_0}$$

22.

Volume Divergence Theorem

22.1 The Divergence Theorem

$$\iiint_V \{ \partial_x P + \partial_y Q + \partial_z R \} dxdydz = \iint_{S=\partial V} PdS_x + QdS_y + RdS_z$$

Proof: The sum of the $\{ \partial_x P + \partial_y Q + \partial_z R \} dxdydz$ over the infinitesimal boxes enclosed in a volume V , equals the sum of the fluxes

$$\iint_{box} \{ P(x, y, z)dS_x + Q(x, y, z)dS_y + R(x, y, z)dS_z \}$$

over the surfaces of the infinitesimal boxes enclosed in the volume V ,

$$\begin{aligned} & \sum_{z=c_1}^{z=c_2} \sum_{y=b_1}^{y=b_2} \sum_{x=a_1}^{x=a_2} \{ \partial_x P + \partial_y Q + \partial_z R \} dxdydz = \\ & = \sum_{z=c_1}^{z=c_2} \sum_{y=b_1}^{y=b_2} \iint_{box} P(x, y, z)(\vec{1}_x \cdot \vec{1}_n) dydz \\ & + \sum_{z=c_1}^{z=c_2} \sum_{x=a_1}^{x=a_2} \iint_{box} Q(x, y, z)(\vec{1}_y \cdot \vec{1}_n) dxdy \end{aligned}$$

$$+ \sum_{y=b_1}^{y=b_2} \sum_{x=a_1}^{x=a_2} \oint_{box} R(x, y, z) (\vec{1}_z \cdot \vec{1}_n) dx dy$$

The sum over the volumes is

$$\iiint_V \{ \partial_x P(x, y, z) + \partial_y Q(x, y, z) + \partial_z R(x, y, z) \} dx dy dz .$$

The Interior area integrals of

$$\oint_{box} P(x, y, z) (\vec{1}_x \cdot \vec{1}_n) dy dz ,$$

$$\oint_{box} Q(x, y, z) (\vec{1}_y \cdot \vec{1}_n) dx dz ,$$

$$\oint_{box} R(x, y, z) (\vec{1}_{xz} \cdot \vec{1}_n) dx dy$$

appear in pairs of opposite signs and cancel, leaving the surface Integral over the boundary surface,

$$\begin{aligned} & \oint_{S=\partial V} P(x, y, z) (\vec{1}_x \cdot \vec{1}_n) dy dz \\ & + \oint_{S=\partial V} Q(x, y, z) (\vec{1}_y \cdot \vec{1}_n) dx dz \\ & + \oint_{S=\partial V} R(x, y, z) (\vec{1}_z \cdot \vec{1}_n) dx dy \\ & = \oint_{S=\partial V} P dS_x + Q dS_y + R dS_z . \square \end{aligned}$$

23.

3-Space Green Identities

23.1 $U(x, y, z)$, and $V(x, y, z)$ are *Hyper-real differentiable* in a 3-space domain D . Then,

$$\nabla \cdot (U[\nabla V]) = U\nabla^2 V + (\nabla U) \cdot (\nabla V)$$

Proof:

$$\begin{aligned} \nabla \cdot (U[\nabla V]) &= \nabla \cdot \begin{bmatrix} U\partial_x V \\ U\partial_y V \\ U\partial_z V \end{bmatrix} \\ &= \partial_x(U\partial_x V) + \partial_y(U\partial_y V) + \partial_z(U\partial_z V) \\ &= U[\partial_x^2 V + \partial_y^2 V + \partial_z^2 V] + (\partial_x U)\partial_x V + (\partial_y U)\partial_y V + (\partial_z U)\partial_z V \\ &= U\nabla^2 V + (\nabla U) \cdot (\nabla V). \square \end{aligned}$$

23.2 1st 3-Space Green Identity

$U(x, y, z)$, $V(x, y, z)$ are *Hyper-real differentiable* in a Volume domain D bounded by a closed Surface ∂D . Then,

$$\iiint_D [U\nabla^2 V + \nabla U \cdot \nabla V] dx dy dz = \iint_{S=\partial D} U \frac{\partial V}{\partial n} dS$$

Proof:

By the 3-Space Divergence Theorem, 22.1,

$$\begin{aligned}
 & \iiint_D \nabla \cdot \underbrace{\begin{bmatrix} U \partial_x V \\ U \partial_y V \\ U \partial_z V \end{bmatrix}}_{\nabla \cdot (U[\nabla V])} dx dy dz = \\
 & = \oint_{S=\partial D} U(\partial_x V) \vec{1}_x \cdot \vec{1}_n dy dz + U(\partial_y V) \vec{1}_y \cdot \vec{1}_n dx dz + U(\partial_z V) \vec{1}_z \cdot \vec{1}_n dx dy \\
 & = \iint_{S=\partial D} U \frac{\partial V}{\partial n} dS
 \end{aligned}$$

By 23.1,

$$\nabla \cdot (U[\nabla V]) = U \nabla^2 V + (\nabla U) \cdot (\nabla V)$$

Therefore,

$$\begin{aligned}
 & \iiint_D [U \nabla^2 V + \nabla U \cdot \nabla V] dx dy dz = \\
 & = \oint_{S=\partial D} U(\partial_x V) \vec{1}_x \cdot \vec{1}_n dy dz + U(\partial_y V) \vec{1}_y \cdot \vec{1}_n dx dz + U(\partial_z V) \vec{1}_z \cdot \vec{1}_n dx dy \\
 & = \iint_{S=\partial D} U \frac{\partial V}{\partial n} dS. \square
 \end{aligned}$$

23.3 2nd 3-Space Green Identity

$U(x, y, z)$, $V(x, y, z)$ are Hyper-real differentiable in a

Volume domain D bounded by a closed Surface ∂D . Then,

$$\iiint_D [U\nabla^2 V - V\nabla^2 U] dx dy dz = \iint_{S=\partial D} [U \frac{\partial V}{\partial n} - \frac{\partial U}{\partial n} V] dS$$

Proof: By 22.2,

$$\iiint_D [\nabla V \cdot \nabla U] dx dy dz = -\iiint_D [V\nabla^2 U] dx dy dz + \iint_{S=\partial D} V \frac{\partial U}{\partial n} dS,$$

$$\iiint_D [\nabla U \cdot \nabla V] dx dy dz = -\iiint_D [U\nabla^2 V] dx dy dz + \iint_{S=\partial D} U \frac{\partial V}{\partial n} dS.$$

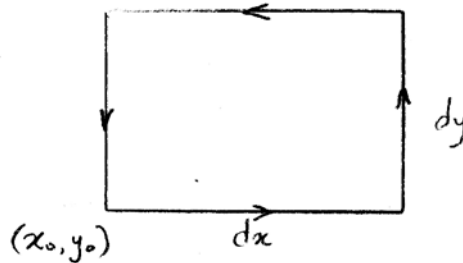
Since $\nabla V \cdot \nabla U = \nabla U \cdot \nabla V$,

$$\iiint_D [U\nabla^2 V - V\nabla^2 U] dx dy dz = \iint_{S=\partial D} [U \frac{\partial V}{\partial n} - \frac{\partial U}{\partial n} V] dS. \square$$

24.

Plane Curl

Let $P(x, y)$, and $Q(x, y)$, be hyper-real differentiable functions, defined on a rectangle with vertex at (x_0, y_0) and sides dx, dy .



24.1 The Area Curl

The **circulation** of $\begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix}$ along the rectangle is

$$\oint_{\text{rectangle}} P(x, y) \vec{i}_x \cdot \vec{i}_l dl + Q(x, y) \vec{i}_y \cdot \vec{i}_l dl.$$

We define the **area density** of that circulation multiplied by

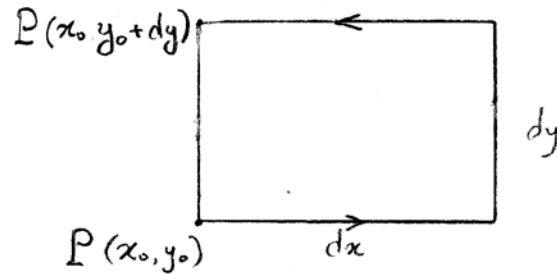
\vec{i}_z as the Curl of $\begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix}$

$$\underbrace{\text{Curl}}_{\nabla \times} \begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix} = \vec{i}_z \frac{1}{dxdy} \oint_{\text{rectangle}} P(x, y) \vec{i}_x \cdot \vec{i}_l dl + Q(x, y) \vec{i}_y \cdot \vec{i}_l dl$$

$$\mathbf{24.2} \quad \underbrace{\text{Curl}}_{\nabla \times} \begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix} \Big|_{x_0, y_0} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P(x, y) & Q(x, y) \end{bmatrix} \Big|_{x_0, y_0} \vec{i}_z$$

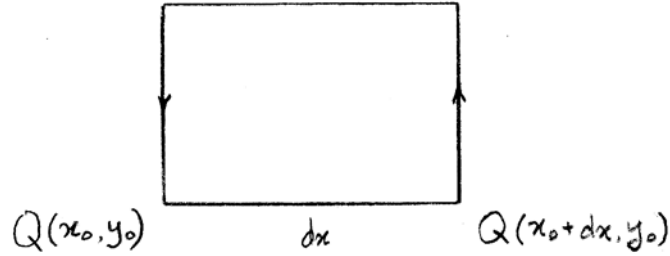
Proof:

Since $P(x, y) \vec{i}_x \cdot \vec{i}_l dl$ is nonzero when \vec{i}_l is along the x axis, the circulation path starts at (x_0, y_0) , and ends at $(x_0, y_0 + dy)$



$$\begin{aligned} \oint_{\text{rectangle}} P(x, y) \vec{i}_x \cdot \vec{i}_l dl &= \int_{(x_0, y_0)}^{(x_0, y_0 + dy)} P(x, y) \vec{i}_x \cdot \vec{i}_l dl \\ &= P(x_0, y_0) \vec{i}_x \cdot \vec{i}_x dx + P(x_0, y_0 + y) \vec{i}_x \cdot (-\vec{i}_x) dx \\ &= \{ P(x_0, y_0) - P(x_0, y_0 + y) \} dx \\ &= \left\{ -\frac{\partial P}{\partial y} \Big|_{x_0, y_0} dy \right\} dx . \end{aligned}$$

Since $Q(x, y)\vec{1}_y \cdot \vec{1}_l dl$ is nonzero when $\vec{1}_l$ is along the y axis, the circulation path starts at $(x_0 + dx, y_0)$, and ends at (x_0, y_0)



$$\begin{aligned}
 \oint_{\text{rectangle}} Q(x, y)\vec{1}_y \cdot \vec{1}_l dl &= \int_{(x_0 + dx, y_0)}^{(x_0, y_0)} Q(x, y)\vec{1}_x \cdot \vec{1}_l dl \\
 &= Q(x_0 + dx, y_0)\vec{1}_y \cdot \vec{1}_y dy + P(x_0, y_0)\vec{1}_y \cdot (-\vec{1}_y)dy \\
 &= \{Q(x_0 + dx, y_0) - Q(x_0, y_0)\} dy. \\
 &= \left[\frac{\partial Q}{\partial x} \Big|_{x_0, y_0} dx \right] dy.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \underbrace{\text{Curl}}_{\nabla \times} \begin{bmatrix} P(x, y) \\ Q(x, y) \end{bmatrix} &= \vec{1}_z \frac{1}{dxdy} \oint_{\text{rectangle}} P(x, y)\vec{1}_x \cdot \vec{1}_l dl + Q(x, y)\vec{1}_y \cdot \vec{1}_l dl \\
 &= \vec{1}_z \frac{1}{dxdy} \left[-\frac{\partial P}{\partial y} \Big|_{x_0, y_0} + \frac{\partial Q}{\partial x} \Big|_{x_0, y_0} \right] dxdy \\
 &= (\partial_x Q - \partial_y P)\vec{1}_z. \square
 \end{aligned}$$

25.

Plane Curl Theorem

Let $P(x, y)$, and $Q(x, y)$, be hyper-real differentiable functions, defined on a plane domain S , enclosed in the loop ∂S .

25.1 Green's Curl Theorem

$$\oint_{\partial S} P(x, y) \vec{1}_x \cdot \vec{1}_l dl + Q(x, y) \vec{1}_y \cdot \vec{1}_l dl = \iint_S \left| \begin{array}{cc} \partial_x & \partial_y \\ P(x, y) & Q(x, y) \end{array} \right| dx dy$$

Proof: The sum of the $\{\partial_x Q - \partial_y P\} dx dy$ over the infinitesimal rectangles enclosed in a plane area S , equals the sum of the circulations

$$\int_{\text{rectangle}} P(x, y) \vec{1}_x \cdot \vec{1}_l dl + Q(x, y) \vec{1}_y \cdot \vec{1}_l dl$$

over the sides of the infinitesimal rectangles enclosed in the area S ,

$$\sum_{y=b_1}^{y=b_2} \sum_{x=a_1}^{x=a_2} \{\partial_x Q - \partial_y P\} dx dy = \sum_{y=b_1}^{y=b_2} P(x, y) \vec{1}_x \cdot \vec{1}_l dl + \sum_{x=a_1}^{x=a_2} Q(x, y) \vec{1}_y \cdot \vec{1}_l dl$$

The sum over the areas is

$$\iint_S \{\partial_x Q(x, y) - \partial_y P(x, y)\} dx dy.$$

The Interior path integrals of

$$\int_{\text{rectangle}} P(x, y) \vec{1}_x \cdot \vec{1}_l dl, \quad \text{and} \quad \int_{\text{rectangle}} Q(x, y) \vec{1}_y \cdot \vec{1}_l dl,$$

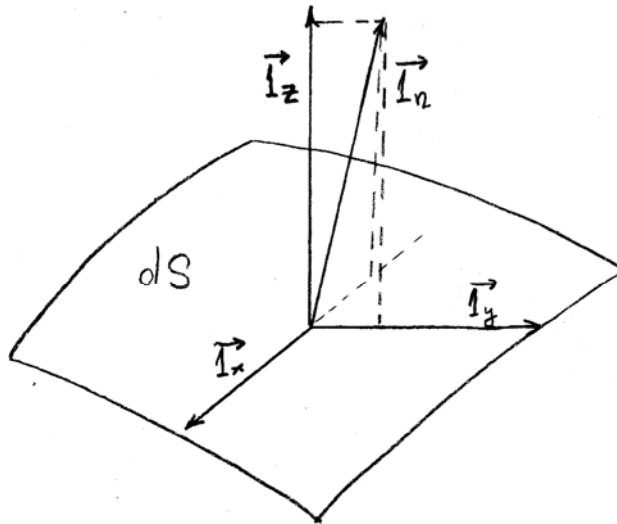
appear in pairs of opposite signs and cancel, leaving the
Integral over the boundary line,

$$\oint_{\partial S} P(x, y) \vec{1}_x \cdot \vec{1}_l dl + Q(x, y) \vec{1}_y \cdot \vec{1}_l dl = \iint_S \{ \partial_x Q - \partial_y P \} dx dy. \square$$

26.

3-Space Curl

Let $P(x, y, z)$, $Q(x, y, z)$, and $R(x, y, z)$ be hyper-real differentiable functions, defined on an infinitesimal area element dS , and its projections on the xy plane, yz plane, and zx plane.



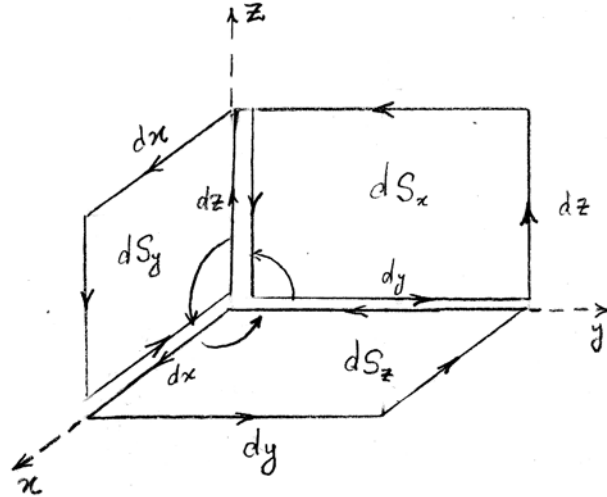
Those projections are

$$dS_x \text{ with area } \vec{i}_x \cdot \vec{i}_n dS = dydz ,$$

$$dS_y \text{ with area } \vec{i}_y \cdot \vec{i}_n dS = dzdx ,$$

$$dS_z \text{ with area } \vec{i}_z \cdot \vec{i}_n dS = dxdy .$$

The projections are walls of a box with vertex at (x_0, y_0, z_0) and sides dx , dy , and dz .



Given positive orientation of a right hand system,

$$\begin{aligned} \nabla \times \begin{bmatrix} P(x, y, z) \\ Q(x, y, z) \end{bmatrix} &= \vec{i}_z \frac{1}{dxdy} \oint_{\partial(dS_z)} P(x, y, z) \vec{i}_x \cdot \vec{i}_l dl + Q(x, y, z) \vec{i}_y \cdot \vec{i}_l dl, \\ &= \begin{vmatrix} \partial_x & \partial_y \\ P(x, y, z) & Q(x, y, z) \end{vmatrix}_{x_0, y_0, z_0} \vec{i}_z \end{aligned}$$

$$\begin{aligned} \nabla \times \begin{bmatrix} Q(x, y, z) \\ R(x, y, z) \end{bmatrix} &= \vec{i}_x \frac{1}{dydz} \oint_{\partial(dS_x)} Q(x, y, z) \vec{i}_y \cdot \vec{i}_l dl + R(x, y, z) \vec{i}_z \cdot \vec{i}_l dl, \\ &= \begin{vmatrix} \partial_y & \partial_z \\ Q(x, y, z) & R(x, y, z) \end{vmatrix}_{x_0, y_0, z_0} \vec{i}_x \end{aligned}$$

$$\begin{aligned} \nabla \times \begin{bmatrix} R(x, y, z) \\ P(x, y, z) \end{bmatrix} &= \vec{1}_y \frac{1}{dzdx} \oint_{\partial(ds_y)} R(x, y, z) \vec{1}_z \cdot \vec{1}_l dl + P(x, y, z) \vec{1}_x \cdot \vec{1}_l dl, \\ &= \begin{vmatrix} \partial_z & \partial_x \\ R(x, y, z) & P(x, y, z) \end{vmatrix}_{x_0, y_0, z_0} \vec{1}_y. \end{aligned}$$

26.1 The 3-space Curl Definition

The 3-space Curl is the sum of the three area curls. That is,

$$\begin{aligned} \nabla \times \begin{bmatrix} P(x, y, z) \\ Q(x, y, z) \\ R(x, y, z) \end{bmatrix} &= \underbrace{\nabla \times \begin{bmatrix} Q \\ R \end{bmatrix}}_{\begin{vmatrix} \partial_y & \partial_z \\ Q & R \end{vmatrix} \vec{1}_x} + \underbrace{\nabla \times \begin{bmatrix} R \\ P \end{bmatrix}}_{\begin{vmatrix} \partial_z & \partial_x \\ R & P \end{vmatrix} \vec{1}_y} + \underbrace{\nabla \times \begin{bmatrix} P \\ Q \end{bmatrix}}_{\begin{vmatrix} \partial_x & \partial_y \\ P & Q \end{vmatrix} \vec{1}_z} \\ &= \left(\begin{vmatrix} \partial_y & \partial_z \\ Q & R \end{vmatrix}, \begin{vmatrix} \partial_z & \partial_x \\ R & P \end{vmatrix}, \begin{vmatrix} \partial_x & \partial_y \\ P & Q \end{vmatrix} \right) \end{aligned}$$

The pairs (y, z) , (z, x) , and (x, y) are a cyclic permutation of (x, y, z) .

The pairs (Q, R) , (R, P) , and (P, Q) are a cyclic permutation of (P, Q, R) .

The infinitesimals are anti-commutative. That is,

$$dydz = -dzdy, \quad dzdx = -dxdz, \quad dxdy = -dydx.$$

Hence, $dxdx = 0$, $dydy = 0$, $dzdz = 0$.

27.

3-Space Curl Theorem

Let $P(x, y, z)$, $Q(x, y, z)$, and $R(x, y, z)$ be hyper-real differentiable functions, defined on an surface S , with closed boundary ∂S .

27.1 Stokes' Curl Theorem

$$\begin{aligned} & \oint_{\partial S} P(x, y, z) \vec{1}_x \cdot \vec{1}_l dl + Q(x, y, z) \vec{1}_y \cdot \vec{1}_l dl + R(x, y, z) \vec{1}_z \cdot \vec{1}_l dl = \\ & = \iint_S (\partial_y R - \partial_z Q) dydz + (\partial_z P - \partial_x R) dzdx + (\partial_x Q - \partial_y P) dxdy \end{aligned}$$

Proof: The sum of the $\{\partial_y R - \partial_z Q\} dydz$ over the areas of the infinitesimal rectangles enclosed in a plane area, S_x projected by S , perpendicular to $\vec{1}_x$, equals the sum of the circulations

$$\int_{\text{rectangle}} Q(x, y, z) \vec{1}_y \cdot \vec{1}_l dl + R(x, y, z) \vec{1}_z \cdot \vec{1}_l dl$$

along the sides of the infinitesimal rectangles enclosed in the area S_x ,

$$\sum_{z=c_1}^{z=c_2} \sum_{y=b_1}^{y=b_2} \{ \partial_y R - \partial_z Q \} dydz = \sum_{y=b_1}^{y=b_2} Q(x, y, z) \vec{1}_y \cdot \vec{1}_l dl + \sum_{z=c_1}^{z=c_2} R(x, y, z) \vec{1}_z \cdot \vec{1}_l dl$$

The sum over the areas is

$$\oiint_{S_x} \{ \partial_y R(x, y, z) - \partial_z Q(x, y, z) \} dydz .$$

The Interior path integrals of

$$\int_{\text{rectangle}} Q(x, y, z) \vec{1}_y \cdot \vec{1}_l dl, \quad \text{and} \quad \int_{\text{rectangle}} R(x, y, z) \vec{1}_z \cdot \vec{1}_l dl,$$

appear in pairs of opposite signs and cancel, leaving the
Integral over the boundary line,

$$\oint_{\partial S_x} Q(x, y, z) \vec{1}_y \cdot \vec{1}_l dl + R(x, y, z) \vec{1}_z \cdot \vec{1}_l dl = \oiint_{S_x} \{ \partial_y R - \partial_z Q \} dydz .$$

Similarly,

$$\oint_{\partial S_y} R(x, y, z) \vec{1}_z \cdot \vec{1}_l dl + P(x, y, z) \vec{1}_x \cdot \vec{1}_l dl = \oiint_{S_y} \{ \partial_z P - \partial_x R \} dzdx ,$$

and

$$\oint_{\partial S_z} P(x, y, z) \vec{1}_x \cdot \vec{1}_l dl + Q(x, y, z) \vec{1}_y \cdot \vec{1}_l dl = \oiint_{S_z} \{ \partial_x Q - \partial_y P \} dx dy .$$

The sum of the left hand sides of the equalities is written as

$$\oint_{\partial S} P(x, y, z) \vec{1}_x \cdot \vec{1}_l dl + Q(x, y, z) \vec{1}_y \cdot \vec{1}_l dl + R(x, y, z) \vec{1}_z \cdot \vec{1}_l dl,$$

or as

$$\oint_{\partial S} P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz .$$

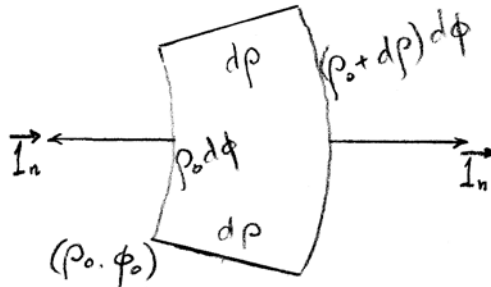
The sum of the right hand sides is written as

$$\oiint_S (\partial_y R - \partial_z Q)dydz + (\partial_z P - \partial_x R)dzdx + (\partial_x Q - \partial_y P)dxdy . \square$$

28.

Polar Gradient

Let $U(\rho, \phi)$ be a hyper-real differentiable Potential function, defined on the polar rectangle with vertex at (ρ_0, ϕ_0) and sides $\rho_0 d\phi$, $d\rho$, and $(\rho_0 + d\rho)d\phi$.



28.1 ρ -Component of Polar Gradient

The potential drop at $\phi = \phi_0$, from $\rho = \rho_0 + d\rho$, to $\rho = \rho_0$ is

$$U(\rho_0 + d\rho, \phi_0) - U(\rho_0, \phi_0).$$

We define the **linear density** of that potential drop as the ρ -component of the Gradient of $U(\rho, \phi)$ at (ρ_0, ϕ_0) ,

$$(\nabla U)_\rho \Big|_{\rho_0, \phi_0} = \frac{1}{d\rho} [U(\rho_0 + d\rho, \phi_0) - U(\rho_0, \phi_0)] = \frac{\partial U}{\partial \rho} \Big|_{\rho_0, \phi_0}.$$

28.2 ϕ -Component of Polar Gradient

The potential drop at $\rho = \rho_0$, from $\phi = \phi_0 + d\phi$, to $\phi = \phi_0$ is

$$U(\rho_0, \phi_0 + d\phi) - U(\rho_0, \phi_0).$$

We define the **linear density** of that potential drop as the ϕ -component of the Gradient of $U(\rho, \phi)$ at (ρ_0, ϕ_0) ,

$$(\nabla U)_\phi \Big|_{\rho_0, \phi_0} = \frac{1}{\rho_0 d\phi} [U(\rho_0, \phi_0 + d\phi) - U(\rho_0, \phi_0)] = \frac{1}{\rho_0} \frac{\partial U}{\partial \phi} \Big|_{\rho_0, \phi_0}.$$

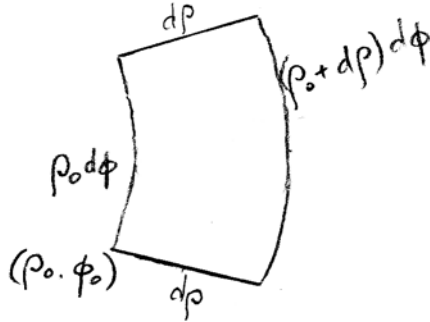
28.3

$$\nabla U \Big|_{\rho_0, \phi_0} = \begin{bmatrix} \partial_\rho U \\ \frac{1}{\rho} \partial_\phi U \end{bmatrix} \Big|_{\rho_0, \phi_0}$$

29.

Polar Divergence

Let $U(\rho, \phi)$, and $V(\rho, \phi)$ be hyper-real differentiable functions, defined on the polar rectangle with vertex at (ρ_0, ϕ_0) and sides $\rho_0 d\phi$, $d\rho$, and $(\rho_0 + d\rho)d\phi$.



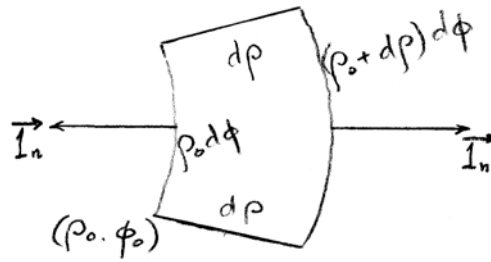
29.1 Polar Divergence due to $U(\rho, \phi)$

The flux of $U(\rho, \phi)$ through the opposite rectangle sides

$$\rho = \rho_0, \text{ with length } \rho_0 d\phi$$

and

$$\rho = \rho_0 + d\rho \text{ with length } (\rho_0 + d\rho)d\phi$$



is

$$\int_{\text{polar rectangle}} U(\rho, \phi) \vec{I}_\rho \cdot \vec{I}_n \rho d\phi.$$

We define the **area density** of that flux as the divergence part of $(U(\rho, \phi), V(\rho, \phi))$ due to $U(\rho, \phi)$

$$\left(\nabla \cdot \begin{bmatrix} U \\ V \end{bmatrix} \right)_U = \frac{1}{\rho_0 d\phi d\rho} \int_{\text{polar rectangle}} U(\rho, \phi) \vec{I}_\rho \cdot \vec{I}_n \rho d\phi$$

$$\mathbf{29.2} \quad \left(\nabla \cdot \begin{bmatrix} U \\ V \end{bmatrix} \right)_U \Big|_{\rho_0, \phi_0} = \frac{1}{\rho_0} \frac{\partial(\rho U)}{\partial \rho} \Big|_{\rho_0, \phi_0}$$

Proof:

The flux of $U(\rho, \phi)$ through the rectangle side $\rho = \rho_0$ is

$$U(\rho_0, \phi_0) \vec{I}_\rho \cdot (-\vec{I}_\rho) \rho_0 d\phi = -U(\rho_0, \phi_0) \rho_0 d\phi.$$

The flux of $U(\rho, \phi)$ through the rectangle side $\rho = \rho_0 + d\rho$ is

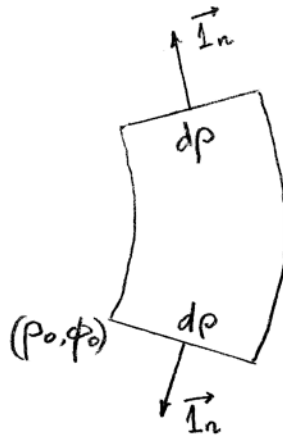
$$U(\rho_0 + d\rho, \phi_0) \vec{I}_\rho \cdot \vec{I}_\rho (\rho_0 + d\rho) d\phi = U(\rho_0 + d\rho, \phi_0) (\rho_0 + d\rho) d\phi.$$

$$\begin{aligned}
(\nabla \cdot \begin{bmatrix} U \\ V \end{bmatrix})_U &= \frac{1}{\rho_0 d\phi d\rho} \int_{\text{polar rectangle}} U(\rho, \phi) \vec{1}_\rho \cdot \vec{1}_n \rho d\phi \\
&= \frac{1}{\rho_0 d\phi d\rho} \{ U(\rho_0 + d\rho, \phi_0) (\rho_0 + d\rho) d\phi - U(\rho_0, \phi_0) \rho_0 d\phi \} \\
&= \frac{1}{\rho_0} \left. \frac{\partial(\rho U)}{\partial \rho} \right|_{\rho_0, \phi_0}
\end{aligned}$$

29.3 Polar Divergence due to $V(\rho, \phi)$

The flux of $V(\rho, \phi)$ through the opposite rectangle sides

$$\phi = \phi_0, \quad \text{and} \quad \phi = \phi_0 + d\phi$$



is

$$\int_{\text{polar rectangle}} V(\rho, \phi) \vec{1}_\phi \cdot \vec{1}_n d\rho.$$

We define the **area density** of that flux as the part of the of the divergence due to $V(\rho, \phi)$

$$(\nabla \cdot \begin{bmatrix} U \\ V \end{bmatrix})_V = \frac{1}{\rho_0 d\phi d\rho} \int_{\text{polar rectangle}} V(\rho, \phi) \vec{1}_\phi \cdot \vec{1}_n d\rho.$$

$$\mathbf{29.4} \quad (\nabla \cdot \begin{bmatrix} U \\ V \end{bmatrix})_V = \frac{1}{\rho_0} \frac{\partial V}{\partial \phi} \Big|_{\rho_0, \phi_0}$$

The flux of $V(\rho, \phi)$ through the polar rectangle side $\phi = \phi_0$ is

$$V(\rho_0, \phi_0) \vec{1}_\phi \cdot (-\vec{1}_\phi) d\rho = -V(\rho_0, \phi_0) d\rho.$$

The flux of $V(\rho, \phi)$ through the rectangle side $\phi = \phi_0 + d\phi$ is

$$V(\rho_0, \phi_0 + d\phi) \vec{1}_\phi \cdot \vec{1}_\phi d\rho = V(\rho_0, \phi_0 + d\phi) d\rho.$$

$$\begin{aligned} (\nabla \cdot \begin{bmatrix} U \\ V \end{bmatrix})_V &= \frac{1}{\rho_0 d\phi d\rho} \int_{\text{polar rectangle}} V(\rho, \phi) \vec{1}_\phi \cdot \vec{1}_n d\rho \\ &= \frac{1}{\rho_0 d\phi d\rho} \{V(\rho_0, \phi_0 + d\phi) d\rho - V(\rho_0, \phi_0) d\rho\} \\ &= \frac{1}{\rho_0} \frac{\partial V}{\partial \phi} \Big|_{\rho_0, \phi_0}. \end{aligned}$$

$$\mathbf{29.5} \quad \nabla \cdot \begin{bmatrix} U(\rho, \phi) \\ V(\rho, \phi) \end{bmatrix} \Big|_{\rho_0, \phi_0} = \frac{1}{\rho_0} [\partial_\rho(\rho U) + \partial_\phi V]_{\rho_0, \phi_0}$$

30.

Polar Green's Identities

In Polar coordinates the Plane Green identities of 20 are

30.1 1st Polar Green Identity

$U(\rho, \phi)$, and $V(\rho, \phi)$ are Hyper-real differentiable in a plane domain D bounded by a closed curve ∂D

$$\iint_D [U\nabla^2 V + \nabla U \cdot \nabla V] \rho d\phi d\rho = \oint_{\partial D} U(\nabla V) \cdot \vec{1}_n dl$$

30.2 2nd Polar Green Identity

$U(\rho, \phi)$, and $V(\rho, \phi)$ are Hyper-real differentiable in a plane domain D bounded by a closed curve ∂D

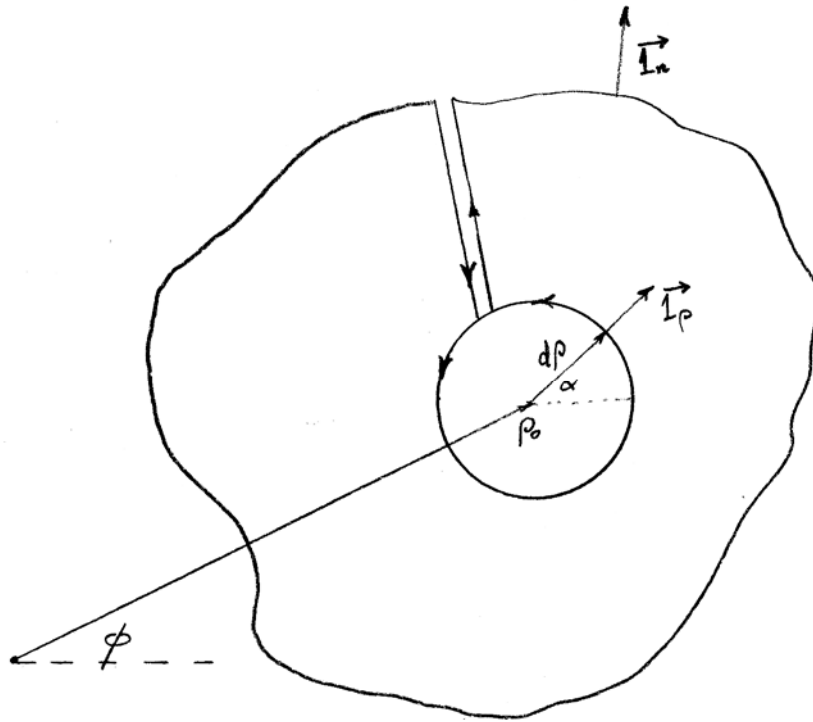
$$\iint_D [U\nabla^2 V - V\nabla^2 U] \rho d\phi d\rho = \oint_{\partial D} [U(\nabla V) \cdot \vec{1}_n - V(\nabla U) \cdot \vec{1}_n] dl$$

31.

Representing Harmonic $V(\rho, \phi)$

Let $V(\rho, \phi)$ be Hyper-real differentiable Harmonic function in a plane domain D , that contains a loop γ .

Use an integration path along γ , that includes an infinitesimal circle of radius $d\rho$ centered at ρ_0 .



$$\mathbf{31.1} \quad V(\rho_0, \phi_0) = \frac{1}{2\pi} \oint_{\gamma} \left\{ V(\rho, \phi) \frac{1}{\rho - \rho_0} \vec{l}_\rho \cdot \vec{l}_n - \log(\rho - \rho_0) \frac{\partial V}{\partial n} \right\} dl$$

Proof: In the 2nd Polar Green Identity, 30.2, put

$$U(\rho, \phi) = \log|\vec{\rho} - \vec{\rho}_0|.$$

Then,

$$\begin{aligned} \nabla \cdot \nabla U(\rho, \phi) &= \nabla \cdot \nabla \log|\vec{\rho} - \vec{\rho}_0| \\ &= \nabla \cdot (\partial_\rho \log|\vec{\rho} - \vec{\rho}_0|) \vec{1}_\rho \\ &= \nabla \cdot \left(\frac{1}{\rho - \rho_0} \vec{1}_\rho \right) \\ &= \frac{1}{\rho - \rho_0} \frac{\partial}{\partial(\rho - \rho_0)} \left((\rho - \rho_0) \frac{1}{\rho - \rho_0} \right) \\ &= 0. \end{aligned}$$

In the domain enclosed between γ and the infinitesimal disk, $\log|\vec{\rho} - \vec{\rho}_0|$ is at most an infinite hyper-real, and since

$$\nabla^2 V(\rho, \phi) = 0,$$

$$\log|\vec{\rho} - \vec{\rho}_0| \nabla^2 V = 0,$$

and

$$\oint\!\!\!\int_{\text{Area between } \gamma \text{ and disk}} [\log|\vec{\rho} - \vec{\rho}_0| \underbrace{\nabla^2 V}_0 - V \underbrace{\nabla^2 \log|\vec{\rho} - \vec{\rho}_0|}_0] \rho d\phi d\rho = 0.$$

The line integrals between the γ loop, and the circle are of opposite signs, and cancel each other. Therefore, by the 2nd Polar Green Identity, this equals to the integrals over the loops. That is,

$$\begin{aligned}
0 &= \oint_{\text{circle}} [\log|\vec{\rho} - \vec{\rho}_0| \nabla V \cdot \vec{\mathbf{1}}_n - V(\rho, \phi)(\nabla \log|\vec{\rho} - \vec{\rho}_0|) \cdot \vec{\mathbf{1}}_n] dl \\
&\quad + \oint_{\gamma} [\log|\vec{\rho} - \vec{\rho}_0| \nabla V \cdot \vec{\mathbf{1}}_n - V(\nabla \log|\vec{\rho} - \vec{\rho}_0|) \cdot \vec{\mathbf{1}}_n] dl
\end{aligned}$$

The first integral along the circle is infinitesimal because

$$\begin{aligned}
\rho - \rho_0 &= d\rho, \\
dl &= d\rho d\alpha, \\
\vec{\mathbf{1}}_n &= \vec{\mathbf{1}}_\rho,
\end{aligned}$$

and,

$$\oint_{|\vec{\rho}-\vec{\rho}_0|=d\rho} \underbrace{\log|\vec{\rho} - \vec{\rho}_0|}_{d\rho} \underbrace{\nabla V \cdot \vec{\mathbf{1}}_n}_{\vec{\mathbf{1}}_\rho} \underbrace{dl}_{d\rho d\alpha} = \oint_{|\vec{\rho}-\vec{\rho}_0|=d\rho} (d\rho) \log(d\rho) \frac{\partial V}{\partial \rho} d\alpha.$$

By Bernoulli-L'Hospital rule, for an infinitesimal ε ,

$$\varepsilon \log \varepsilon = \frac{\log \varepsilon}{\frac{1}{\varepsilon}} = \frac{D_\varepsilon(\log \varepsilon)}{D_\varepsilon(\frac{1}{\varepsilon})} = \frac{\frac{1}{\varepsilon}}{-\frac{1}{\varepsilon^2}} = -\varepsilon.$$

That is,

$$(d\rho) \log(d\rho) = \text{infinitesimal}.$$

Hence,

$$\oint_{\text{circle}} \left(\frac{\partial V}{\partial \rho}\right) (d\rho) \log(d\rho) d\alpha = (\text{infinitesimal}) \times \int_{\alpha=0}^{\alpha=2\pi} d\alpha = \text{infinitesimal}.$$

The second integral along the circle is

$$- \oint_{|\vec{\rho}-\vec{\rho}_0|=d\rho} V(\rho, \phi) \underbrace{(\nabla \log|\vec{\rho} - \vec{\rho}_0|)}_{\frac{1}{|\vec{\rho}-\vec{\rho}_0|} = \frac{1}{d\rho}} \cdot \underbrace{\vec{\mathbf{1}}_n}_{\vec{\mathbf{1}}_\rho} \underbrace{dl}_{d\rho d\alpha} = -V(\rho_0, \phi_0) \underbrace{\int_{\alpha=0}^{\alpha=2\pi} d\alpha}_{2\pi}$$

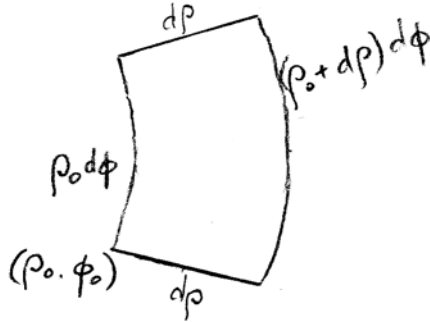
Therefore,

$$\begin{aligned}
 V(\rho_0, \phi_0) &= -\frac{1}{2\pi} \oint_{\gamma} \left\{ \log |\vec{\rho} - \vec{\rho}_0| \underbrace{\nabla V \cdot \vec{\mathbf{1}}_n}_{\frac{\partial V}{\partial n}} - V(\rho, \phi) \underbrace{(\nabla \log |\vec{\rho} - \vec{\rho}_0|)}_{\frac{1}{|\vec{\rho} - \vec{\rho}_0|} \vec{\mathbf{1}}_\rho} \cdot \vec{\mathbf{1}}_n \right\} dl \\
 &= \frac{1}{2\pi} \oint_{\gamma} \left\{ V(\rho, \phi) \frac{1}{|\vec{\rho} - \vec{\rho}_0|} \vec{\mathbf{1}}_\rho \cdot \vec{\mathbf{1}}_n - \log |\vec{\rho} - \vec{\rho}_0| \frac{\partial V}{\partial n} \right\} dl
 \end{aligned}$$

32.

Polar Curl

Let $U(\rho, \phi)$, and $V(\rho, \phi)$ be hyper-real differentiable functions, defined on the polar rectangle with vertex at (ρ_0, ϕ_0) and sides $\rho_0 d\phi$, $d\rho$, and $(\rho_0 + d\rho)d\phi$.



32.1 The Polar Curl

The **circulation** of $\begin{bmatrix} U(\rho, \phi) \\ V(\rho, \phi) \end{bmatrix}$ along the polar rectangle is

$$\oint_{\text{polar rectangle}} U(\rho, \phi) \vec{1}_\rho \cdot \vec{1}_l dl + V(\rho, \phi) \vec{1}_\phi \cdot \vec{1}_l dl.$$

We define the **area density** of that circulation as the z -

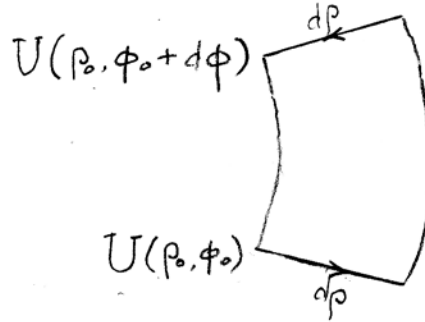
Component of the Curl of $\begin{bmatrix} U(\rho, \phi) \\ V(\rho, \phi) \end{bmatrix}$

$$\nabla \times \begin{bmatrix} U(\rho, \phi) \\ V(\rho, \phi) \end{bmatrix} = \vec{\mathbf{i}}_z \frac{1}{\rho_0 d\phi d\rho} \oint_{\text{rectangle}} U(\rho, \phi) \vec{\mathbf{i}}_\rho \cdot \vec{\mathbf{i}}_\ell dl + V(\rho, \phi) \vec{\mathbf{i}}_\phi \cdot \vec{\mathbf{i}}_\ell dl$$

32.2
$$\nabla \times \begin{bmatrix} U(\rho, \phi) \\ V(\rho, \phi) \end{bmatrix}_{\rho_0, \phi_0} = \frac{1}{\rho} \begin{vmatrix} \partial_\rho & \partial_\phi \\ U(\rho, \phi) & \rho V(\rho, \phi) \end{vmatrix}_{\rho_0, \phi_0} \vec{\mathbf{i}}_z$$

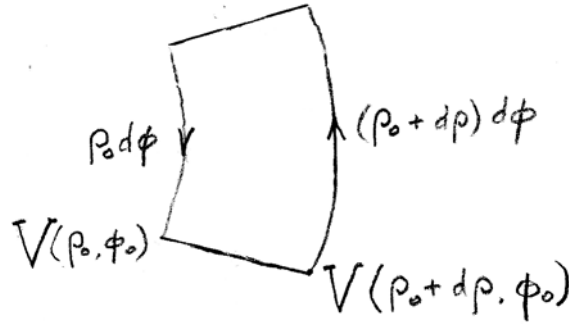
Proof:

Since $U(\rho, \phi) \vec{\mathbf{i}}_\rho \cdot \vec{\mathbf{i}}_\ell dl$ is nonzero when $\vec{\mathbf{i}}_\ell$ is along ρ , the circulation path starts at (ρ_0, ϕ_0) , and ends at $(\rho_0, \phi_0 + d\phi)$



$$\begin{aligned} \oint_{\text{rectangle}} U(\rho, \phi) \vec{\mathbf{i}}_\rho \cdot \vec{\mathbf{i}}_\ell dl &= \int_{(\rho_0, \phi_0)}^{(\rho_0, \phi_0 + d\phi)} U(\rho, \phi) \vec{\mathbf{i}}_\rho \cdot \vec{\mathbf{i}}_\ell dl \\ &= U(\rho_0, \phi_0) \vec{\mathbf{i}}_\rho \cdot \vec{\mathbf{i}}_\rho d\rho + U(\rho_0, \phi_0 + d\phi) \vec{\mathbf{i}}_\rho \cdot (-\vec{\mathbf{i}}_\rho) d\rho \\ &= \{U(\rho_0, \phi_0) - U(\rho_0, \phi_0 + d\phi)\} d\rho \\ &= \{-\partial_\phi U\}_{\rho_0, \phi_0} d\phi d\rho. \end{aligned}$$

Since $V(\rho, \phi) \vec{1}_\phi \cdot \vec{1}_l dl$ is nonzero when $\vec{1}_l$ is along ϕ , the circulation path starts at $(\rho_0 + d\rho, \phi_0)$, and ends at (ρ_0, ϕ_0)



$$\begin{aligned}
 \oint_{\text{rectangle}} V(\rho, \phi) \vec{1}_\phi \cdot \vec{1}_l dl &= \int_{(\rho_0 + d\rho, \phi_0)}^{(\rho_0, \phi_0)} V(\rho, \phi) \vec{1}_\phi \cdot \vec{1}_l dl \\
 &= V(\rho_0 + d\rho, \phi_0) \vec{1}_\phi \cdot \vec{1}_\phi (\rho_0 + d\rho) d\phi + V(\rho_0, \phi_0) \vec{1}_\phi \cdot (-\vec{1}_\phi) \rho_0 d\phi \\
 &= \{V(\rho_0 + d\rho, \phi_0)(\rho_0 + d\rho) - V(\rho_0, \phi_0)\rho_0\} d\phi \\
 &= \left\{ \partial_\rho (\rho U) \right\}_{\rho_0, \phi_0} d\rho d\phi.
 \end{aligned}$$

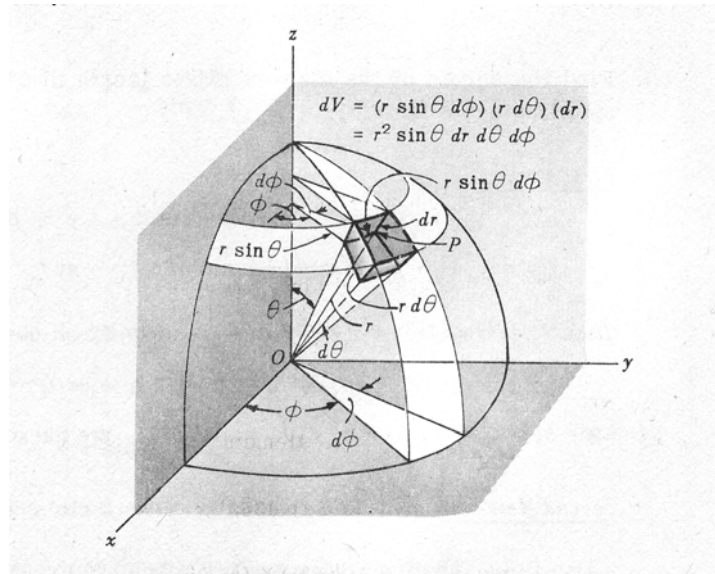
Therefore,

$$\begin{aligned}
 \nabla \times \begin{bmatrix} U(\rho, \phi) \\ V(\rho, \phi) \end{bmatrix} &= \vec{1}_z \frac{1}{\rho_0 d\phi d\rho} \oint_{\text{rectangle}} U(\rho, \phi) \vec{1}_\rho \cdot \vec{1}_l dl + V(\rho, \phi) \vec{1}_\phi \cdot \vec{1}_l dl \\
 &= \vec{1}_z \frac{1}{\rho_0 d\rho d\phi} \left\{ -\partial_\phi U + \partial_\rho (\rho V) \right\}_{\rho_0, \phi_0} d\rho d\phi \\
 &= \frac{1}{\rho_0} \left\{ \partial_\rho (\rho V) - \partial_\phi U \right\} \vec{1}_z. \square
 \end{aligned}$$

33.

Spherical Gradient

Let $U(r, \theta, \phi)$ be a hyper-real differentiable Potential function, defined on an infinitesimal spherical box



The box has vertex at (r_0, θ_0, ϕ_0) ,

base with sides $r_0 d\theta$, $r_0 \sin \theta_0 d\phi$,

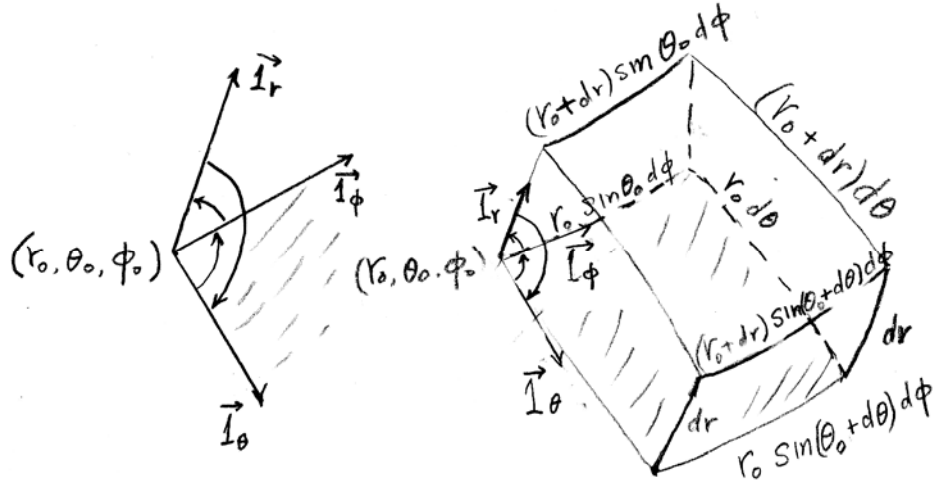
$r_0 d\theta$, $r_0 \sin(\theta_0 + d\theta) d\phi$,

height dr ,

and top with sides

$(r_0 + dr) \sin \theta_0 d\phi$, $(r_0 + dr) d\theta$,

$(r_0 + dr) \sin(\theta_0 + d\theta) d\phi$, $(r_0 + dr) d\theta$,



33.1 r -Component of Spherical Gradient

The potential drop at θ_0, ϕ_0 , from $r = r_0 + dr$, to $r = r_0$ is

$$U(r_0 + dr, \theta_0, \phi_0) - U(r_0, \theta_0, \phi_0).$$

We define the **linear density** of that potential drop as the r -component of the Gradient of $U(r, \theta, \phi)$ at (r_0, θ_0, ϕ_0) ,

$$(\nabla U)_r \Big|_{r_0, \theta_0, \phi_0} = \frac{1}{dr} [U(r_0 + dr, \theta_0, \phi_0) - U(r_0, \theta_0, \phi_0)] = \frac{\partial U}{\partial r} \Big|_{r_0, \theta_0, \phi_0}.$$

33.2 θ -Component of Spherical Gradient

The potential drop at r_0, ϕ_0 , from $\theta = \theta_0 + d\theta$, to $\theta = \theta_0$ is

$$U(r_0, \theta_0 + d\theta, \phi_0) - U(r_0, \theta_0, \phi_0).$$

We define the **linear density** of that potential drop as the θ -component of the Gradient of $U(r, \theta, \phi)$ at (r_0, θ_0, ϕ_0) ,

$$(\nabla U)_\theta \Big|_{r_0, \theta_0, \phi_0} = \frac{1}{r_0 d\theta} [U(r_0, \theta_0 + d\theta, \phi_0) - U(r_0, \theta_0, \phi_0)] = \frac{1}{r_0} \frac{\partial U}{\partial \theta} \Big|_{r_0, \theta_0, \phi_0} .$$

33.3 ϕ -Component of Spherical Gradient

The potential drop at r_0, θ_0 , from $\phi = \phi_0 + d\phi$, to $\phi = \phi_0$ is

$$U(r_0, \theta_0, \phi_0 + d\phi) - U(r_0, \theta_0, \phi_0).$$

We define the **linear density** of that potential drop as the ϕ -component of the Gradient of $U(r, \theta, \phi)$ at (r_0, θ_0, ϕ_0) ,

$$\begin{aligned} (\nabla U)_\phi \Big|_{r_0, \theta_0, \phi_0} &= \frac{1}{r_0 \sin \theta_0 d\phi} [U(r_0, \theta_0, \phi_0 + d\phi) - U(r_0, \theta_0, \phi_0)] \\ &= \frac{1}{r_0 \sin \theta_0} \frac{\partial U}{\partial \phi} \Big|_{r_0, \theta_0, \phi_0} . \end{aligned}$$

33.4

$$\nabla U \Big|_{r_0, \theta_0, \phi_0} = \begin{bmatrix} \frac{\partial U}{\partial r} \\ \frac{1}{r_0} \frac{\partial U}{\partial \theta} \\ \frac{1}{r_0 \sin \theta_0} \frac{\partial U}{\partial \phi} \end{bmatrix} \Big|_{r_0, \theta_0, \phi_0}$$

34.

Spherical Divergence

Let $U(r, \theta, \phi)$, $V(r, \theta, \phi)$, and $W(r, \theta, \phi)$ be hyper-real differentiable functions, defined on an infinitesimal spherical box with

vertex at (r_0, θ_0, ϕ_0) ,

base with sides $r_0 d\theta$, $r_0 \sin \theta_0 d\phi$,

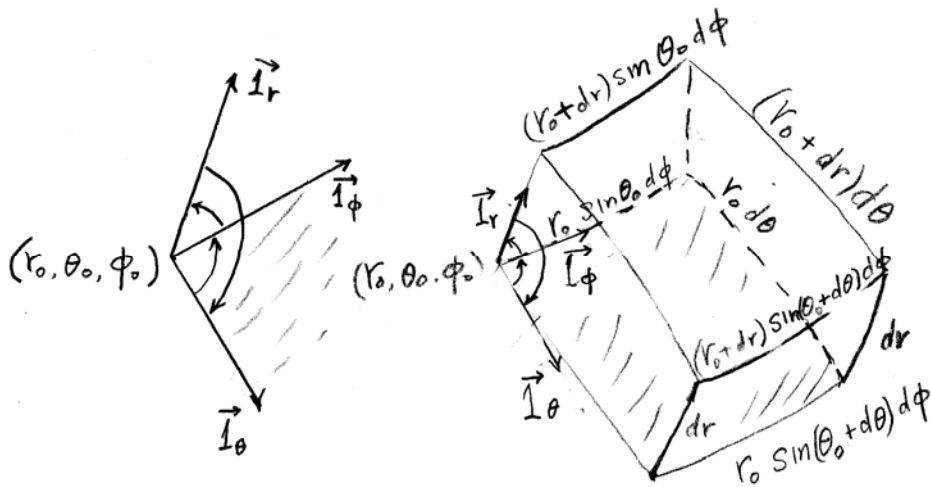
$r_0 d\theta$, $r_0 \sin(\theta_0 + d\theta) d\phi$,

height dr ,

and top with sides

$(r_0 + dr) \sin \theta_0 d\phi$, $(r_0 + dr) d\theta$,

$(r_0 + dr) \sin(\theta_0 + d\theta) d\phi$, $(r_0 + dr) d\theta$,



34.1 The Volume Divergence due to $U(r, \theta, \phi)$

The flux of $U(r, \theta, \phi)$ through the box base and top is

$$\iint_{\text{spherical box}} U(r, \theta, \phi) dS_r = \iint_{\text{spherical box}} U(r, \theta, \phi) \vec{\mathbf{1}}_r \cdot \vec{\mathbf{1}}_n (r d\theta)(r \sin \theta_0 d\phi).$$

We define the **volume density** of that flux as the part of the Divergence due to $U(r, \theta, \phi)$

$$\left(\nabla \cdot \begin{bmatrix} U \\ V \\ W \end{bmatrix} \right)_U = \frac{1}{(dr)(r_0 d\theta)(r_0 \sin \theta_0 d\phi)} \iint_{\text{spherical box}} U(r, \theta, \phi) \vec{\mathbf{1}}_r \cdot \vec{\mathbf{1}}_n (r_0 d\theta)(r_0 \sin \theta_0 d\phi)$$

$$\mathbf{34.2} \quad \left(\nabla \cdot \begin{bmatrix} U \\ V \\ W \end{bmatrix} \right)_U = \frac{1}{r^2} \frac{\partial(r^2 U)}{\partial r}$$

Proof:

The flux of $U(r, \theta, \phi)$ through the base is

$$U(r_0, \theta_0, \phi_0) \vec{\mathbf{1}}_r \cdot (-\vec{\mathbf{1}}_r)(r_0 d\theta)(r_0 \sin \theta_0 d\phi) = -U(r_0, \theta_0, \phi_0) r_0^2 \sin \theta_0 d\theta d\phi.$$

The flux of $U(r, \theta, \phi)$ through the top is

$$\begin{aligned} U(r_0 + dr, \theta_0, \phi_0) \vec{\mathbf{1}}_r \cdot \vec{\mathbf{1}}_r (r_0 + dr) d\theta (r_0 + dr) \sin \theta_0 d\phi = \\ = U(r_0 + dr, \theta_0, \phi_0) (r_0 + dr)^2 \sin \theta_0 d\theta d\phi. \end{aligned}$$

Hence, the part of the Divergence due to $U(r, \theta, \phi)$ is

$$\begin{aligned}
& \frac{1}{(dr)(r_0 d\theta)(r_0 \sin \theta_0 d\phi)} \iint_{\text{spherical box}} U(r, \theta, \phi) \vec{1}_r \cdot \vec{1}_n (r_0 d\theta)(r_0 \sin \theta_0 d\phi) = \\
& = \frac{1}{r_0^2 dr \sin \theta_0 d\theta d\phi} \left\{ U(r_0 + dr, \theta_0, \phi_0) (r_0 + dr)^2 \sin \theta_0 d\theta d\phi \right. \\
& \qquad \qquad \qquad \left. - U(r_0, \theta_0, \phi_0) r_0^2 \sin \theta_0 d\theta d\phi \right\} \\
& = \frac{1}{r_0^2 dr} \left\{ (r_0 + dr)^2 U(r_0 + dr, \theta_0, \phi_0) - r_0^2 U(r_0, \theta_0, \phi_0) \right\} \\
& = \frac{1}{r^2} \frac{\partial(r^2 U)}{\partial r} \Big|_{r_0, \theta_0, \phi_0} . \square
\end{aligned}$$

34.3 The Volume Divergence due to $V(r, \theta, \phi)$

The flux of $V(r, \theta, \phi)$ through the spherical box walls $\theta = \theta_0$,
and $\theta = \theta_0 + d\theta$ is

$$\iint_{\text{spherical box}} V(r, \theta, \phi) dS_\theta = \iint_{\text{spherical box}} V(r, \theta, \phi) \vec{1}_\theta \cdot \vec{1}_n (dr)(r_0 \sin \theta d\phi).$$

We define the **volume density** of that flux as the part of the
Divergence due to $V(r, \theta, \phi)$

$$(\nabla \cdot \begin{bmatrix} U \\ V \\ W \end{bmatrix})_V = \frac{1}{(dr)(r_0 d\theta)(r_0 \sin \theta_0 d\phi)} \iint_{\text{spherical box}} V(r, \theta, \phi) \vec{1}_\theta \cdot \vec{1}_n (dr)(r_0 \sin \theta d\phi)$$

$$\mathbf{34.4} \quad \left(\nabla \cdot \begin{bmatrix} U \\ V \\ W \end{bmatrix} \right)_V \Big|_{r_0, \theta_0, \phi_0} = \frac{1}{r \sin \theta} \frac{\partial(V \sin \theta)}{\partial \theta} \Big|_{r_0, \theta_0, \phi_0}$$

Proof:

The flux of $V(r, \theta, \phi)$ through the wall $\theta = \theta_0$ is

$$V(r_0, \theta_0, \phi_0) \vec{1}_\theta \cdot (-\vec{1}_\theta)(dr)(r_0 \sin \theta_0 d\phi) = -V(r_0, \theta_0, \phi_0) r_0 \sin \theta_0 d\phi dr.$$

The flux of $V(r, \theta, \phi)$ through the wall $\theta = \theta_0 + d\theta$ is

$$\begin{aligned} V(r_0, \theta_0 + d\theta, \phi_0) \vec{1}_\theta \cdot \vec{1}_\theta (dr) r_0 \sin(\theta_0 + d\theta) d\phi &= \\ &= V(r_0, \theta_0 + d\theta, \phi_0) (dr) r_0 \sin(\theta_0 + d\theta) d\phi. \end{aligned}$$

Hence, the part of the Divergence due to $V(r, \theta, \phi)$ is

$$\begin{aligned} & \frac{1}{(dr)(r_0 d\theta)(r_0 \sin \theta_0 d\phi)} \iint_{\text{spherical box}} V(r, \theta, \phi) \vec{1}_\theta \cdot \vec{1}_n (dr) (r_0 \sin \theta d\phi) = \\ &= \frac{1}{r_0^2 (dr) \sin \theta_0 d\theta d\phi} \left\{ V(r_0, \theta_0 + d\theta, \phi_0) (dr) r_0 \sin(\theta_0 + d\theta) d\phi \right. \\ & \quad \left. - V(r_0, \theta_0, \phi_0) (dr) r_0 \sin \theta_0 d\phi \right\} \\ &= \frac{1}{r_0 \sin \theta_0} \left\{ V(r_0, \theta_0 + d\theta, \phi_0) \sin(\theta_0 + d\theta) - V(r_0, \theta_0, \phi_0) \sin \theta_0 \right\} \\ &= \frac{1}{r_0 \sin \theta_0} \frac{\partial(V \sin \theta)}{\partial \theta} \Big|_{r_0, \theta_0, \phi_0} . \square \end{aligned}$$

34.5 The Volume Divergence due to $W(r, \theta, \phi)$

The flux of $W(r, \theta, \phi)$ through the spherical box walls $\phi = \phi_0$,
and $\phi = \phi_0 + d\phi$ is

$$\iint_{\text{spherical box}} W(r, \theta, \phi) dS_\phi = \iint_{\text{spherical box}} W(r, \theta, \phi) \vec{1}_\phi \cdot \vec{1}_n (dr) (r_0 d\theta).$$

We define the **volume density** of that flux as the part of the
Divergence due to $W(r, \theta, \phi)$

$$\left(\nabla \cdot \begin{bmatrix} U \\ V \\ W \end{bmatrix} \right)_W = \frac{1}{(dr)(r_0 d\theta)(r_0 \sin \theta d\phi)} \iint_{\text{spherical box}} W(r, \theta, \phi) \vec{1}_\phi \cdot \vec{1}_n (dr) r_0 d\theta$$

$$\mathbf{34.6} \quad \left(\nabla \cdot \begin{bmatrix} U \\ V \\ W \end{bmatrix} \right)_W \Big|_{r_0, \theta_0, \phi_0} = \frac{1}{r \sin \theta} \frac{\partial W}{\partial \phi} \Big|_{r_0, \theta_0, \phi_0}$$

Proof:

The flux of $W(r, \theta, \phi)$ through the wall $\phi = \phi_0$ is

$$W(r_0, \theta_0, \phi_0) \vec{1}_\phi \cdot (-\vec{1}_\phi) (dr) r_0 d\theta = -W(r_0, \theta_0, \phi_0) (dr) r_0 d\theta.$$

The flux of $W(r, \theta, \phi)$ through the wall $\phi = \phi_0 + d\phi$ is

$$W(r_0, \theta_0, \phi_0 + d\phi) \vec{1}_\phi \cdot \vec{1}_\phi (dr) r_0 d\theta = W(r_0, \theta_0, \phi_0 + d\phi) (dr) r_0 d\theta.$$

Hence, the part of the Divergence due to $W(r, \theta, \phi)$ is

$$\begin{aligned}
& \frac{1}{(dr)(r_0 d\theta)(r_0 \sin \theta_0 d\phi)} \iint_{\text{spherical box}} W(r, \theta, \phi) \vec{\mathbf{1}}_\phi \cdot \vec{\mathbf{1}}_n (dr) r_0 d\theta = \\
& = \frac{1}{r_0^2 (dr) \sin \theta_0 d\theta d\phi} \left\{ W(r_0, \theta_0, \phi_0 + d\phi) (dr) r_0 d\theta - W(r_0, \theta_0, \phi_0) (dr) r_0 d\theta \right\} \\
& = \frac{1}{r_0 \sin \theta_0} \left. \frac{\partial W}{\partial \phi} \right|_{r_0, \theta_0, \phi_0} . \square
\end{aligned}$$

34.7 Divergence of $(U(r, \theta, \phi), V(r, \theta, \phi), W(r, \theta, \phi))$

We define the *volume density of the total flux through the walls of the spherical box* as the **Divergence** of the vector

$$(U(r, \theta, \phi), V(r, \theta, \phi), W(r, \theta, \phi))$$

$$\begin{aligned}
\nabla \cdot \begin{bmatrix} U(r, \theta, \phi) \\ V(r, \theta, \phi) \\ W(r, \theta, \phi) \end{bmatrix} &= \frac{1}{(dr)(r_0 d\theta)(r_0 \sin \theta_0 d\phi)} \times \\
& \times \left\{ \begin{aligned} & \iint_{\text{spherical box}} U(r, \theta, \phi) \vec{\mathbf{1}}_r \cdot \vec{\mathbf{1}}_n (r_0 d\theta) (r_0 \sin \theta_0 d\phi) \\ & + \iint_{\text{spherical box}} V(r, \theta, \phi) \vec{\mathbf{1}}_\theta \cdot \vec{\mathbf{1}}_n (dr) (r_0 \sin \theta d\phi) \\ & + \iint_{\text{spherical box}} W(r, \theta, \phi) \vec{\mathbf{1}}_\phi \cdot \vec{\mathbf{1}}_n (dr) r_0 d\theta \end{aligned} \right\}
\end{aligned}$$

$$\mathbf{34.8} \quad \nabla \cdot \begin{bmatrix} U(r, \theta, \phi) \\ V(r, \theta, \phi) \\ W(r, \theta, \phi) \end{bmatrix} = \frac{1}{r^2} \frac{\partial(r^2 U)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(V \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial W}{\partial \phi}$$

35.

Spherical Green Identities

In spherical coordinates the 3-space Green's Identities of 23 are

35.1 1st Spherical Green Identity

$U(r, \theta, \phi), V(r, \theta, \phi)$ are Hyper-real differentiable in a Volume domain D bounded by a closed Surface ∂D . Then,

$$\iiint_D [U \nabla^2 V + \nabla U \cdot \nabla V] r^2 \sin \theta dr d\theta d\phi = \iint_{S=\partial D} U \frac{\partial V}{\partial n} r^2 \sin \theta d\theta d\phi$$

35.2 2nd Spherical Green Identity

$U(r, \theta, \phi), V(r, \theta, \phi)$ are Hyper-real differentiable in a Volume domain D bounded by a closed Surface ∂D . Then,

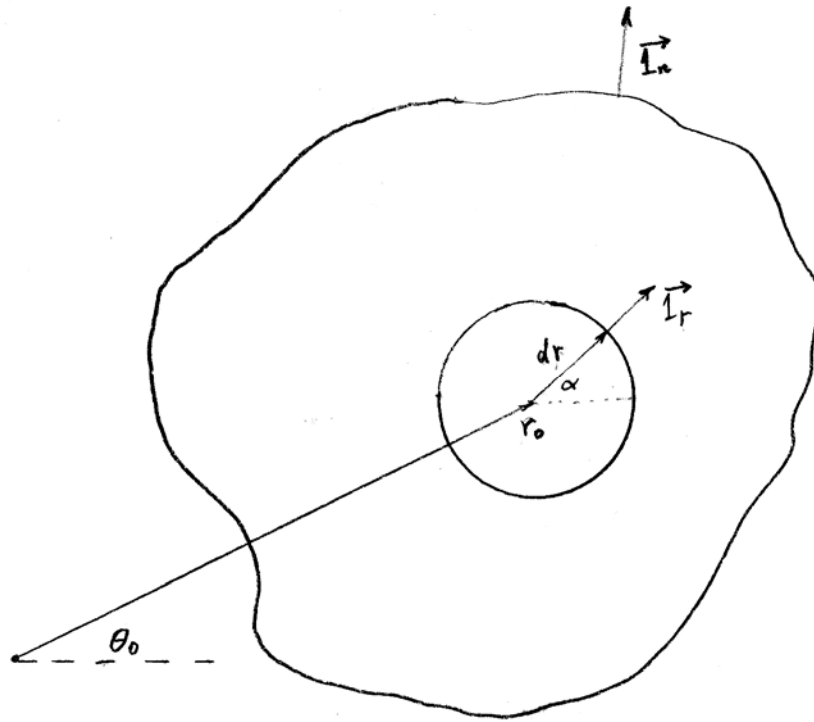
$$\iiint_D [U \nabla^2 V - V \nabla^2 U] r^2 \sin \theta dr d\theta d\phi = \iint_{S=\partial D} [U \frac{\partial V}{\partial n} - \frac{\partial U}{\partial n} V] r^2 \sin \theta d\theta d\phi$$

36.

Representing Harmonic $V(r, \theta, \phi)$

Let $V(r, \theta, \phi)$ be Hyper-real differentiable Harmonic function in a volume D , bounded by the closed surface $S = \partial D$.

Integrate over a surface that includes $S = \partial D$, and an infinitesimal sphere of radius dr centered at r_0 .



$$36.1 \quad V(r_0, \theta_0, \phi_0) = \frac{1}{4\pi} \oint_S \left\{ \frac{1}{|\vec{r} - \vec{r}_0|} \frac{\partial V}{\partial n} + V(r, \theta, \phi) \frac{1}{(r - r_0)^2} \vec{1}_r \cdot \vec{1}_n \right\} dS$$

Proof: In the 2nd Spherical Green Identity, 35.2, put

$$U(r, \theta, \phi) = \frac{1}{|\vec{r} - \vec{r}_0|}.$$

Then,

$$\begin{aligned} \nabla \cdot \nabla U(r, \theta, \phi) &= \nabla \cdot \nabla \frac{1}{|\vec{r} - \vec{r}_0|} \\ &= \nabla \cdot \left(\partial_r \frac{1}{|\vec{r} - \vec{r}_0|} \vec{1}_r \right) \\ &= \nabla \cdot \left(\frac{-1}{|\vec{r} - \vec{r}_0|^2} \vec{1}_r \right) \\ &= \frac{1}{|\vec{r} - \vec{r}_0|^2} \frac{\partial}{\partial(r - r_0)} \left(|\vec{r} - \vec{r}_0|^2 \frac{-1}{|\vec{r} - \vec{r}_0|^2} \right) \\ &= 0. \end{aligned}$$

In the domain enclosed between S and the infinitesimal sphere, $\frac{1}{|\vec{r} - \vec{r}_0|}$ is at most an infinite hyper-real, and since

$$\nabla^2 V(r, \theta, \phi) = 0,$$

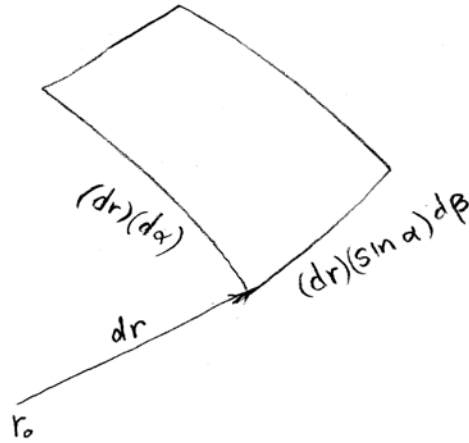
$$\underbrace{\iiint}_{\text{Volume between S and sphere}} \left\{ \frac{1}{|\vec{r} - \vec{r}_0|} \underbrace{\nabla^2 V}_0 - V \underbrace{\nabla^2 \frac{1}{|\vec{r} - \vec{r}_0|}}_0 \right\} r^2 \sin \theta dr d\theta d\phi = 0.$$

By the 2nd Spherical Green Identity, this equals to the integrals over S , and the Sphere. That is,

$$0 = \iint_{\text{sphere}} \left\{ \frac{1}{|\vec{r} - \vec{r}_0|} \nabla V \cdot \vec{1}_n - V(r, \theta, \phi) \left(\nabla \frac{1}{|\vec{r} - \vec{r}_0|} \right) \cdot \vec{1}_n \right\} dS$$

$$+ \iint_S \left\{ \frac{1}{|\vec{r} - \vec{r}_0|} \nabla V \cdot \vec{1}_n - V(r, \theta, \phi) \left(\nabla \frac{1}{|\vec{r} - \vec{r}_0|} \right) \cdot \vec{1}_n \right\} dS$$

The first integral over the sphere is infinitesimal. Indeed,



$$|\vec{r} - \vec{r}_0| = dr,$$

$$dS = [dr d\beta][dr(\sin \beta) d\alpha],$$

$$\vec{1}_n = \vec{1}_r,$$

where β is the elevation angle, and α is the azimuth angle on the sphere. Hence,

$$\begin{aligned}
\oint_{|\vec{r}-\vec{r}_0|=dr} \frac{1}{|\vec{r}-\vec{r}_0|} \underbrace{\nabla V \cdot \vec{1}_n}_{\frac{\partial V}{\partial r} \vec{1}_r} \underbrace{dS}_{(dr)^2 (d\alpha)(\sin \alpha)(d\beta)} &= \oint_{|\vec{r}-\vec{r}_0|=dr} \underbrace{\partial_r V dr}_{\text{infinitesimal}} (\sin \alpha) d\alpha d\beta. \\
&= (\text{infinitesimal}) \times \underbrace{\int_{\alpha=0}^{\alpha=\pi} (\sin \alpha) d\alpha}_2 \underbrace{\int_{\beta=0}^{\beta=2\pi} d\beta}_{2\pi}
\end{aligned}$$

The second integral over the sphere surface is

$$\begin{aligned}
- \oint_{|\vec{r}-\vec{r}_0|=dr} V(r, \theta, \phi) \underbrace{\left(\nabla \frac{1}{|\vec{r}-\vec{r}_0|} \right)}_{\frac{-1}{|\vec{r}-\vec{r}_0|^2} \vec{1}_r} \cdot \vec{1}_n \underbrace{dS}_{(dr)^2 (\sin \alpha)(d\alpha)(d\beta)} &= \\
= \left(V(r_0, \theta_0, \phi_0) + \underbrace{dV}_{\text{infinitesimal}} \Big|_{r_0, \theta_0, \phi_0} \right) \underbrace{\int_{\alpha=0}^{\alpha=\pi} \sin \alpha d\alpha}_2 \underbrace{\int_{\beta=0}^{\beta=2\pi} d\beta}_{2\pi} \\
= 4\pi V(r_0, \theta_0, \phi_0) + \text{infinitesimal}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
V(r_0, \theta_0, \phi_0) &= \frac{1}{4\pi} \oint_S \left\{ \frac{1}{|\vec{r}-\vec{r}_0|} \nabla V \cdot \vec{1}_n - V(r, \theta, \phi) \left(\nabla \frac{1}{|\vec{r}-\vec{r}_0|} \right) \cdot \vec{1}_n \right\} dS \\
&= \frac{1}{4\pi} \oint_S \left\{ \frac{1}{|\vec{r}-\vec{r}_0|} \frac{\partial V}{\partial n} + V(r, \theta, \phi) \frac{1}{|\vec{r}-\vec{r}_0|^2} \vec{1}_r \cdot \vec{1}_n \right\} dS. \square
\end{aligned}$$

37.

Spherical Curl

Let $U(r, \theta, \phi)$, $V(r, \theta, \phi)$, and $W(r, \theta, \phi)$ be hyper-real differentiable functions, defined on an infinitesimal area element dS . dS projects onto the walls of an infinitesimal spherical box with vertex at (r_0, θ_0, ϕ_0) ,

base with sides $r_0 d\theta$, $r_0 \sin \theta_0 d\phi$,

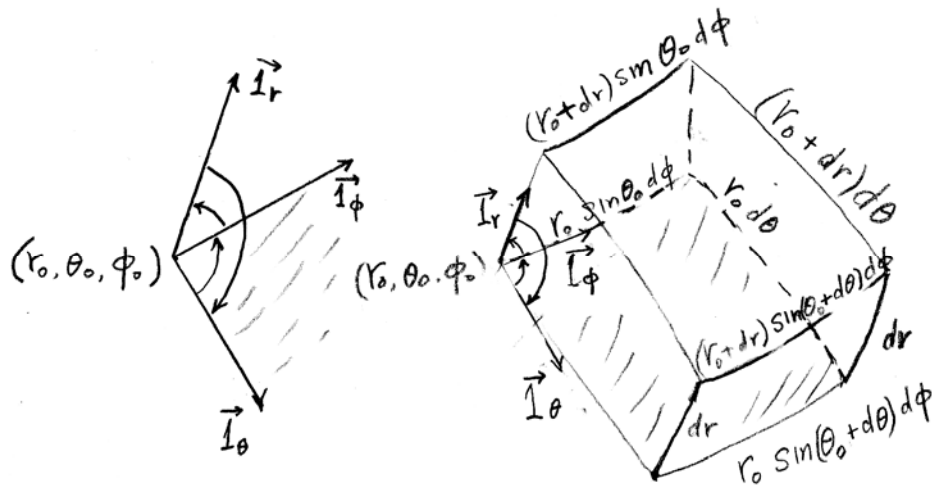
$r_0 d\theta$, $r_0 \sin(\theta_0 + d\theta) d\phi$,

height dr ,

and top with sides

$(r_0 + dr) \sin \theta_0 d\phi$, $(r_0 + dr) d\theta$,

$(r_0 + dr) \sin(\theta_0 + d\theta) d\phi$, $(r_0 + dr) d\theta$,

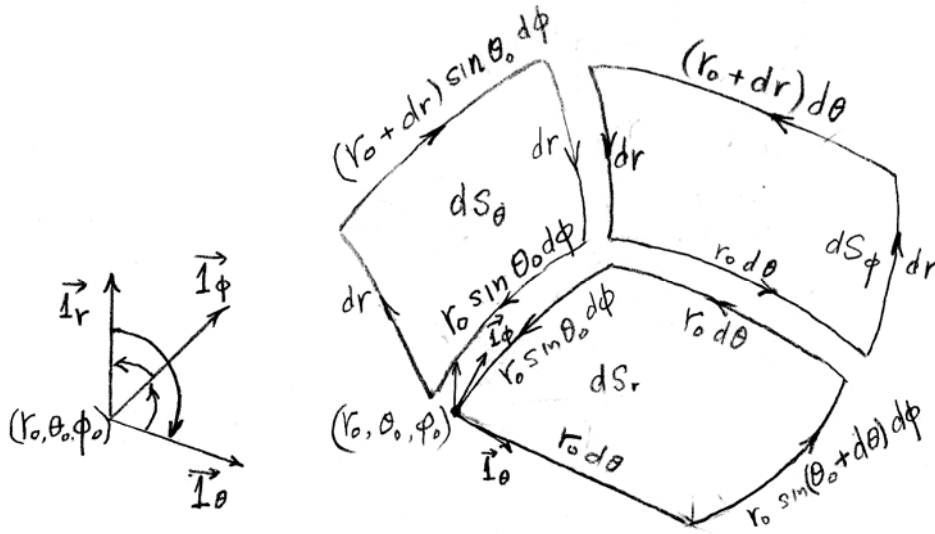


The projections are

$$dS_r \text{ with area } \vec{1}_r \cdot \vec{1}_n dS = (r_0 d\theta)(r_0 \sin \theta_0 d\phi),$$

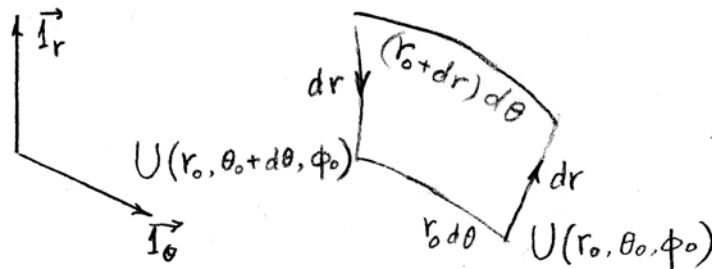
$$dS_\theta \text{ with area } \vec{1}_\theta \cdot \vec{1}_n dS = r_0 \sin \theta_0 d\phi dr,$$

$$dS_\phi \text{ with area } \vec{1}_\phi \cdot \vec{1}_n dS = r_0 d\theta dr.$$



37.1 ϕ - component

On the rectangle that varies by dr , and $d\theta$,

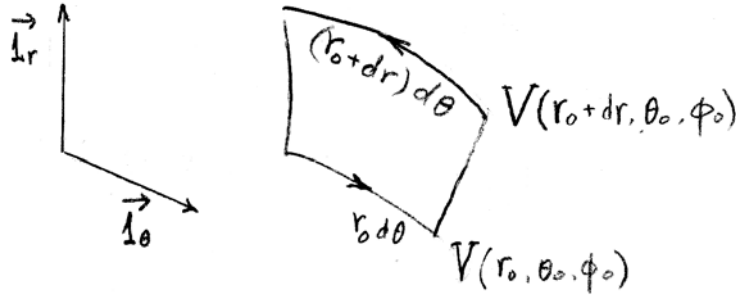


$U(r, \theta, \phi) \vec{1}_r \cdot \vec{1}_l dl$ is nonzero when $\vec{1}_l$ is along r ,

the path starts at (r_0, θ_0, ϕ_0) , and ends at $(r_0, \theta_0 + d\theta, \phi_0)$, and the circulation is

$$\begin{aligned}
 \oint_{dS_\phi \text{ rectangle}} U(r, \theta, \phi) \vec{1}_r \cdot \vec{1}_l dl &= \int_{\theta_0}^{\theta_0 + d\theta} U(r, \theta, \phi) \vec{1}_r \cdot \vec{1}_l dl \\
 &= U(r_0, \theta_0, \phi_0) \vec{1}_r \cdot \vec{1}_r dr + U(r_0, \theta_0 + d\theta, \phi_0) \vec{1}_r \cdot (-\vec{1}_r) dr \\
 &= \{U(r_0, \theta_0, \phi_0) - U(r_0, \theta_0 + d\theta, \phi_0)\} dr \\
 &= \{-\partial_\theta U\}_{r_0, \theta_0, \phi_0} d\theta dr.
 \end{aligned}$$

On that rectangle, $V(r, \theta, \phi) \vec{1}_\theta \cdot \vec{1}_l dl$ is nonzero when $\vec{1}_l$ is along θ , the path starts at $(r_0 + dr, \theta_0, \phi_0)$, and ends at (r_0, θ_0, ϕ_0) ,



and the circulation is

$$\begin{aligned}
 \oint_{dS_\phi \text{ rectangle}} V(r, \theta, \phi) \vec{1}_\theta \cdot \vec{1}_l dl &= \int_{r_0 + dr}^{r_0} V(r, \theta, \phi) \vec{1}_\theta \cdot \vec{1}_l dl \\
 &= V(r_0 + dr, \theta_0, \phi_0) \vec{1}_\theta \cdot \vec{1}_\theta (r_0 + dr) d\theta + V(r_0, \theta_0, \phi_0) \vec{1}_\theta \cdot (-\vec{1}_\theta) r_0 d\theta
 \end{aligned}$$

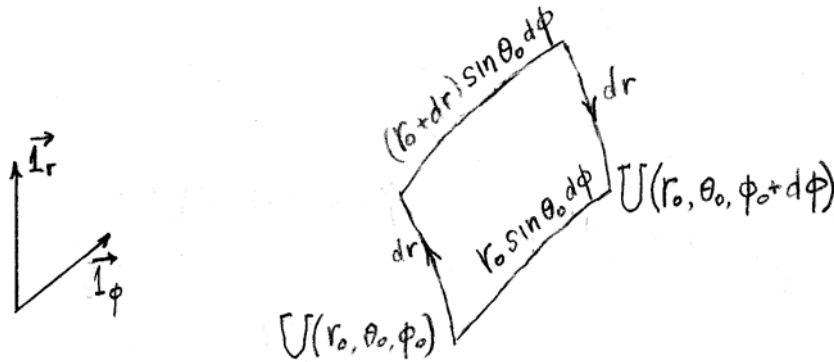
$$\begin{aligned}
 &= \left\{ V(r_0 + dr, \theta_0, \phi_0)(r_0 + dr) - V(r_0, \theta_0, \phi_0)r_0 \right\} d\theta \\
 &= \left\{ \partial_r (rV) \right\}_{r_0, \theta_0, \phi_0} dr d\theta.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \nabla \times \begin{bmatrix} U(r, \theta, \phi) \\ V(r, \theta, \phi) \end{bmatrix} &= \vec{i}_\phi \frac{1}{r_0} \oint_{dS_\phi \text{ rectangle}} U(r, \theta, \phi) \vec{i}_r \cdot \vec{i}_l dl + V(r, \theta, \phi) \vec{i}_\theta \cdot \vec{i}_l dl \\
 &= \vec{i}_\phi \frac{1}{r_0} \left\{ -\partial_\theta U + \partial_r (rV) \right\}_{r_0, \theta_0, \phi_0} dr d\theta \\
 &= \vec{i}_\phi \left\{ \frac{1}{r_0} \partial_r (rV) - \frac{1}{r_0} \partial_\theta U \right\}_{r_0, \theta_0, \phi_0}
 \end{aligned}$$

37.2 θ – component

On the rectangle that varies by dr , and $d\phi$,

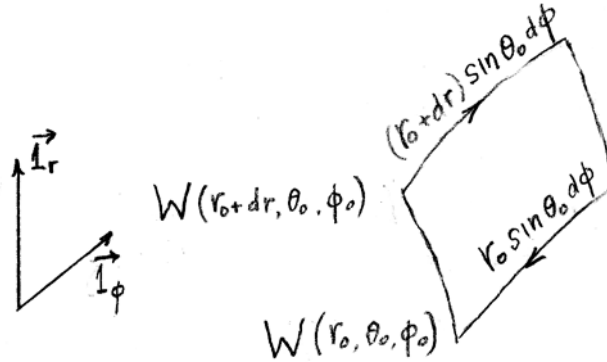


$U(r, \theta, \phi) \vec{i}_r \cdot \vec{i}_l dl$ is nonzero when \vec{i}_l is along r ,

the path starts at (r_0, θ_0, ϕ_0) , and ends at $(r_0, \theta_0, \phi_0 + d\phi)$, and the circulation is

$$\begin{aligned} \oint_{dS_\theta \text{ rectangle}} U(r, \theta, \phi) \vec{l}_r \cdot \vec{l}_l dl &= \int_{\phi_0}^{\phi_0 + d\phi} U(r, \theta, \phi) \vec{l}_r \cdot \vec{l}_l dl \\ &= U(r_0, \theta_0, \phi_0) \vec{l}_r \cdot \vec{l}_r dr + U(r_0, \theta_0, \phi_0 + d\phi) \vec{l}_r \cdot (-\vec{l}_r) dr \\ &= \{U(r_0, \theta_0, \phi_0) - U(r_0, \theta_0, \phi_0 + d\phi)\} dr \\ &= \{-\partial_\phi U\}_{r_0, \theta_0, \phi_0} d\phi dr. \end{aligned}$$

On that rectangle, $W(r, \theta, \phi) \vec{l}_\phi \cdot \vec{l}_l dl$ is nonzero when \vec{l}_l is along ϕ , the path starts at $(r_0 + dr, \theta_0, \phi_0)$, and ends at (r_0, θ_0, ϕ_0)



and the circulation is

$$\begin{aligned} \oint_{dS_\theta \text{ rectangle}} W(r, \theta, \phi) \vec{l}_\phi \cdot \vec{l}_l dl &= \int_{r_0 + dr}^{r_0} W(r, \theta, \phi) \vec{l}_\phi \cdot \vec{l}_l dl \\ &= W(r_0 + dr, \theta_0, \phi_0) \vec{l}_\phi \cdot \vec{l}_\phi (r_0 + dr) \sin \theta_0 d\phi + W(r_0, \theta_0, \phi_0) \vec{l}_\phi \cdot (-\vec{l}_\phi) r_0 \sin \theta_0 d\phi \end{aligned}$$

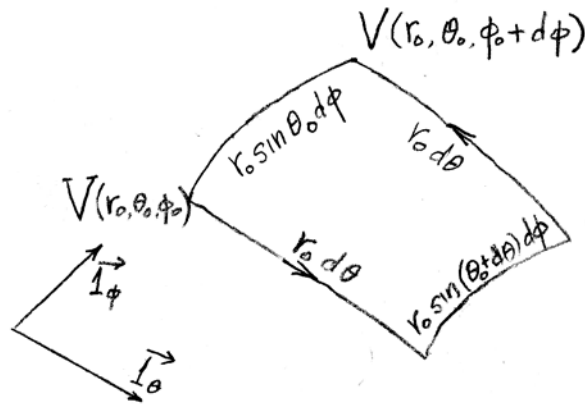
$$\begin{aligned}
 &= \{W(r_0 + dr, \theta_0, \phi_0)(r_0 + dr) - W(r_0, \theta_0, \phi_0)r_0\} \sin \theta_0 d\phi \\
 &= \left\{ \partial_r (rW) \right\}_{r_0, \theta_0, \phi_0} dr (\sin \theta_0) d\phi.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \nabla \times \begin{bmatrix} U(r, \theta, \phi) \\ W(r, \theta, \phi) \end{bmatrix} &= \vec{1}_\theta \frac{1}{r_0 \sin \theta_0 d\phi dr} \oint_{dS_\theta \text{ rectangle}} U(r, \theta, \phi) \vec{1}_r \cdot \vec{1}_l dl + W(r, \theta, \phi) \vec{1}_\phi \cdot \vec{1}_l dl \\
 &= \vec{1}_\theta \frac{1}{r_0 \sin \theta_0 d\phi dr} \left\{ -\partial_\phi U dr d\phi + \partial_r (rW) \sin \theta_0 d\phi dr \right\}_{r_0, \theta_0, \phi_0} \\
 &= \vec{1}_\theta \left\{ -\frac{1}{r_0 \sin \theta_0} \partial_\phi U + \frac{1}{r_0} \partial_r (rW) \right\}_{r_0, \theta_0, \phi_0}
 \end{aligned}$$

37.3 r - component

On the rectangle that varies by $d\theta$, and $d\phi$,

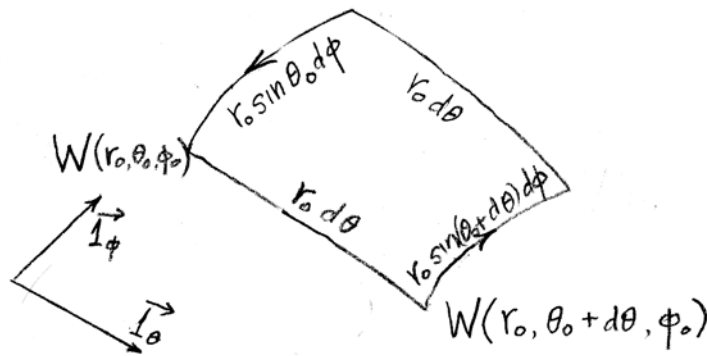


$V(r, \theta, \phi) \vec{1}_\theta \cdot \vec{1}_l dl$ is nonzero when $\vec{1}_l$ is along θ ,

the path starts at (r_0, θ_0, ϕ_0) , and ends at $(r_0, \theta_0, \phi_0 + d\phi)$, and the circulation is

$$\begin{aligned}
 \oint_{dS_r \text{ rectangle}} V(r, \theta, \phi) \vec{1}_\theta \cdot \vec{1}_l dl &= \int_{\phi_0}^{\phi_0 + d\phi} V(r, \theta, \phi) \vec{1}_\theta \cdot \vec{1}_l dl \\
 &= V(r_0, \theta_0, \phi_0) \vec{1}_\theta \cdot \vec{1}_\theta r_0 d\theta + V(r_0, \theta_0, \phi_0 + d\phi) \vec{1}_\theta \cdot (-\vec{1}_\theta) r_0 d\theta \\
 &= V(r_0, \theta_0, \phi_0) r_0 d\theta - V(r_0, \theta_0, \phi_0 + d\phi) r_0 d\theta \\
 &= \{V(r_0, \theta_0, \phi_0) - V(r_0, \theta_0, \phi_0 + d\phi)\} r_0 d\theta \\
 &= \{-\partial_\phi V\}_{r_0, \theta_0, \phi_0} d\phi (r_0 d\theta).
 \end{aligned}$$

On that rectangle, $W(r, \theta, \phi) \vec{1}_\phi \cdot \vec{1}_l dl$ is nonzero when $\vec{1}_l$ is along ϕ , the path starts at $(r_0, \theta_0 + d\theta, \phi_0)$, and ends at (r_0, θ_0, ϕ_0)



and the circulation is

$$\oint_{dS_r \text{ rectangle}} W(r, \theta, \phi) \vec{1}_\phi \cdot \vec{1}_l dl = \int_{\theta_0 + d\theta}^{\theta_0} W(r, \theta, \phi) \vec{1}_\phi \cdot \vec{1}_l dl$$

$$\begin{aligned}
&= W(r_0, \theta_0 + d\theta, \phi_0) \vec{1}_\phi \cdot \vec{1}_\phi r_0 \sin(\theta_0 + d\theta) d\phi + W(r_0, \theta_0, \phi_0) \vec{1}_\phi \cdot (-\vec{1}_\phi) r_0 \sin \theta_0 d\phi \\
&= \{W(r_0, \theta_0 + d\theta, \phi_0) \sin(\theta_0 + d\theta) - W(r_0, \theta_0, \phi_0) \sin \theta_0\} r_0 d\phi \\
&= \left\{ \partial_\theta (W \sin \theta) \right\}_{r_0, \theta_0, \phi_0} r_0 d\theta d\phi.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\nabla \times \begin{bmatrix} V(r, \theta, \phi) \\ W(r, \theta, \phi) \end{bmatrix} &= \vec{1}_r \frac{1}{r_0 \sin \theta_0 d\phi (r_0 d\theta)} \oint_{dS, \text{rectangle}} V(r, \theta, \phi) \vec{1}_\theta \cdot \vec{1}_l dl + W(r, \theta, \phi) \vec{1}_\phi \cdot \vec{1}_l dl \\
&= \vec{1}_r \frac{1}{r_0^2 \sin \theta_0 d\theta d\phi} \left\{ (-\partial_\phi V) r_0 d\theta d\phi + \partial_\theta (W \sin \theta) r_0 d\theta d\phi \right\}_{r_0, \theta_0, \phi_0} \\
&= \vec{1}_r \left\{ \frac{1}{r_0 \sin \theta_0} (-\partial_\phi V) + \frac{1}{r_0 \sin \theta_0} \partial_\theta (W \sin \theta) \right\}_{r_0, \theta_0, \phi_0}
\end{aligned}$$

37.4 The Spherical Curl

$$\nabla \times \begin{bmatrix} U(r, \theta, \phi) \\ V(r, \theta, \phi) \\ W(r, \theta, \phi) \end{bmatrix} = \frac{1}{r} \begin{bmatrix} \partial_r (rV) - \partial_\theta U \\ \partial_r (rW) - \frac{1}{\sin \theta} \partial_\phi U \\ \frac{1}{\sin \theta} \partial_\theta (W \sin \theta) - \frac{1}{\sin \theta} \partial_\phi V \end{bmatrix}$$

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