Space-Time Curves and Surfaces

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Abstract The Differential Geometry of Space-Time Curves and surfaces, requires the Cross-Product of 4-vectors.

A 3-space curve $\vec{x}(s)$, parametrized by its arc-length s, is characterized by two curvature functions $\kappa(s)$, and $\tau(s)$. At each point along $\vec{x}(s)$, the Tangent unit vector

$$\vec{T}(s) \equiv \frac{\vec{x}\,'(s)}{\left|\vec{x}\,'(s)\right|} = \vec{1}_{\vec{x}\,'(s)},$$

and the Normal unit vector

$$\vec{N}(s) \equiv \frac{\vec{T}\,'(s)}{\left|\vec{T}\,'(s)\right|} = \vec{1}_{\vec{T}\,'(s)},$$

define the Binormal unit vector

$$\vec{B}(s) \equiv \vec{T} \times \vec{N} = \begin{vmatrix} \vec{1}_x & \vec{1}_y & \vec{1}_z \\ T_1 & T_2 & T_3 \\ N_1 & N_2 & N_3 \end{vmatrix}.$$

The Derivatives $\vec{T}\,'(s)$, $\vec{N}\,'(s)$, $\vec{B}\,'(s)$, in the Frenet Frame, $\vec{T}$, $\vec{N}$, $\vec{B}$, satisfy the Frenet Differential Equation.

A 4-space curve $\vec{x}(s)$, parametrized by its arc-length s, is characterized by three curvature functions $\kappa(s)$, $\tau(s)$, and $\omega(s)$. At each point along $\vec{x}(s)$, the Tangent unit vector

$$\vec{T}(s) \equiv \frac{\vec{x}\,'(s)}{\left|\vec{x}\,'(s)\right|} = \vec{1}_{\vec{x}\,'(s)},$$

and the Normal unit vector

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define the Binormal unit vector

$$\vec{B}(s) \equiv \vec{T} \times \vec{N},$$

which is believed to have <u>six</u> components.

Clearly, the Curve Equations cannot be written as a mixture of 4-vectors, and 6-vectors.

Thus, the Fundamental Differential Equations of Space-time Curves were not developed.

A 3-space surface $\vec{x}(u,v)$, with components along the axes $x, y, z,$ is parametrized by two linearly independent families of curves, u, and v.

At each point on $\vec{x}(u,v)$, the Tangent Vector along u

$$\vec{x}_u(u,v) \equiv \partial_u \vec{x}(u,v),$$

and the Tangent Vector along v

$$\vec{x}_v(u,v) \equiv \partial_v \vec{x}(u,v),$$

define the Normal to the surface,

$$\vec{x}_u(u,v) \times \vec{x}_v(u,v) \equiv \vec{n}(u,v).$$

The Derivatives $\vec{x}_{uu}(u,v)$, $\vec{x}_{uv}(u,v) = \vec{x}_{vu}(u,v)$, and $\vec{x}_{vv}(u,v)$ in the Gauss Frame, $\vec{x}_u$, $\vec{x}_v$, $\vec{x}_u \times \vec{x}_v$, satisfy the Gauss Differential Equations.

The Derivatives $\partial_u \vec{n}$, and $\partial_v \vec{n}$ in the Gauss Frame satisfy the Weingarten Differential Equations.

A Space-time surface $\vec{x}(u,v)$, with components along the axes x, y, z, t is parametrized by two linearly independent families of curves, u, and v.

At each point on $\vec{x}(u,v)$, the Tangent Vector along u

$$\vec{x}_u(u,v) \equiv \partial_u \vec{x}(u,v),$$

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Clearly, the Space-time Differential Equations cannot be written as a mixture of 4-vectors, and 6-vectors.

Thus, the Fundamental Differential Equations of Space-time Surfaces were not developed.

Recently, we showed that the cross product of 4-vectors is a 4-vector, and supplied the correct formula for it.

Applying this formula to Space-time Vectors, we obtain the Differential Equations for Space-time Curves, for Space-time surfaces, and for the Normals to a surface in Space time.

Keywords: Infinitesimal, Infinite-Hyper-real, Hyper-real, Cardinal, Infinity. Non-Archimedean, Calculus, Limit, Continuity, Derivative, Integral, Gradient, Divergence, Curl, Space-time Vectors Fields

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Introduction

0.1 Space-time Curve

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and the Normal unit vector

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define the Binormal unit vector

$$\vec{B}(s) \equiv \vec{T} \times \vec{N} = \begin{vmatrix} \vec{1}_x & \vec{1}_y & \vec{1}_z \\ T_1 & T_2 & T_3 \\ N_1 & N_2 & N_3 \end{vmatrix}.$$

The Derivatives $\vec{T}\,'(s)$, $\vec{N}\,'(s)$, $\vec{B}\,'(s)$, in the Frenet Frame $\vec{T}$, $\vec{N}$, $\vec{B}$, satisfy the Frenet Differential Equations.

$$\begin{bmatrix} \vec{T}\,'(s) \\ \vec{N}\,'(s) \\ \vec{B}\,'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \vec{T}(s) \\ \vec{N}(s) \\ \vec{B}(s) \end{bmatrix}.$$

The Differential Geometry of Space-time Curves requires the

Cross-Product of 4-vectors.

A space-time curve $\vec{x}(s)$, parametrized by its arc-length s, is characterized by three curvature functions $\kappa(s)$, $\tau(s)$, and $\omega(s)$. At each point along $\vec{x}(s)$, the Tangent unit vector

$$\vec{T}(s) \equiv \frac{\vec{x}\,'(s)}{\left|\vec{x}\,'(s)\right|} = \vec{1}_{\vec{x}\,'(s)},$$

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define the Binormal unit vector

$$\vec{B}(s) \equiv \vec{T} \times \vec{N},$$

provided the cross product of space-time vectors is a space-time vector.

It is not self evident how the Spatial Cross-Product, with its $\vec{1}_x$, $\vec{1}_y$, and $\vec{1}_z$ components may be generalized to a Space-time Cross-product with $\vec{1}_x$, $\vec{1}_y$, $\vec{1}_z$, and $\vec{1}_t$ components.

For instance, we can add a column to obtain

$$\begin{vmatrix} \vec{1}_x & \vec{1}_y & \vec{1}_z & \vec{1}_t \\ A_x & A_y & A_z & A_t \\ B_x & B_y & B_z & B_t \\ ? & ? & ? & ? \end{vmatrix},$$

but what will be a fourth raw of that 4×4 determinant?

It is less evident how the fact that there are six terms of the form

$$A_i B_j - A_j B_i$$

that determine the 4-space Cross Product, lead to the belief that the 4-Space cross product is six dimensional.

Clearly, the Curve Differential Equations cannot be written as a mixture of 4-vectors, and 6-vectors.

Thus, the Fundamental Differential Equations of Space-time Curves were not developed.

0.2 Space-time Surface

A 3-space surface $\vec{x}(u,v)$, with components along the axes $x, y, z,$ is parametrized by two linearly independent families of curves, u, and v.

At each point on $\vec{x}(u,v)$, the Tangent Vector along u

$$\vec{x}_u(u,v) \equiv \partial_u \vec{x}(u,v),$$

and the Tangent Vector along v

$$\vec{x}_v(u,v) \equiv \partial_v \vec{x}(u,v),$$

define the Normal to the surface,

$$\vec{x}_u(u,v) \times \vec{x}_v(u,v) \equiv \vec{n}(u,v).$$

The Derivatives $\vec{x}_{uu}(u,v)$, $\vec{x}_{uv}(u,v) = \vec{x}_{vu}(u,v)$, and $\vec{x}_{vv}(u,v)$ in the Gauss Frame, $\vec{x}_u$, $\vec{x}_v$, $\vec{x}_u \times \vec{x}_v$, satisfy the Gauss Differential Equations.

The Derivatives $\partial_u \vec{n}$, and $\partial_v \vec{n}$ in the Gauss Frame satisfy the Weingarten Differential Equations.

The Differential Geometry of Space-time Surfaces requires the Cross-Product of 4-vectors.

A Space-time surface $\vec{x}(u,v)$, with components along the axes x, y, z, t is parametrized by two linearly independent families of curves, u, and v.

At each point on $\vec{x}(u,v)$, the Tangent Vector along u

$$\vec{x}_u(u,v) \equiv \partial_u \vec{x}(u,v),$$

and the Tangent Vector along v

$$\vec{x}_v(u,v) \equiv \partial_v \vec{x}(u,v),$$

define the Normal to the surface,

$$\vec{x}_u(u,v) \times \vec{x}_v(u,v),$$

which is believed to have <u>six</u> components.

Clearly, the Surface Differential Equations cannot be written as a mixture of 4-vectors, and 6-vectors.

Thus, the Fundamental Differential Equations of Space-time Surfaces were not developed.

0.3 Correct Space-time Cross Product

Recently, we showed that the cross product of 4-vectors is a 4-vector, and supplied the correct formula for it.

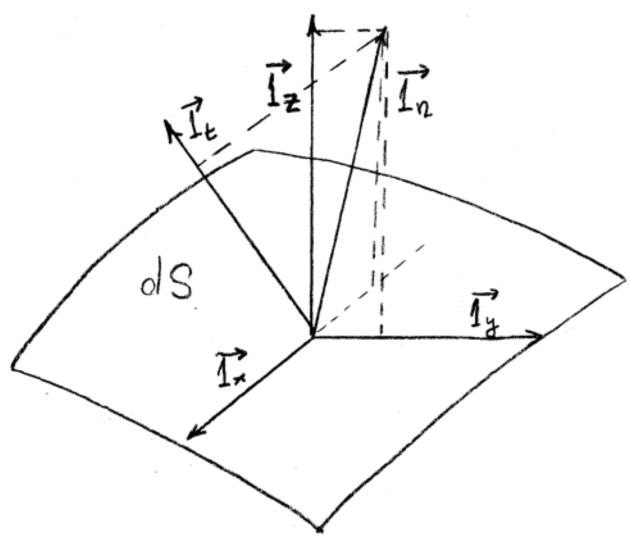
In [Dan4], we showed that the 4-space curl is a 4-vector, and supplied the correct formula for it. In [Dan5], we obtained the Cross-Product of 4-vectors, for space-time Electromagnetic Vector Fields.

Applying this formula to Space-time Vectors, we obtain the Differential Equations for Space-time Curves, for Space-time surfaces, and for the Normals to a surface in Space time.

1.

Curl of 4-Vectors

Let $P(x,y,z,t)$, $Q(x,y,z,t)$, $R(x,y,z,t)$, and $S(x,y,z,t)$ be hyper-real differentiable functions, defined on an infinitesimal area element dS. dS projects onto six 2-planes generated by the unit vectors $\vec{1}_x, \vec{1}_y, \vec{1}_z$, and $\vec{1}_t$.



x-y projection with area $dxdy$ and normal $\vec{1}_x \times \vec{1}_y = \vec{1}_z$

y-z projection with area $dydz$ and normal $\vec{1}_y \times \vec{1}_z = \vec{1}_t$

z-t projection with area $dzdt$ and normal $\vec{1}_z \times \vec{1}_t = \vec{1}_x$

t-x projection with area $dtdx$ and normal $\vec{1}_t \times \vec{1}_x = \vec{1}_y$

<u>t-y projection</u> with area $dtdy$ and normal

$$\vec{1}_t \times \vec{1}_y = (\vec{1}_y \times \vec{1}_z) \times \vec{1}_y = \vec{1}_z \underbrace{(\vec{1}_y \cdot \vec{1}_y)}_{1} - \vec{1}_y \underbrace{(\vec{1}_z \cdot \vec{1}_y)}_{0} = \vec{1}_z$$

<u>z-x projection</u> with area $dzdx$ and normal

$$\vec{1}_z \times \vec{1}_x = (\vec{1}_x \times \vec{1}_y) \times \vec{1}_x = \vec{1}_y \underbrace{(\vec{1}_x \cdot \vec{1}_x)}_{1} - \vec{1}_x \underbrace{(\vec{1}_y \cdot \vec{1}_x)}_{0} = \vec{1}_y$$

The projected areas are

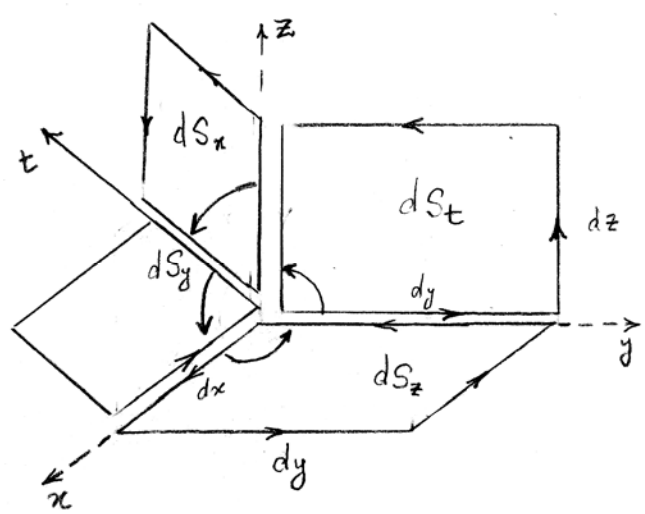
$$dS_x = \vec{1}_x \cdot \vec{1}_n dS = dzdt\,,$$

$$dS_y = \vec{1}_y \cdot \vec{1}_n dS = dtdx + dzdx\,,$$

$$dS_z = \vec{1}_z \cdot \vec{1}_n dS = dxdy + dtdy\,,$$

$$dS_t = \vec{1}_t \cdot \vec{1}_n dS = dydz$$

The projections areas are walls of a box with vertex at (x_0, y_0, z_0, t_0) and sides dx, dy, dz, and dt.



Given positive orientation of a right hand system,

$$\nabla \times \begin{bmatrix} P(x,y,z,t) \\ Q(x,y,z,t) \end{bmatrix} = \vec{1}_z \frac{1}{dxdy} \oint_{\partial(dS_z)} P(x,y,z,t)\vec{1}_x \cdot \vec{1}_l dl + Q(x,y,z,t)\vec{1}_y \cdot \vec{1}_l dl \,,$$

$$= \begin{vmatrix} \partial_x & \partial_y \\ P(x,y,z,t) & Q(x,y,z,t) \end{vmatrix}_{x_0,y_0,z_0,t_0} \vec{1}_z$$

$$\nabla \times \begin{bmatrix} Q(x,y,z,t) \\ R(x,y,z,t) \end{bmatrix} = \vec{1}_t \frac{1}{dydz} \oint_{\partial(dS_t)} Q(x,y,z,t)\vec{1}_y \cdot \vec{1}_l dl + R(x,y,z,t)\vec{1}_z \cdot \vec{1}_l dl \,,$$

$$= \begin{vmatrix} \partial_y & \partial_z \\ Q(x,y,z,t) & R(x,y,z,t) \end{vmatrix}_{x_0,y_0,z_0,t_0} \vec{1}_t$$

$$\nabla \times \begin{bmatrix} R(x,y,z,t) \\ S(x,y,z,t) \end{bmatrix} = \vec{1}_x \frac{1}{dzdt} \oint_{\partial(dS_x)} R(x,y,z,t)\vec{1}_z \cdot \vec{1}_l dl + S(x,y,z,t)\vec{1}_t \cdot \vec{1}_l dl \,,$$

$$= \begin{vmatrix} \partial_z & \partial_t \\ R(x,y,z,t) & S(x,y,z,t) \end{vmatrix}_{x_0,y_0,z_0,t_0} \vec{1}_x \,.$$

$$\nabla \times \begin{bmatrix} S(x,y,z,t) \\ P(x,y,z,t) \end{bmatrix} = \vec{1}_y \frac{1}{dtdx} \oint_{\partial(dS_y)} S(x,y,z,t)\vec{1}_t \cdot \vec{1}_l dl + P(x,y,z,t)\vec{1}_x \cdot \vec{1}_l dl \,,$$

$$= \begin{vmatrix} \partial_t & \partial_x \\ S(x,y,z,t) & P(x,y,z,t) \end{vmatrix}_{x_0,y_0,z_0,t_0} \vec{1}_y$$

$$\nabla \times \begin{bmatrix} S(x,y,z,t) \\ Q(x,y,z,t) \end{bmatrix} = \vec{1}_z \frac{1}{dtdy} \oint_{\partial(dS_z)} S(x,y,z,t)\vec{1}_t \cdot \vec{1}_l dl + Q(x,y,z,t)\vec{1}_y \cdot \vec{1}_l dl \,,$$

$$= \begin{vmatrix} \partial_t & \partial_y \\ S(x,y,z,t) & Q(x,y,z,t) \end{vmatrix}_{x_0,y_0,z_0,t_0} \vec{1}_z$$

$$\nabla \times \begin{bmatrix} P(x,y,z,t) \\ R(x,y,z,t) \end{bmatrix} = \vec{1}_y \frac{1}{dzdx} \oint_{\partial(dS_y)} P(x,y,z,t)\vec{1}_x \cdot \vec{1}_l dl + R(x,y,z,t)\vec{1}_z \cdot \vec{1}_l dl\,,$$

$$= \begin{vmatrix} \partial_x & \partial_z \\ P(x,y,z,t) & R(x,y,z,t) \end{vmatrix}_{x_0,y_0,z_0,t_0} \vec{1}_y$$

The 4-space Curl is the sum of the six area curls. That is,

$$\nabla \times \begin{bmatrix} P(x,y,z,t) \\ Q(x,y,z,t) \\ R(x,y,z,t) \\ S(x,y,z,t) \end{bmatrix} =$$

$$= \underbrace{\nabla \times \begin{bmatrix} R \\ S \end{bmatrix}}_{\begin{vmatrix} \partial_z & \partial_t \\ R & S \end{vmatrix}\vec{1}_x} + \underbrace{\nabla \times \begin{bmatrix} S \\ P \end{bmatrix}}_{\begin{vmatrix} \partial_t & \partial_x \\ S & P \end{vmatrix}\vec{1}_y} + \underbrace{\nabla \times \begin{bmatrix} P \\ R \end{bmatrix}}_{\begin{vmatrix} \partial_x & \partial_z \\ P & R \end{vmatrix}\vec{1}_y} + \underbrace{\nabla \times \begin{bmatrix} P \\ Q \end{bmatrix}}_{\begin{vmatrix} \partial_x & \partial_y \\ P & Q \end{vmatrix}\vec{1}_z} + \underbrace{\nabla \times \begin{bmatrix} S \\ Q \end{bmatrix}}_{\begin{vmatrix} \partial_t & \partial_y \\ S & Q \end{vmatrix}\vec{1}_z} + \underbrace{\nabla \times \begin{bmatrix} Q \\ R \end{bmatrix}}_{\begin{vmatrix} \partial_y & \partial_z \\ Q & R \end{vmatrix}\vec{1}_t}$$

$$\boxed{= \begin{pmatrix} S_z - R_t \\ P_t - S_x + R_x - P_z \\ Q_x - P_y + Q_t - S_y \\ R_y - Q_z \end{pmatrix}}.$$

2.

Cross-Product of 4-Vectors

2.1 *The Cross-product of 4-vectors is the sum of six cross-products of 2-vectors, That is,*

$$\begin{bmatrix} A_x \\ A_y \\ A_z \\ A_t \end{bmatrix} \times \begin{bmatrix} B_x \\ B_y \\ B_z \\ B_t \end{bmatrix} = \underbrace{\begin{bmatrix} A_z \\ A_t \end{bmatrix} \times \begin{bmatrix} B_z \\ B_t \end{bmatrix}}_{\begin{vmatrix} A_z & A_t \\ B_z & B_t \end{vmatrix} \vec{1}_x} + \underbrace{\begin{bmatrix} A_t \\ A_x \end{bmatrix} \times \begin{bmatrix} B_t \\ B_x \end{bmatrix}}_{\begin{vmatrix} A_t & A_x \\ B_t & B_x \end{vmatrix} \vec{1}_y} + \underbrace{\begin{bmatrix} A_x \\ A_z \end{bmatrix} \times \begin{bmatrix} B_x \\ B_z \end{bmatrix}}_{\begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} \vec{1}_y} +$$

$$+ \underbrace{\begin{bmatrix} A_x \\ A_y \end{bmatrix} \times \begin{bmatrix} B_x \\ B_y \end{bmatrix}}_{\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \vec{1}_z} + \underbrace{\begin{bmatrix} A_t \\ A_y \end{bmatrix} \times \begin{bmatrix} B_t \\ B_y \end{bmatrix}}_{\begin{vmatrix} A_t & A_y \\ B_t & B_y \end{vmatrix} \vec{1}_z} + \underbrace{\begin{bmatrix} A_y \\ A_z \end{bmatrix} \times \begin{bmatrix} B_y \\ B_z \end{bmatrix}}_{\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} \vec{1}_t}$$

$$\boxed{= \begin{bmatrix} \begin{vmatrix} A_z & A_t \\ B_z & B_t \end{vmatrix} \\ \begin{vmatrix} A_t & A_x \\ B_t & B_x \end{vmatrix} + \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} \\ \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} + \begin{vmatrix} A_t & A_y \\ B_t & B_y \end{vmatrix} \\ \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} \end{bmatrix}}.$$

2.2 $$\vec{1}_x \times \vec{1}_y = \vec{1}_z$$

$$\vec{1}_y \times \vec{1}_z = \vec{1}_t$$

$$\vec{1}_z \times \vec{1}_t = \vec{1}_x$$

$$\vec{1}_t \times \vec{1}_x = \vec{1}_y$$

Proof: $$\vec{1}_x \times \vec{1}_y = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \vec{1}_z = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \vec{1}_z . \square$$

2.3 $$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

2.4 $$\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

Proof: $$\vec{1}_y \times (\vec{1}_x \times \vec{1}_y) = \vec{1}_y \times \vec{1}_z = \vec{1}_t$$

$$(\vec{1}_y \cdot \vec{1}_y)\vec{1}_x - (\vec{1}_y \cdot \vec{1}_x)\vec{1}_y = \vec{1}_x$$

2.5 $$\frac{d}{ds}\left\{ \vec{A}(s) \cdot \vec{B}(s) \right\} = \vec{A}'(s) \cdot \vec{B}(s) + \vec{A}(s) \cdot \vec{B}'(s)$$

$$\frac{d}{ds}\left\{ \vec{A}(s) \times \vec{B}(s) \right\} = \vec{A}'(s) \times \vec{B}(s) + \vec{A}(s) \times \vec{B}'(s)$$

3.

Space-time Curve Frame

Let $\vec{x}(s)$ be a Curve parametrized by its arc-length s,

$$\boxed{\vec{x}(s) = \begin{bmatrix} x_1(s) \\ x_2(s) \\ x_3(s) \\ x_4(s) \end{bmatrix}},$$

with continuous linearly independent derivative functions,

$$\vec{x}\,'(s) = \frac{d\vec{x}(s)}{ds},$$

$$\vec{x}\,''(s) = \frac{d^2\vec{x}(s)}{ds^2},$$

$$\vec{x}\,'''(s) = \frac{d^3\vec{x}(s)}{ds^3},$$

$$\vec{x}\,''''(s) = \frac{d^4\vec{x}(s)}{ds^4}.$$

3.1 The <u>Tangent</u> unit vector At s,

$$\boxed{\vec{T}(s) \equiv \frac{\vec{x}\,'(s)}{\left|\vec{x}\,'(s)\right|} = \vec{1}_{\vec{x}\,'(s)}}.$$

3.2 $$\boxed{\vec{T}\,'(s) \perp \vec{T}(s)}$$

$\vec{T}\,'(s)$ is Normal to the Tangent.

Proof: $$\vec{T}(s) \cdot \vec{T}(s) = 1\,,$$

$$\vec{T}\,'(s) \cdot \vec{T}(s) + \vec{T}(s) \cdot \vec{T}\,'(s) = 0\,,$$

$$\vec{T}\,'(s) \cdot \vec{T}(s) = 0\,,$$

3.3 The Normal unit vector at s

$$\boxed{\vec{N}(s) \equiv \vec{1}_{\vec{T}\,'(s)}}$$

$$\boxed{\vec{T}\,'(s) = \left|\vec{T}\,'(s)\right| \vec{N}(s)}$$

3.4 The Curvature of $\vec{x}(s)$ at s

$$\boxed{\kappa(s) = \left|\vec{T}\,'(s)\right|}$$

3.5 The First Frenet Equation

$$\boxed{\vec{T}\,'(s) = \kappa(s)\vec{N}(s)}$$

3.6 The Bi-Normal unit vector at s

$$\boxed{\vec{B}(s) \equiv \vec{T}(s) \times \vec{N}(s)}$$

3.7 The <u>Tri-Normal</u> unit vector at s

$$\boxed{\vec{W}(s) \equiv \vec{N}(s) \times \vec{B}(s)}$$

3.8 $\{\vec{T}, \vec{N}, \vec{B}, \vec{W}\}$ is an Orthonormal Vector System at s

$$\vec{B}(s) \times \vec{W}(s) = \vec{T}(s)$$

$$\vec{W}(s) \times \vec{T}(s) = \vec{N}(s)$$

3.10 $\vec{T}$, *and* $\vec{N}$ *span the <u>Osculating (=Tangent) Plane</u>*

$$\text{Span}\{\vec{T}, \vec{N}\} = \{\lambda\vec{T} + \mu\vec{N} : \lambda, \mu \in \mathbb{R}\}$$

3.11 $\vec{N}$, *and* $\vec{B}$ *span the <u>Normal Plane</u>*

$$\text{Span}\{\vec{N}, \vec{B}\} = \{\lambda\vec{N} + \mu\vec{B} : \lambda, \mu \in \mathbb{R}\}$$

3.12 $\vec{B}$, *and* $\vec{W}$ *span the <u>Bi-Normal Plane</u>*

$$\text{Span}\{\vec{B}, \vec{W}\} = \{\lambda\vec{B} + \mu\vec{W} : \lambda, \mu \in \mathbb{R}\}$$

3.13 $\vec{W}$, *and* $\vec{T}$ *span the <u>Tri-Normal Plane</u>*

$$\text{Span}\{\vec{W}, \vec{T}\} = \{\lambda\vec{W} + \mu\vec{T} : \lambda, \mu \in \mathbb{R}\}$$

4.

Space-time Curve Equations

The Vector Functions

$$\vec{T}\,'(s),\ \ \vec{N}\,'(s),\ \ \vec{B}\,'(s), \text{ and } \vec{W}\,'(s)$$

are in

$$\text{Span}\{\vec{T}(s), \vec{N}(s), \vec{B}(s), \vec{W}(s)\}.$$

From 3.2,

4.1 $\vec{T}\,'(s) \perp \vec{T}(s)$, *and*

$\vec{T}\,'(s)$ *has no component along* $\vec{T}(s)$.

Similarly,

4.2 $\vec{N}\,'(s) \perp \vec{N}(s)$, *and*

$\vec{N}\,'(s)$ *has no component along* $\vec{N}(s)$,

$\vec{B}\,'(s) \perp \vec{B}(s)$, *and*

$\vec{B}\,'(s)$ *has no component along* $\vec{B}(s)$,

$\vec{W}\,'(s) \perp \vec{W}(s)$, *and*

$\vec{W}\,'(s)$ *has no component along* $\vec{W}(s)$.

By 3.3, $\vec{T}\,'(s)$ only component is along $\vec{N}(s)$. That is,

4.3 *$\vec{T}\,'(s)$ has no component along $\vec{B}(s)$,*

$$\boxed{\vec{T}\,'(s)\cdot\vec{B}(s)=0}.$$

$\vec{T}\,'(s)$ has no component along $\vec{W}(s)$,

$$\boxed{\vec{T}\,'(s)\cdot\vec{W}(s)=0}.$$

$\vec{T}\,'(s)$ has no component along $\vec{T}(s)$,

$$\boxed{\vec{T}\,'(s)\cdot\vec{T}(s)=0}.$$

and by 3.5, the Frenet Equation for $\vec{T}\,'(s)$ is

$$\boxed{\vec{T}\,'(s)=\kappa(s)\vec{N}(s)},$$

Similarly, we obtain the Differential Equations for $\vec{N}\,'(s)$, $\vec{B}\,'(s)$, and $\vec{W}\,'(s)$.

4.4 *The component of $\vec{N}\,'(s)$ along $\vec{T}(s)$ is*

$$\boxed{\vec{N}\,'(s)\cdot\vec{T}(s)=-\kappa(s)}.$$

<u>Proof</u>:

$$\vec{N}\cdot\vec{T}=0,$$

$$\vec{N}\,'(s)\cdot\vec{T}(s)+\underbrace{\vec{N}(s)\cdot\underbrace{\vec{T}\,'(s)}_{\kappa(s)\vec{N}(s)}}_{\kappa(s)}=0.\square$$

4.5 *The component of* $\vec{N}\,'(s)$ *along* $\vec{B}(s)$ *is*

$$\vec{N}\,'(s)\cdot\vec{B}(s) = -\vec{N}(s)\cdot\vec{B}\,'(s).$$

<u>*Proof*</u>:
$$\vec{N}(s)\cdot\vec{B}(s) = 0\,,$$

$$\vec{N}\,'(s)\cdot\vec{B}(s) + \vec{N}(s)\cdot\vec{B}\,'(s) = 0\,.\square$$

4.6 The <u>Bi-Curvature (=Torsion)</u> of $\vec{x}(s)$ **at** s

$$\boxed{\tau(s) = \vec{N}\,'(s)\cdot\vec{B}(s)}$$

4.7 *The component of* $\vec{N}\,'(s)$ *along* $\vec{W}(s)$ *is*

$$\boxed{\vec{N}\,'(s)\cdot\vec{W}(s) = 0}.$$

<u>*Proof*</u>: By the Schmidt Orthogonalization Process

$$\vec{T} = \vec{x}\,',$$

$$\vec{N} = \frac{\vec{x}\,''}{\left|\vec{x}\,''\right|} \in \mathrm{Span}\{\vec{x}\,',\vec{x}\,''\},$$

$$\vec{B} = \frac{\vec{x}\,''' - (\vec{x}\,'''\cdot\vec{T})\vec{T} - (\vec{x}\,'''\cdot\vec{N})\vec{N}}{\left|\vec{x}\,''' - (\vec{x}\,'''\cdot\vec{T})\vec{T} - (\vec{x}\,'''\cdot\vec{N})\vec{N}\right|} \in \mathrm{Span}\{\vec{x}\,',\vec{x}\,'',\vec{x}\,'''\},$$

$$\vec{N}\,' \in \mathrm{Span}\{\vec{x}\,',\vec{x}\,'',\vec{x}\,'''\} = \mathrm{Span}\{\vec{T},\vec{N},\vec{B}\},$$

The projection of $\vec{N}\,'$ on $\vec{W}$ is zero.$\square$

4.8 The Differential Equation for $\vec{N}\,'(s)$

$$\boxed{\vec{N}\,'(s) = -\kappa(s)\vec{T}(s) + \tau(s)\vec{B}(s)},$$

4.9 *The component of $\vec{B}\,'(s)$ along $\vec{T}(s)$ is*

$$\boxed{\vec{B}\,'(s)\cdot\vec{T}(s) = 0}.$$

<u>*Proof*</u>:
$$\vec{B}\cdot\vec{T} = 0,$$

$$\vec{B}\,'\cdot\vec{T} + \underbrace{\vec{B}\cdot\vec{T}\,'}_{\vec{T}\,'\cdot\vec{B}=0} = 0.\square$$

4.10 *The component of $\vec{B}\,'(s)$ along $\vec{N}(s)$ is*

$$\boxed{\vec{B}\,'(s)\cdot\vec{N}(s) = -\tau(s)}.$$

<u>*Proof*</u>:
$$\vec{B}(s)\cdot\vec{N}(s) = 0,$$

$$\vec{B}\,'\cdot\vec{N} + \vec{B}\cdot\vec{N}\,' = 0$$

$$\vec{B}\,'\cdot\vec{N} = -\underbrace{\vec{B}\cdot\vec{N}\,'}_{\tau(s)}.\square$$

4.11 *The component of $\vec{B}\,'(s)$ along $\vec{W}(s)$ is*

$$\vec{B}\,'(s)\cdot\vec{W}(s) = -\vec{W}\,'(s)\cdot\vec{B}(s).$$

<u>*Proof*</u>:
$$\vec{B}(s)\cdot\vec{W}(s) = 0,$$

$$\vec{B}\,'\cdot\vec{W} + \underbrace{\vec{B}\cdot\vec{W}\,'}_{\vec{W}\,'\cdot\vec{B}} = 0\,.\square$$

4.12 The <u>Tri-Curvature</u> of $\vec{x}(s)$ at s

$$\boxed{\omega(s) = \vec{B}\,'(s)\cdot\vec{W}(s)}$$

4.13 The Differential Equation for $\vec{B}\,'(s)$

$$\boxed{\vec{B}\,'(s) = -\tau(s)\vec{N}(s) + \omega(s)\vec{W}(s)},$$

4.14 *The component of $\vec{W}\,'(s)$ along $\vec{T}(s)$ is*

$$\boxed{\vec{W}\,'(s)\cdot\vec{T}(s) = 0}$$

<u>Proof</u>:

$$\vec{W}\cdot\vec{T} = 0\,,$$

$$\vec{W}\,'\cdot\vec{T} + \underbrace{\vec{W}\cdot\vec{T}\,'}_{\vec{T}\,'\cdot\vec{W}=0} = 0\,.\square$$

4.15 *The component of $\vec{W}\,'(s)$ along $\vec{N}(s)$ is*

$$\boxed{\vec{W}\,'(s)\cdot\vec{N}(s) = 0}.$$

<u>Proof</u>:

$$\vec{W}(s)\cdot\vec{N}(s) = 0\,,$$

$$\vec{W}\,'\cdot\vec{N} + \underbrace{\vec{W}\cdot\vec{N}\,'}_{\vec{N}\,'\cdot\vec{W}=0,\ by\ (4.7)} = 0\,.\square$$

4.16 *The component of* $\vec{W}\,'(s)$ *along* $\vec{B}(s)$ *is*

$$\boxed{\vec{W}\,'(s)\cdot\vec{B}(s) = -\omega(s)}.$$

<u>*Proof*</u>:

$$\vec{W}(s)\cdot\vec{B}(s) = 0\,,$$

$$\vec{W}\,'\cdot\vec{B} + \underbrace{\vec{W}\cdot\vec{B}\,'}_{\vec{B}\,'\cdot\vec{W}=\omega(s)} = 0\,.\square$$

4.17 **The Differential Equation for** $\vec{W}\,'(s)$

$$\boxed{\vec{W}\,'(s) = -\omega(s)\vec{B}(s)},$$

4.18 **The Space-time Curve Differential Equations**

$$\begin{bmatrix}\vec{T}\,'(s)\\ \vec{N}\,'(s)\\ \vec{B}\,'(s)\\ \vec{W}\,'(s)\end{bmatrix} = \begin{bmatrix}0 & \kappa(s) & 0 & 0\\ -\kappa(s) & 0 & \tau(s) & 0\\ 0 & -\tau(s) & 0 & \omega(s)\\ 0 & 0 & -\omega(s) & 0\end{bmatrix}\begin{bmatrix}\vec{T}(s)\\ \vec{N}(s)\\ \vec{B}(s)\\ \vec{W}(s)\end{bmatrix}$$

5.

Space-Time Surface Frame

Let $\vec{x}(u,v)$ be a Space-time Surface Patch parametrized by two linearly independent families of curves, u, and v,

$$\boxed{\vec{x}(u,v) = \begin{bmatrix} x_1(u,v) \\ x_2(u,v) \\ x_3(u,v) \\ x_4(u,v) \end{bmatrix}},$$

with continuously differentiable partial derivatives of any necessary order.

5.1 Tangent along the u-Curves

At each point on $\vec{x}(u,v)$*, the Tangent* Vector *along* u *is*

$$\boxed{\vec{x}_u(u,v) \equiv \partial_u \vec{x}(u,v)},$$

and the Unit Tangent Vector along u *is*

$$\boxed{\vec{1}_{\vec{x}_u} = \frac{\partial_u \vec{x}(u,v)}{\left|\partial_u \vec{x}(u,v)\right|}}$$

5.2 Tangent along the v-Curves

At each point on $\vec{x}(u,v)$*, the Tangent* Vector *along* v *is*

$$\boxed{\vec{x}_v(u,v) \equiv \partial_v \vec{x}(u,v)},$$

and the Unit Tangent Vector along v *is*

$$\boxed{\vec{1}_{\vec{x}_v} = \frac{\partial_v \vec{x}(u,v)}{\left|\partial_v \vec{x}(u,v)\right|}}$$

5.3 The Metric Tensor at (u,v)

$$\boxed{g_{ij}(u,v) \equiv \begin{bmatrix} \vec{x}_u \cdot \vec{x}_u & \vec{x}_u \cdot \vec{x}_v \\ \vec{x}_v \cdot \vec{x}_u & \vec{x}_v \cdot \vec{x}_v \end{bmatrix}}$$

5.4 The Inverse Metric Tensor at (u,v)

$$\boxed{(g_{ij})^{-1}(u,v) = \begin{bmatrix} \vec{x}_v \cdot \vec{x}_v & -\vec{x}_u \cdot \vec{x}_v \\ -\vec{x}_v \cdot \vec{x}_u & \vec{x}_u \cdot \vec{x}_u \end{bmatrix} \equiv g^{ij}(u,v)}$$

5.5 Area of parallelogram generated by $\vec{x}_u$, and $\vec{x}_u$

$$Area(\vec{x}_u, \vec{x}_v) = \left|\vec{x}_u \times \vec{x}_v\right|$$

$$\boxed{\left|\vec{x}_u \times \vec{x}_v\right|^2 = \det(g_{ij}) = g_{11}g_{22} - (g_{12})^2}$$

Proof: $\left|\vec{x}_u \times \vec{x}_v\right|^2$ expanded into its components by 2.1, equals

$g_{11}g_{22} - (g_{12})^2$ expanded into components.□

5.6 The Normal Unit Vector at (u,v)

$$\boxed{\vec{N}(u,v) \equiv \frac{\vec{x}_u \times \vec{x}_v}{\left|\vec{x}_u \times \vec{x}_v\right|} = \vec{1}_{\vec{x}_u \times \vec{x}_v} = \frac{\vec{x}_u \times \vec{x}_v}{\sqrt{g_{11}g_{22} - (g_{12})^2}}}$$

5.7 The Bi-Normal Unit Vector at (u,v)

$$\boxed{\vec{B}(u,v) \equiv \vec{1}_{\vec{x}_u} \times \vec{N}(u,v)}$$

5.8 $\vec{B}(u,v)$ *is in the Tangent Plane*

$$\text{Span}\{\vec{x}_u, \vec{x}_v\} = \{\lambda\vec{x}_u + \mu\vec{x}_v : \lambda, \mu \in \mathbb{R}\}.$$

Proof: $\vec{B}(u,v)$ is normal to $\vec{N}(u,v)$.□

5.10 The Tri-Normal unit vector at (u,v)

$$\boxed{\vec{W}(u,v) \equiv \vec{N}(u,v) \times \vec{B}(u,v)}$$

5.11 $\{\vec{1}_{\vec{x}_u}, \vec{N}, \vec{B}, \vec{W}\}$ is an Orthonormal Vector System

$$\vec{N}(u,v) \times \vec{B}(u,v) = \vec{1}_{\vec{x}_u}(u,v)$$

$$\vec{B}(u,s)\times\vec{1}_{\vec{x}_u}(u,v) = \vec{N}(u,v)$$

5.12 $\vec{1}_{\vec{x}_u}$ *, and* $\vec{1}_{\vec{x}_v}$ *span the* *<u>Tangent Plane</u>*

$$\mathrm{Span}\{\vec{1}_{\vec{x}_u},\vec{1}_{\vec{x}_v}\} = \{\lambda\vec{1}_{\vec{x}_u} + \mu\vec{1}_{\vec{x}_v} : \lambda,\mu\in\mathbb{R}\}$$

6.

Space-time Surface Curvatures

Since $\vec{N} \cdot \vec{N} = 1$,

$$\partial_u \vec{N} \cdot \vec{N} = 0$$

Thus,

6.1 $\vec{N}_u(u,v) \equiv \partial_u \vec{N} \perp \vec{N}(u,v)$, ***and***

$\vec{N}_u(u,v)$ ***has no component along*** $\vec{N}(u,v)$,

$\vec{N}_v(u,v) \perp \vec{N}(u,v)$, ***and***

$\vec{N}_v(u,v)$ ***has no component along*** $\vec{N}(u,v)$,

Similarly,

6.2 $\vec{W}_u(u,v) \perp \vec{W}(u,v)$, ***and***

$\vec{W}_u(u,v)$ ***has no component along*** $\vec{W}(u,v)$,

$\vec{W}_v(u,v) \perp \vec{W}(u,v)$, ***and***

$\vec{W}_v(u,v)$ ***has no component along*** $\vec{W}(u,v)$,

Since $\vec{x}_u \cdot \vec{N} = 0$,

$$\vec{x}_{uu} \cdot \vec{N} + \vec{x}_u \cdot \vec{N}_u = 0.$$

Therefore,

6.3
$$\boxed{\vec{x}_{uu}(u,v)\cdot\vec{N}(u,v) = -\vec{x}_u(u,v)\cdot\vec{N}_u(u,v)}$$

$$\boxed{\vec{x}_{uv}(u,v)\cdot\vec{N}(u,v) = -\vec{x}_u(u,v)\cdot\vec{N}_v(u,v)}$$

$$\boxed{\vec{x}_{vu}(u,v)\cdot\vec{N}(u,v) = -\vec{x}_v(u,v)\cdot\vec{N}_u(u,v)}$$

$$\boxed{\vec{x}_{vv}(u,v)\cdot\vec{N}(u,v) = -\vec{x}_v(u,v)\cdot\vec{N}_v(u,v)}$$

Since $\vec{x}_u \cdot \vec{W} = 0$,

$$\vec{x}_{uu}\cdot\vec{W} + \vec{x}_u\cdot\vec{W}_u = 0.$$

Therefore,

6.4
$$\boxed{\vec{x}_{uu}(u,v)\cdot\vec{W}(u,v) = -\vec{x}_u(u,v)\cdot\vec{W}_u(u,v)}$$

$$\boxed{\vec{x}_{uv}(u,v)\cdot\vec{W}(u,v) = -\vec{x}_u(u,v)\cdot\vec{W}_v(u,v)}$$

$$\boxed{\vec{x}_{uv}(u,v)\cdot\vec{W}(u,v) = -\vec{x}_v(u,v)\cdot\vec{W}_u(u,v)}$$

$$\boxed{\vec{x}_{vv}(u,v)\cdot\vec{W}(u,v) = -\vec{x}_v(u,v)\cdot\vec{W}_v(u,v)}$$

Since $\vec{N}\cdot\vec{W} = 0$,

$$\vec{N}_u\cdot\vec{W} + \vec{N}\cdot\vec{W}_u = 0.$$

$$\vec{N}_v\cdot\vec{W} + \vec{N}\cdot\vec{W}_v = 0$$

Therefore,

6.5
$$\boxed{\vec{N}_u(u,v)\cdot\vec{W}(u,v) = -\vec{N}(u,v)\cdot\vec{W}_u(u,v)}$$

$$\boxed{\vec{N}_v(u,v)\cdot\vec{W}(u,v) = -\vec{N}(u,v)\cdot\vec{W}_v(u,v)}$$

7.

Space-time Surface Equations, and Christoffel Curvatures

Since the Space-time Vector Functions

$$\vec{x}_{uu}, \quad \vec{x}_{uv} = \vec{x}_{vu}, \quad \vec{x}_{vv},$$

are in $\mathrm{Span}\{\vec{x}_u, \vec{x}_v, \vec{N}, \vec{W}\}$, we have

7.1 $$\boxed{\vec{x}_{uu}(u,v) = \Gamma_{11}^1 \vec{x}_u + \Gamma_{11}^2 \vec{x}_v + \underbrace{(\vec{x}_{uu} \cdot \vec{N})}_{-\vec{x}_u \cdot \vec{N}_u} \vec{N} + \underbrace{(\vec{x}_{uu} \cdot \vec{W})}_{-\vec{x}_u \cdot \vec{W}_u} \vec{W}}$$

7.2 $$\boxed{\vec{x}_{uv}(u,v) = \Gamma_{12}^1 \vec{x}_u + \Gamma_{12}^2 \vec{x}_v + + \underbrace{(\vec{x}_{uv} \cdot \vec{N})}_{-\vec{x}_u \cdot \vec{N}_v} \vec{N} + \underbrace{(\vec{x}_{uv} \cdot \vec{W})}_{-\vec{x}_u \cdot \vec{W}_v} \vec{W}}$$

7.3 $$\boxed{\vec{x}_{vu}(u,v) = \Gamma_{21}^1 \vec{x}_u + \Gamma_{21}^2 \vec{x}_v + + \underbrace{(\vec{x}_{vu} \cdot \vec{N})}_{-\vec{x}_v \cdot \vec{N}_u} \vec{N} + \underbrace{(\vec{x}_{vu} \cdot \vec{W})}_{-\vec{x}_v \cdot \vec{W}_u} \vec{W}}$$

7.4 $$\boxed{\vec{x}_{vv}(u,v) = \Gamma_{22}^1 \vec{x}_u + \Gamma_{22}^2 \vec{x}_v + + \underbrace{(\vec{x}_{vv} \cdot \vec{N})}_{-\vec{x}_v \cdot \vec{N}_v} \vec{N} + \underbrace{(\vec{x}_{vv} \cdot \vec{W})}_{-\vec{x}_v \cdot \vec{W}_v} \vec{W}}$$

where Γ_{ij}^k are **Christoffel Curvatures**.

Since $\partial_{uv}\vec{x} = \partial_{vu}\vec{x}$, the equations for $\vec{x}_{uv}(u,v)$, and $\vec{x}_{vu}(u,v)$ are the same, and

$$\Gamma_{ij}^{k} = \Gamma_{ji}^{k}.$$

The Christoffel Curvatures can be written in terms of the Metric Tensor. Multiplying each equation by $\vec{x}_u$, and by $\vec{x}_v$,

<u>The First Equation</u> gives

$$\underbrace{\vec{x}_{uu} \cdot \vec{x}_u}_{\frac{1}{2}\partial_u \underbrace{(\vec{x}_u \cdot \vec{x}_u)}_{g_{11}}} = \Gamma_{11}^{1} \underbrace{\vec{x}_u \cdot \vec{x}_u}_{g_{11}} + \Gamma_{11}^{2} \underbrace{\vec{x}_v \cdot \vec{x}_u}_{g_{12}},$$

$$\underbrace{\vec{x}_{uu} \cdot \vec{x}_v}_{\partial_u \underbrace{(\vec{x}_u \cdot \vec{x}_v)}_{g_{12}} - \frac{1}{2}\partial_v \underbrace{(\vec{x}_u \cdot \vec{x}_u)}_{g_{11}}} = \Gamma_{11}^{1} \underbrace{\vec{x}_u \cdot \vec{x}_v}_{g_{12}} + \Gamma_{11}^{2} \underbrace{\vec{x}_v \cdot \vec{x}_v}_{g_{22}},$$

7.5

$$\boxed{\Gamma_{11}^{1} = \frac{\begin{vmatrix} \frac{1}{2} g_{11,u} & g_{12} \\ g_{12,u} - \frac{1}{2} g_{11,v} & g_{22} \end{vmatrix}}{g_{11}g_{22} - (g_{12})^2}}$$

7.6

$$\boxed{\Gamma_{11}^{2} = \frac{\begin{vmatrix} g_{11} & \frac{1}{2} g_{11,u} \\ g_{12} & g_{12,u} - \frac{1}{2} g_{11,v} \end{vmatrix}}{g_{11}g_{22} - (g_{12})^2}}$$

<u>The Second (or Third) Equation</u> gives

$$\underbrace{\vec{x}_{uv} \cdot \vec{x}_u}_{\frac{1}{2}\partial_v \underbrace{(\vec{x}_u \cdot \vec{x}_u)}_{g_{11}}} = \Gamma_{12}^{1} \underbrace{\vec{x}_u \cdot \vec{x}_u}_{g_{11}} + \Gamma_{12}^{2} \underbrace{\vec{x}_v \cdot \vec{x}_u}_{g_{12}},$$

$$\underbrace{\vec{x}_{uv}\cdot\vec{x}_v}_{\frac{1}{2}\partial_u\underbrace{(\vec{x}_v\cdot\vec{x}_v)}_{g_{22}}} = \Gamma_{12}^1\underbrace{\vec{x}_u\cdot\vec{x}_v}_{g_{12}} + \Gamma_{12}^2\underbrace{\vec{x}_v\cdot\vec{x}_v}_{g_{22}},$$

7.7
$$\boxed{\Gamma_{12}^1 = \frac{\begin{vmatrix} \frac{1}{2}g_{11,v} & g_{12} \\ \frac{1}{2}g_{22,u} & g_{22} \end{vmatrix}}{g_{11}g_{22}-(g_{12})^2} = \Gamma_{21}^1}$$

7.8
$$\boxed{\Gamma_{12}^2 = \frac{\begin{vmatrix} g_{11} & \frac{1}{2}g_{11,v} \\ g_{12} & \frac{1}{2}g_{22,u} \end{vmatrix}}{g_{11}g_{22}-(g_{12})^2} = \Gamma_{21}^2}$$

<u>The Fourth Equation</u> gives

$$\underbrace{\vec{x}_{vv}\cdot\vec{x}_u}_{\partial_v\underbrace{(\vec{x}_u\cdot\vec{x}_v)}_{g_{12}}-\frac{1}{2}\partial_u\underbrace{(\vec{x}_v\cdot\vec{x}_v)}_{g_{22}}} = \Gamma_{22}^1\underbrace{\vec{x}_u\cdot\vec{x}_u}_{g_{11}} + \Gamma_{22}^2\underbrace{\vec{x}_v\cdot\vec{x}_u}_{g_{12}},$$

$$\underbrace{\vec{x}_{vv}\cdot\vec{x}_v}_{\frac{1}{2}\partial_v\underbrace{(\vec{x}_v\cdot\vec{x}_v)}_{g_{22}}} = \Gamma_{22}^1\underbrace{\vec{x}_u\cdot\vec{x}_v}_{g_{12}} + \Gamma_{22}^2\underbrace{\vec{x}_v\cdot\vec{x}_v}_{g_{22}},$$

7.9
$$\boxed{\Gamma_{22}^1 = \frac{\begin{vmatrix} g_{12,v}-\frac{1}{2}g_{22,u} & g_{12} \\ \frac{1}{2}g_{22,v} & g_{22} \end{vmatrix}}{g_{11}g_{22}-(g_{12})^2}}$$

7.10

$$\boxed{\Gamma_{22}^{2} = \frac{\begin{vmatrix} g_{11} & g_{12,v} - \frac{1}{2} g_{22,u} \\ g_{12} & \frac{1}{2} g_{22,v} \end{vmatrix}}{g_{11}g_{22} - (g_{12})^2}}$$

8.

Space-time Normals Equations, and Curvatures

Since the Space-time Vector Functions

$$\vec{N}_u, \ \vec{N}_v,$$

are in $\mathrm{Span}\{\vec{x}_u, \vec{x}_v, \vec{W}\}$, we have

8.1 $$\boxed{\vec{N}_u(u,v) = \Omega_{11}\vec{x}_u + \Omega_{12}\vec{x}_v + M_1\vec{W}}$$

8.2 $$\boxed{\vec{N}_v(u,v) = \Omega_{21}\vec{x}_u + \Omega_{22}\vec{x}_v + M_2\vec{W}}$$

Multiplying each equation by $\vec{x}_u$, and by $\vec{x}_v$,

<u>The First Equation</u> gives

$$\vec{N}_u \cdot \vec{x}_u = \Omega_{11}(\underbrace{\vec{x}_u \cdot \vec{x}_u}_{g_{11}}) + \Omega_{12}(\underbrace{\vec{x}_v \cdot \vec{x}_u}_{g_{12}})$$

$$\vec{N}_u \cdot \vec{x}_v = \Omega_{11}(\underbrace{\vec{x}_u \cdot \vec{x}_v}_{g_{12}}) + \Omega_{12}(\underbrace{\vec{x}_v \cdot \vec{x}_v}_{g_{22}})$$

8.3 $$\boxed{\Omega_{11} = \frac{\begin{vmatrix} \vec{N}_u \cdot \vec{x}_u & g_{12} \\ \vec{N}_u \cdot \vec{x}_v & g_{22} \end{vmatrix}}{g_{11}g_{22} - (g_{12})^2}}$$

8.4
$$\boxed{\Omega_{12} = \frac{\begin{vmatrix} g_{11} & \vec{N}_u \cdot \vec{x}_u \\ g_{12} & \vec{N}_u \cdot \vec{x}_v \end{vmatrix}}{g_{11}g_{22} - (g_{12})^2}}$$

<u>The Second Equation</u> gives

$$\vec{N}_v \cdot \vec{x}_u = \Omega_{21}(\underbrace{\vec{x}_u \cdot \vec{x}_u}_{g_{11}}) + \Omega_{22}(\underbrace{\vec{x}_v \cdot \vec{x}_u}_{g_{12}})$$

$$\vec{N}_v \cdot \vec{x}_v = \Omega_{21}(\underbrace{\vec{x}_u \cdot \vec{x}_v}_{g_{12}}) + \Omega_{22}(\underbrace{\vec{x}_v \cdot \vec{x}_v}_{g_{22}})$$

8.5
$$\boxed{\Omega_{21} = \frac{\begin{vmatrix} \vec{N}_v \cdot \vec{x}_u & g_{12} \\ \vec{N}_v \cdot \vec{x}_v & g_{22} \end{vmatrix}}{g_{11}g_{22} - (g_{12})^2}}$$

8.6
$$\boxed{\Omega_{22} = \frac{\begin{vmatrix} g_{11} & \vec{N}_v \cdot \vec{x}_u \\ g_{12} & \vec{N}_v \cdot \vec{x}_v \end{vmatrix}}{g_{11}g_{22} - (g_{12})^2}}$$

Since the Space-time Vector Functions

$$\vec{W}_u, \ \vec{W}_v,$$

are in $\mathrm{Span}\{\vec{x}_u, \vec{x}_v, \vec{N}\}$, we have

8.7
$$\boxed{\vec{W}_u(u,v) = \Lambda_{11}\vec{x}_u + \Lambda_{12}\vec{x}_v - M_1\vec{N}}$$

8.8
$$\boxed{\vec{W}_v(u,v) = \Lambda_{21}\vec{x}_u + \Lambda_{22}\vec{x}_v - M_2\vec{N}}$$

<u>The First Equation</u> gives

$$\vec{W}_u \cdot \vec{x}_u = \Lambda_{11}(\underbrace{\vec{x}_u \cdot \vec{x}_u}_{g_{11}}) + \Lambda_{12}(\underbrace{\vec{x}_v \cdot \vec{x}_u}_{g_{12}})$$

$$\vec{W}_u \cdot \vec{x}_v = \Lambda_{11}(\underbrace{\vec{x}_u \cdot \vec{x}_v}_{g_{12}}) + \Lambda_{12}(\underbrace{\vec{x}_v \cdot \vec{x}_v}_{g_{22}})$$

8.9
$$\boxed{\Lambda_{11} = \frac{\begin{vmatrix} \vec{W}_u \cdot \vec{x}_u & g_{12} \\ \vec{W}_u \cdot \vec{x}_v & g_{22} \end{vmatrix}}{g_{11}g_{22} - (g_{12})^2}}$$

8.10
$$\boxed{\Lambda_{12} = \frac{\begin{vmatrix} g_{11} & \vec{W}_u \cdot \vec{x}_u \\ g_{12} & \vec{W}_u \cdot \vec{x}_v \end{vmatrix}}{g_{11}g_{22} - (g_{12})^2}}$$

<u>The Second Equation</u> gives

$$\vec{W}_v \cdot \vec{x}_u = \Lambda_{21}(\underbrace{\vec{x}_u \cdot \vec{x}_u}_{g_{11}}) + \Lambda_{22}(\underbrace{\vec{x}_v \cdot \vec{x}_u}_{g_{12}})$$

$$\vec{W}_v \cdot \vec{x}_v = \Lambda_{21}(\underbrace{\vec{x}_u \cdot \vec{x}_v}_{g_{12}}) + \Lambda_{22}(\underbrace{\vec{x}_v \cdot \vec{x}_v}_{g_{22}})$$

8.11
$$\boxed{\Lambda_{21} = \frac{\begin{vmatrix} \vec{W}_v \cdot \vec{x}_u & g_{12} \\ \vec{W}_v \cdot \vec{x}_v & g_{22} \end{vmatrix}}{g_{11}g_{22} - (g_{12})^2}}$$

8.12

$$\boxed{\Lambda_{22} = \frac{\begin{vmatrix} g_{11} & \vec{W}_v \cdot \vec{x}_u \\ g_{12} & \vec{W}_v \cdot \vec{x}_v \end{vmatrix}}{g_{11}g_{22} - (g_{12})^2}}$$

8.13 Differential Equations of a Space-Time Surface

$$\begin{bmatrix} \vec{x}_{uu}(u,v) \\ \vec{x}_{uv}(u,v) \\ \vec{x}_{vu}(u,v) \\ \vec{x}_{vv}(u,v) \\ \vec{N}_u(u,v) \\ \vec{N}_v(u,v) \\ \vec{W}_u(u,v) \\ \vec{W}_v(u,v) \end{bmatrix} = \begin{bmatrix} \Gamma_{11}^1 & \Gamma_{11}^2 & -\vec{x}_u \cdot \vec{N}_u & -\vec{x}_u \cdot \vec{W}_u \\ \Gamma_{12}^1 & \Gamma_{12}^2 & -\vec{x}_u \cdot \vec{N}_v & -\vec{x}_u \cdot \vec{W}_v \\ \Gamma_{21}^1 & \Gamma_{21}^2 & -\vec{x}_v \cdot \vec{N}_u & -\vec{x}_v \cdot \vec{W}_u \\ \Gamma_{22}^1 & \Gamma_{22}^2 & -\vec{x}_v \cdot \vec{N}_v & -\vec{x}_v \cdot \vec{W}_v \\ \Omega_{11} & \Omega_{12} & 0 & \vec{N}_u \cdot \vec{W} \\ \Omega_{21} & \Omega_{22} & 0 & \vec{N}_v \cdot \vec{W} \\ \Lambda_{11} & \Lambda_{12} & -\vec{N}_u \cdot \vec{W} & 0 \\ \Lambda_{21} & \Lambda_{22} & -\vec{N}_v \cdot \vec{W} & 0 \end{bmatrix} \begin{bmatrix} \vec{x}_u \\ \vec{x}_v \\ \vec{N} \\ \vec{W} \end{bmatrix}$$

References

[Dan] Victor Dannon, "*Integral Characterizations and the Theory of Curves*", Proceedings of the American Mathematical Society, Volume 81, Number 4, April 1981;

[Dan1] H. Vic Dannon, "*Infinitesimals*" in Gauge Institute Journal Vol.6 No 4, November 2010;

[Dan2] H. Vic Dannon, "*Infinitesimal Calculus*" in Gauge Institute Journal Vol.7 No 4, November 2011;

[Dan3] H. Vic Dannon, "*Infinitesimal Vector Calculus*" posted to www.gauge-institute.org December, 2011;

[Dan4] H. Vic Dannon, "*4-space Curl is a 4-Vector*" posted to www.gauge-institute.org, January, 2012;

[Dan5] H. Vic Dannon, "*Space-time Electrodynamics, and Magnetic Monopoles*" posted to www.gauge-institute.org, June, 2013;

[Gauss] Karl Friedrich Gauss, "*General Investigations of Curved Surfaces*" Raven Press, 1965.

[Kuhnel] Wolfgang Kuhnel, "*Differential Geometry Curves-Surfaces-Manifolds*" AMS, 2002.

[Lipschutz] Martin Lipschutz, "*Differential Geometry*" Schaum's, 1969.

[Spiegel] Murray R. Spiegel, "*Mathematical Handbook of Formulas and Tables*", Schaum's, 1968.