The Circulation Theorem

H. Vic Dannon
vic0@comcast.net
January, 2012

Abstract The Curl Theorem involves 3 projected areas in 3-space, 6 in 4-space, 10 in 5-space, 15 in 6-space, etc.

Infinitesimal Calculus yields the Circulation Theorem. A version of the Curl Theorem that depends on 3 areas in 3-space, 4 areas in 4-space, 5 in 5-space, etc.

The Circulation Theorem is easier to state, and comprehend, than the Curl Theorem.

It requires Infinitesimals, and does not exist in the Calculus of Limits, where infinitesimals are not defined.

Thus, The Circulation Theorem is another demonstration of the power of the Infinitesimal Calculus, and its advantage over the Calculus of Limits.

Keywords: Infinitesimal, Infinite-Hyper-real, Hyper-real, Cardinal, Infinity. Non-Archimedean, Calculus, Limit, Continuity, Derivative, Integral, Gradient, Divergence, Curl,

2000 Mathematics Subject Classification 26E35; 26E30; 26E15; 26E20; 26A06; 26A12; 03E10; 03E55; 03E17; 03H15;46S20; 97I40; 97I30.

Introduction

The Curl Theorem over an infinitesimal rectangle in the Plane says that the circulation of $\left[P(x,y), \quad Q(x,y)\right]$ around the rectangle depends on one area

$$\oint_{\substack{\text{infinitesimal}\\ \text{rectangle}}} P(x,y)dx + Q(x,y)dy = \begin{vmatrix} \partial_x & \partial_y \\ P & Q \end{vmatrix} dxdy$$

The Curl Theorem Over an infinitesimal 3-box says that the circulation of $\left[P(x,y,z), \quad Q(x,y,z), \quad R(x,y,z)\right]$ around the sides of the 3-box depends on 3 areas.

$$\oint_{\substack{\text{sides of}\\ \text{infinitesimal 3-box}}} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz =$$

$$= \begin{vmatrix} \partial_x & \partial_y \\ P & Q \end{vmatrix} dxdy + \begin{vmatrix} \partial_z & \partial_x \\ R & Q \end{vmatrix} dzdx + \begin{vmatrix} \partial_y & \partial_z \\ Q & R \end{vmatrix} dydz$$

The Curl Theorem Over an infinitesimal 4-box says that the circulation of $\left[P(x,y,z,t), \quad Q(x,y,z,t), \quad R(x,y,z,t), \quad S(x,y,z,t)\right]$ around the sides of the 3-box rectangle depend on 6 areas.

$$\oint_{\substack{\text{sides of}\\ \text{infinitesimal 4-box}}} P(x,y,z,t)dx + Q(x,y,z,t)dy + R(x,y,z,t)dz + S(x,y,z,t)dt =$$

$$= \begin{vmatrix} \partial_z & \partial_t \\ R & S \end{vmatrix} dzdt + \begin{vmatrix} \partial_t & \partial_x \\ S & P \end{vmatrix} dtdx + \begin{vmatrix} \partial_x & \partial_z \\ P & R \end{vmatrix} dxdz +$$

$$+ \begin{vmatrix} \partial_x & \partial_y \\ P & Q \end{vmatrix} dxdy + \begin{vmatrix} \partial_t & \partial_y \\ S & Q \end{vmatrix} dtdy + \begin{vmatrix} \partial_y & \partial_z \\ Q & R \end{vmatrix} dydz$$

The Curl Theorem Over an infinitesimal 5-box says that the circulation of

$$\left[P(x,y,z,t,u), \quad Q(x,y,z,t,u), \quad R(x,y,z,t,u), \quad S(x,y,z,t,u), \quad T(x,y,z,t,u)\right]$$

around the sides of the 5-box rectangle depends on 10 areas.

The number of different areas is the number of combinations of pairs in the space,

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix} = 1, \quad \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 3, \quad \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 6, \quad \begin{pmatrix} 5 \\ 2 \end{pmatrix} = 10, \quad \begin{pmatrix} 6 \\ 2 \end{pmatrix} = 15, \ldots$$

The false belief that the 6 areas in 4-space are all independent, lead to the definition of a 6 dimensional Curl in 4-space. But as we pointed out in [Dan4], the 4-space Curl is 4 dimensional.

While the number of circulation terms in the Curl Theorem equals the space dimension, the number of areas depends on the space dimension indirectly.

Here, we seek, and obtain a Circulation Theorem in which the number of areas involved is the dimension of the space.

The Circulation Theorem is easier to state, and comprehend, than the Curl Theorem.

For space dimension greater than 3, the Circulation Theorem holds, while the Curl Theorem holds only if one accepts an erroneous definition of the Curl.

For instance, in 4 dimensions the Curl is 4 dimensional, and not 6-dimensional

The derivation, and the statement of the Circulation Theorem requires Infinitesimals.

Consequently, the Circulation Theorem does not exist in the Calculus of Limits, where infinitesimals are not defined.

Thus, the Circulation Theorem is another demonstration of the power of the Infinitesimal Calculus, and its advantage over the Calculus of Limits.

We need to use Infinitesimals, Infinitesimal Calculus, and Infinitesimal Vector Calculus.

We have constructed infinitesimals in [Dan1], established the Infinitesimal Calculus in [Dan2], and derived the Infinitesimal Vector Calculus in [Dan3]. The Hyper-real Plane, Hyper-real Vector Functions, Continuity, Derivatives, and Hyper-real Integration are presented in [Dan3].

1.

Hyper-real Plane

We present the Hyper-real 2-Space, which is a cross product of two Hyper-real lines.

Each 2-vector of real numbers (α, β) can be represented by a Cauchy sequence of rational numbers, $(r_1, q_1), (r_2, q_2), (r_3, q_3)...$ so that $(r_n, q_n) \to (\alpha, \beta)$.

The constant sequence $(\alpha, \beta), (\alpha, \beta), (\alpha, \beta)...$ is a constant hyper-real 2-vector.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to $(0, 0)$ sequences of 2-vectors $(\iota_1, o_1)(\iota_2, o_2), (\iota_3, o_3)...$ constitutes a family of infinitesimal hyper-real 2-vectors.
2. The infinitesimal 2-vectors are smaller than any real 2-vector, yet strictly greater than the zero 2-vector.
3. Their reciprocals $(\frac{1}{\iota_1}, \frac{1}{o_1}), (\frac{1}{\iota_2}, \frac{1}{o_2}), (\frac{1}{\iota_3}, \frac{1}{o_3}), ...$ are the infinite hyper-real 2-vectors.

4. The infinite hyper-real 2-vectors are greater than any real 2-vector, yet strictly smaller than the infinity 2-vector.
5. The infinite hyper-real 2-vectors with negative signs are smaller than any real 2-vector, yet strictly greater than $(-\infty, -\infty)$.
6. The sum of a real 2-vector with an infinitesimal 2-vector is a non-constant hyper-real 2-vector.
7. The Hyper-real 2-vectors are the totality of

   constant hyper-real 2-vectors,

   a family of infinitesimal 2-vectors, with signs that may be $(+,+)$, $(+,-)$, $(-,+)$, or $(-,-)$,

   a family of infinite hyper-real 2-vectors with signs that may be $(+,+)$, $(+,-)$, $(-,+)$, or $(-,-)$, and

   non-constant hyper-real 2-vectors.
8. The hyper-real 2-vectors constitute the <u>Hyper-real Plane</u>.
9. That plane includes the real 2-vectors separated by the non-constant hyper-real 2-vectors. Each real 2-vector is the center of a disk of infinitesimal radius of hyper-real 2-vectors, that includes no other real 2-vector.

10. In particular, the zero 2-vector is separated from any real 2-vector by infinitesimal 2-vectors that lie in a disk of infinitesimal radius around the zero.

11. The Zero 2-vector is not an infinitesimal 2-vector, because zero is not strictly greater than zero.

12. We do not add the infinity 2-vector to the hyper-real Plane.

13. The infinitesimal 2-vectors, and the infinite hyper-real 2-vectors, are semi-groups with respect to addition. Neither set includes zero.

14. The hyper-real Plane is embedded in $\mathbb{R}^{\infty} \times \mathbb{R}^{\infty}$, and is not homeomorphic to the real Plane. There is no bi-continuous one-one mapping from the hyper-real Plane onto the real plane.

15. In particular, there are no points in the real Plane that can be assigned uniquely to the infinitesimal hyper-real 2-vectors, or to the infinite hyper-real 2-vectorss, or to the non-constant hyper-real 2-vectors.

16. No neighbourhood of a hyper-real 2-vector is homeomorphic to an $\mathbb{R}^n \times \mathbb{R}^n$ ball. Therefore, the hyper-real plane is not a manifold.

17. The hyper-real plane is not spanned by two elements, and it is not two-dimensional.

2.

Hyper-real Vector Function

2.1 Definition of a hyper-real function

$f(x,y)$ is a hyper-real function, iff it is from the hyper-real 2-vectors into the hyper-reals.

This means that any number in the domain, or in the range of a hyper-real $f(x,y)$ is either one of the following

real vector

real vector + infinitesimal vector

infinitesimal vector

infinite hyper-real vector

Clearly,

2.2 *Every function from the real plane into the reals is a hyper-real function.*

3.

2-Space Path Integral

The definition of Path Integration extends to a vector of two Hyper-real functions

3.1 2-space Path Integral Definition

the Hyper-real Path Integral of $\left[P(x,y), \quad Q(x,y)\right]$ over a path $(x(t),y(t))$, $\tau \in [\alpha,\beta]$, is the sum of the areas

$$\sum_{t\in[\alpha,\beta]}\left\{P(x(t),y(t))dx(t)+Q(x(t),y(t))dy(t)\right\}$$

If for any infinitesimal dt, the Integration Sum equals the same hyper-real number, then $\left[P(x,y), \quad Q(x,y)\right]$ is Hyper-Real Integrable over the path $\gamma(a,b)$.

Then, we call the Integration Sum the Hyper-Real Path Integral of $\left[P(x,y), \quad Q(x,y)\right]$ over $\gamma(a,b)$, and denote it by

$$\int_{\gamma(a,b)} P(x,y)dx+Q(x,y)dy\,.$$

Since there are countably many real numbers in $[\alpha,\beta]$,

5.2 The Integration Sum has countably many terms.

5.3 Continuous $\left[P(x,y), \quad Q(x,y)\right]$ is Path-Integrable

If $P(x,y)$, $Q(x,y)$,*are Continuous on a domain* D

Then $\left[P(x,y), \quad Q(x,y)\right]$ *is Path-Integrable in* D

4.

Hyper-real Area Integral

4.1 Area Integral of $P(x,y)$ Definition

Let $P(x,y)$ be a hyper-real function, defined on a bounded domain in the Hyper-Real Plane.

$P(x,y)$ may take infinite hyper-real values.

An area element is

$$dA = dxdy,$$

For each (x,y), there is an infinitesimal rectangular box with base area $dA = dxdy$, height $P(x,y)$, and volume $P(x,y)dxdy$.

We form the **Double Sum** of all the volumes that are enclosed between the surface of $P(x,y)$, and the $P(x,y)$ domain in the plane

$$\sum_{y=b_1}^{y=b_2} \sum_{x=a_1}^{x=a_2} P(x,y)dxdy.$$

If for any infinitesimals dx, and dy, the Double Sum equals the same hyper-real number, then $P(x,y)$ is Hyper-Real Integrable over the plain domain.

Then, we call the Double Integration Sum the Hyper-Real Area Integral of $P(x,y)$ over the Domain, and denote it by

$$\int_{y=b_1}^{y=b_2}\int_{x=a_1}^{x=a_2} P(x,y)dxdy\,.$$

If the number is an infinite hyper-real, then it equals $\int_{y=b_1}^{y=b_2}\int_{x=a_1}^{x=a_2} P(x,y)dxdy\,.$

If the number is a finite hyper-real, then its constant part equals $\int_{y=b_1}^{y=b_2}\int_{x=a_1}^{x=a_2} P(x,y)dxdy\,.\square$

The Integration Sum may take infinite hyper-real values, such as $\frac{1}{(dx)(dy)}$, but may not equal to ∞.

Since there are countably many real numbers in the plane,

4.2 The Integration Sum is countable.

4.3 Continuous $P(x,y)$ is Area-Integrable

5.

3-Space Surface Integral

5.1 The Surface Area Element

A point on a surface is determined by two parameters u, and v, so that in the x, y, z coordinate system,

$$\vec{r}(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}.$$

At the point $\vec{r}(u,v)$,

the tangent to the surface in the direction of u is

$$\frac{\partial \vec{r}}{\partial u},$$

the tangent to the surface in the direction of v is

$$\frac{\partial \vec{r}}{\partial v},$$

and the normal to the tangent plane is

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{bmatrix} \vec{1}_x & \vec{1}_y & \vec{1}_z \\ \partial_u x & \partial_u y & \partial_u z \\ \partial_v x & \partial_v y & \partial_v z \end{bmatrix}$$

$$= \left(\frac{\partial(y,z)}{\partial(u,v)}, \quad \frac{\partial(z,x)}{\partial(u,v)} \quad \frac{\partial(x,y)}{\partial(u,v)} \right).$$

The unit normal is

$$\vec{1}_n = \frac{\dfrac{\partial \vec{r}}{\partial u} \times \dfrac{\partial \vec{r}}{\partial v}}{\left| \dfrac{\partial \vec{r}}{\partial u} \times \dfrac{\partial \vec{r}}{\partial v} \right|},$$

$$= \begin{bmatrix} (\vec{1}_x, \vec{1}_n) \\ (\vec{1}_y, \vec{1}_n) \\ (\vec{1}_z, \vec{1}_n) \end{bmatrix},$$

$$= \begin{bmatrix} \cos(\vec{1}_x, \vec{1}_n) \\ \cos(\vec{1}_y, \vec{1}_n) \\ \cos(\vec{1}_z, \vec{1}_n) \end{bmatrix}.$$

The surface area element is

$$d\vec{S} = \left(\frac{\partial \vec{r}}{\partial u} du\right) \times \left(\frac{\partial \vec{r}}{\partial v} dv\right)$$

$$= \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} dudv = \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| \vec{1}_n dudv$$

$$= \begin{bmatrix} \dfrac{\partial(y,z)}{\partial(u,v)} dudv \\ \dfrac{\partial(x,z)}{\partial(u,v)} dudv \\ \dfrac{\partial(x,y)}{\partial(u,v)} dudv \end{bmatrix} = \begin{bmatrix} dydz \\ dxdz \\ dxdy \end{bmatrix} = \begin{bmatrix} dS_x \\ dS_y \\ dS_z \end{bmatrix} = \begin{bmatrix} \vec{1}_x \cdot d\vec{S} \\ \vec{1}_y \cdot d\vec{S} \\ \vec{1}_z \cdot d\vec{S} \end{bmatrix}$$

5.2 Surface Integral of $\varphi(x,y,z)$ Definition

Let

$$\varphi(x,y,z) = \varphi(x(u,v), y(u,v), z(u,v))$$
$$= \varphi(u,v),$$

be hyper-real function, defined on a bounded surface $\vec{r}(u,v)$ in the Hyper-Real 3-space.

$\varphi(u,v)$ may take infinite hyper-real values.

For each (u,v), there is an infinitesimal volume

$$\varphi(u,v)\vec{1}_x \cdot d\vec{S} = \varphi(x,y,z)dydz\,.$$

We form the **Double Sum**

$$\sum_{z=c_1}^{z=c_2}\sum_{y=b_1}^{y=b_2}\varphi(x,y,z)dydz\,.$$

If for any infinitesimals dy, and dz, the Double Sum equals the same hyper-real number, then $\varphi(x,y,z)$ is Hyper-Real Integrable over the surface.

Then, we call the Double Integration Sum the Hyper-Real Surface Integral of $\varphi(u,v)$ over the surface, and denote it by

$$\int_{z=c_1}^{z=c_2}\int_{y=b_1}^{y=b_2}\varphi(x,y,z)dydz\,.$$

If the number is an infinite hyper-real, then it equals $\int_{z=c_1}^{z=c_2}\int_{y=b_1}^{y=b_2}\varphi(x,y,z)dydz\,.$

If the number is a finite hyper-real, then its constant part equals $\int\limits_{z=c_1}^{z=c_2}\int\limits_{y=b_1}^{y=b_2}\varphi(x,y,z)dydz$.□

The Integration Sum may take infinite hyper-real values, such as $\frac{1}{(dy)(dz)}$, but may not equal to ∞.

Since there are countably many real numbers in the plane,

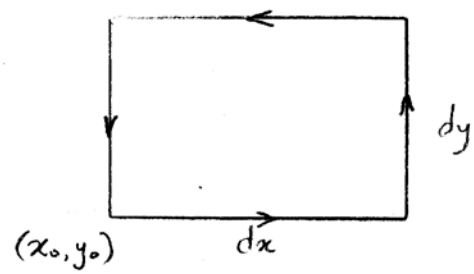
5.3 The Integration Sum is countable.

5.4 Continuous $\varphi(x,y,z)$ is Surface-Integrable.

6.

Plane Curl

Let $P(x,y)$, and $Q(x,y)$, be hyper-real differentiable functions, defined on a rectangle with vertex at (x_0, y_0) and sides dx, dy.



6.1 The Area Curl

The **circulation** of $\left[P(x,y), \quad Q(x,y)\right]$ along the rectangle is

$$\oint_{\text{rectangle}} P(x,y)\vec{1}_x \cdot \vec{1}_l dl + Q(x,y)\vec{1}_y \cdot \vec{1}_l dl .$$

We define the **area density** of that circulation multiplied by $\vec{1}_z$ as the Curl of $\begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}$

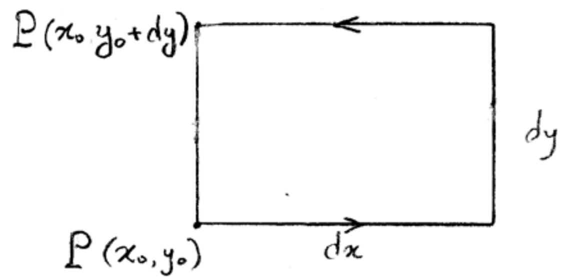
$$\underbrace{\text{Curl}}_{\nabla\times}\begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix} = \vec{1}_z \frac{1}{dxdy} \oint_{\text{rectangle}} P(x,y)\vec{1}_x \cdot \vec{1}_l dl + Q(x,y)\vec{1}_y \cdot \vec{1}_l dl$$

6.2 $$\underbrace{\text{Curl}}_{\nabla\times}\begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix}_{x_0,y_0} = \begin{vmatrix} \partial_x & \partial_y \\ P(x,y) & Q(x,y) \end{vmatrix}_{x_0,y_0} \vec{1}_z$$

Proof:

Since $P(x,y)\vec{1}_x \cdot \vec{1}_l dl$ is nonzero when $\vec{1}_l$ is along the x axis,

the circulation path starts at (x_0, y_0), and ends at $(x_0, y_0 + dy)$



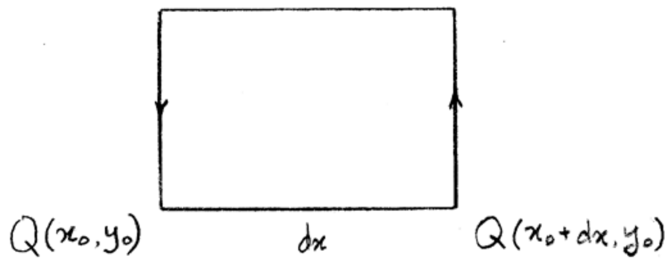
$$\oint_{\text{rectangle}} P(x,y)\vec{1}_x \cdot \vec{1}_l dl = \int_{(x_0,y_0)}^{(x_0,y_0+dy)} P(x,y)\vec{1}_x \cdot \vec{1}_l dl$$

$$= P(x_0,y_0)\vec{1}_x \cdot \vec{1}_x dx + P(x_0, y_0 + y)\vec{1}_x \cdot (-\vec{1}_x) dx$$

$$= \left\{ P(x_0,y_0) - P(x_0,y_0+y) \right\} dx .$$

$$= \left\{ -\frac{\partial P}{\partial y}\bigg|_{x_0,y_0} dy \right\} dx .$$

Since $Q(x,y)\vec{1}_y \cdot \vec{1}_l dl$ is nonzero when $\vec{1}_l$ is along the y axis, the circulation path starts at $(x_0 + dx, y_0)$, and ends at (x_0, y_0)



$$\oint_{\text{rectangle}} Q(x,y)\vec{1}_y \cdot \vec{1}_l dl = \int_{(x_0+dx,y_0)}^{(x_0,y_0)} Q(x,y)\vec{1}_x \cdot \vec{1}_l dl$$

$$= Q(x_0 + dx, y_0)\vec{1}_y \cdot \vec{1}_y dy + P(x_0, y_0)\vec{1}_y \cdot (-\vec{1}_y)dy$$

$$= \left\{ Q(x_0 + dx, y_0) - Q(x_0, y_0) \right\} dy\,.$$

$$= \left\{ \frac{\partial Q}{\partial x}\bigg|_{x_0,y_0} dx \right\} dy\,.$$

Therefore,

$$\underbrace{\text{Curl}}_{\nabla\times}\begin{bmatrix} P(x,y) \\ Q(x,y) \end{bmatrix} = \vec{1}_z \frac{1}{dxdy} \oint_{\text{rectangle}} P(x,y)\vec{1}_x \cdot \vec{1}_l dl + Q(x,y)\vec{1}_y \cdot \vec{1}_l dl$$

$$= \vec{1}_z \frac{1}{dxdy} \left\{ -\frac{\partial P}{\partial y}\bigg|_{x_0,y_0} + \frac{\partial Q}{\partial x}\bigg|_{x_0,y_0} \right\} dxdy$$

$$= (\partial_x Q - \partial_y P)\vec{1}_z\,.\square$$

7.

Plane Curl Theorem

Let $P(x,y)$, and $Q(x,y)$, be hyper-real differentiable functions, defined on a plane domain S, enclosed in the loop ∂S.

7.1 Green's Curl Theorem

$$\oint_{\partial S} P(x,y)\vec{1}_x \cdot \vec{1}_l dl + Q(x,y)\vec{1}_y \cdot \vec{1}_l dl = \iint_S \begin{vmatrix} \partial_x & \partial_y \\ P(x,y) & Q(x,y) \end{vmatrix} dxdy$$

Proof: The sum of the $\left\{\partial_x Q - \partial_y P\right\} dxdy$ over the infinitesimal rectangles enclosed in a plane area S,

equals the sum of the circulations

$$\int_{\text{rectangle}} P(x,y)\vec{1}_x \cdot \vec{1}_l dl + Q(x,y)\vec{1}_y \cdot \vec{1}_l dl$$

over the sides of the infinitesimal rectangles enclosed in the area S,

$$\sum_{y=b_1}^{y=b_2} \sum_{x=a_1}^{x=a_2} \left\{\partial_x Q - \partial_y P\right\} dxdy = \sum_{y=b_1}^{y=b_2} P(x,y)\vec{1}_x \cdot \vec{1}_l dl + \sum_{x=a_1}^{x=a_2} Q(x,y)\vec{1}_y \cdot \vec{1}_l dl$$

The sum over the areas is

$$\iint_S \left\{\partial_x Q(x,y) - \partial_y P(x,y)\right\} dxdy.$$

The Interior path integrals of

$$\int_{\text{rectangle}} P(x,y)\vec{1}_x \cdot \vec{1}_l dl\,, \quad \text{and} \quad \int_{\text{rectangle}} Q(x,y)\vec{1}_y \cdot \vec{1}_l dl\,,$$

appear in pairs of opposite signs and cancel, leaving the Integral over the boundary line,

$$\oint_{\partial S} P(x,y)\vec{1}_x \cdot \vec{1}_l dl + Q(x,y)\vec{1}_y \cdot \vec{1}_l dl = \iint_{S} \left\{ \partial_x Q - \partial_y P \right\} dxdy\,.\square$$

8.

The Infinitesimal Circulation Theorem

Let $P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$ be hyper-real differentiable functions, defined on an infinitesimal area element dS.

8.1 Infinitesimal Circulation Theorem in 3-Space

$$\oint_{\partial(dS)} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = \oiint_{dS} dPdx + dQdy + dRdz$$

Proof:

A point on dS is determined by two parameters u, and v, that define a uv plane

Hence,

$$x = x(u,v),\ \ y = y(u,v),\ \ z = z(u,v)$$

$$P(x,y,z) = P(x(u),y(u),z(u)) = P(u,v)$$

$$Q(x,y,z) = Q(x(u),y(u),z(u)) = Q(u,v)$$

$$R(x,y,z) = R(x(u),y(u),z(u)) = R(u,v)$$

Thus,

$$P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz =$$

$$= P(u,v)[x_u du + x_v dv] + Q(u,v)[y_u du + y_v dv] + R(u,v)[z_u du + z_v dv]$$

$$= \left\{ P(u,v)x_u + Q(u,v)y_u + R(u,v)z_u \right\} du + \left\{ P(u,v)x_v + Q(u,v)y_v + R(u,v)z_v \right\} dv$$

Therefore,

$$\oint_{\partial(dS)} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz =$$

$$= \oint_{\partial(dS)} \left\{ P(u,v)x_u + Q(u,v)y_u + R(u,v)z_u \right\} du + \left\{ P(u,v)x_v + Q(u,v)y_v + R(u,v)z_v \right\} dv$$

By the Curl Theorem in the plane,

$$= \oiint_{dS} \begin{vmatrix} \partial_u & \partial_v \\ P(u,v)x_u + Q(u,v)y_u + R(u,v)z_u & P(u,v)x_v + Q(u,v)y_v + R(u,v)z_v \end{vmatrix} dudv$$

Now,

$$\begin{vmatrix} \partial_u & \partial_v \\ Px_u + Qy_u + Rz_u & P(u,v)x_v + Qy_v + Rz_v \end{vmatrix} dudv =$$

$$= \left\{ \left[P_u x_v + Px_{vu} + Q_u y_v + Qy_{vu} + R_u z_v + Rz_{vu} \right] \right.$$

$$\left. - \left[P_v x_u + Px_{uv} + Q_v y_u + Qy_{uv} + R_v z_u + Rz_{uv} \right] \right\} dudv =$$

$$= \underbrace{(P_u x_v - P_v x_u)}_{\frac{\partial(P,x)}{\partial(u,v)}} dudv + \underbrace{(Q_u y_v - Q_v y_u)}_{\frac{\partial(Q,y)}{\partial(u,v)}} dudv + \underbrace{(R_u z_v - R_v z_u)}_{\frac{\partial(R,z)}{\partial(u,v)}} dudv$$

$$= \frac{\partial(P,x)}{\partial(u,v)} dudv + \frac{\partial(Q,y)}{\partial(u,v)} dudv + \frac{\partial(R,z)}{\partial(u,v)} dudv$$

$$= dPdx + dQdy + dRdz.$$

Consequently,

$$\oint_{\partial(dS)} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = \oiint_{dS} dPdx + dQdy + dRdz\,.\square$$

The Proof shows that the Circulation Theorem holds for any number of any dimensions in this form.

That is,

8.2 Infinitesimal Circulation Theorem in n-Space

$$\oint_{\partial(dS)} P_i(x_1,\ldots x_n)dx_i = \oiint_{dS} dP_i dx_i\,, \qquad \textit{for any } n \geq 2$$

9.

Meaning of $\oiint\limits_{dS} dPdx + dQdy$

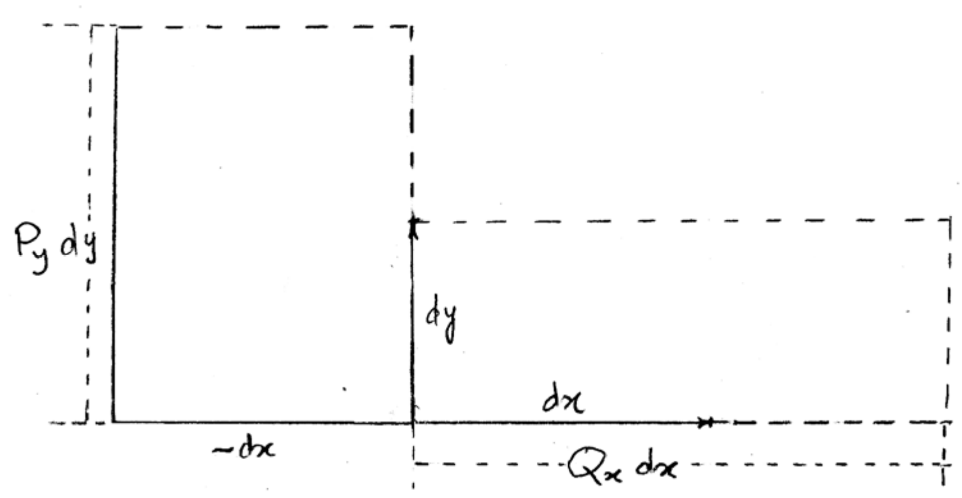
Under positive right hand orientation, the infinitesimals are anti-commutative. That is,

$$dxdy = -dydx\,.$$

And

$$dxdx = 0\,, \quad dydy = 0\,.$$

$$\begin{aligned} dPdx + dQdy &= (P_x dx + P_y dy)dx + (Q_x dx + Q_y dy)dy \\ &= P_x \underbrace{dxdx}_{0} + P_y \underbrace{dydx}_{-dxdy} + Q_x dxdy + Q_y \underbrace{dydy}_{0} \\ &= Q_x dxdy - P_y dxdy \end{aligned}$$



10.

Meaning of $\oiint\limits_{dS} dPdx + dQdy + dRdz$

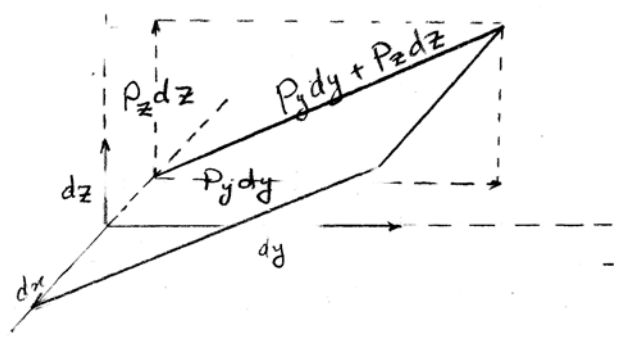
Under positive right hand orientation, the infinitesimals are anti-commutative. That is,

$$dydz = -dzdy\,, \quad dzdx = -dxdz\,, \quad dxdy = -dydx\,.$$

And

$$dxdx = 0\,, \quad dydy = 0\,, \quad dzdz = 0\,.$$

$$dPdx = (P_x dx + P_y dy + P_z dz)dx$$



That is,

10.1 $P_y dy + P_z dz$ *is the diagonal in the rectangle with sides* $P_y dy$, *and* $P_z dz$. *That diagonal is perpendicular to* dx

$dPdx$ *is the area of the rectangle with sides* dx, *and* $P_y dy + P_z dz$.

10.2 *The Infinitesimal 3-Space Circulation Theorem implies the Infinitesimal 3-Space Curl Theorem.*

Proof:

$$dPdx + dQdy + dRdz = (P_x dx + P_y dy + P_z dz)dx +$$

$$+(Q_x dx + Q_y dy + Q_z dz)dy +$$

$$+(R_x dx + R_y dy + R_z dz)dz$$

$$= P_x \underbrace{dxdx}_{0} + P_y \underbrace{dydx}_{-dxdy} + P_z dzdx +$$

$$+Q_x dxdy + Q_y \underbrace{dydy}_{0} + Q_z \underbrace{dzdy}_{-dydz} +$$

$$+R_x \underbrace{dxdz}_{-dzdx} + R_y dydz + R_z \underbrace{dzdz}_{0}$$

$$= (Q_x - P_y)dxdy + (P_z - R_x)dzdx + (R_y - Q_z)dydz$$

11.

Meaning of $\oint\!\!\oint_{dS} dPdx + dQdy + dRdz + dSdt$

Under positive right hand orientation, the infinitesimals are anti-commutative. That is,

$$dydz = -dzdy\,, \quad dzdx = -dxdz\,, \quad dxdy = -dydx\,,$$

$$dzdy = -dydz\,, \quad dzdt = -dtdz\,, \quad dtdx = -dxdt\,.$$

$$dxdx = 0\,, \quad dydy = 0\,, \quad dzdz = 0\,, \quad dtdt = 0$$

$$dPdx = (P_x dx + P_y dy + P_z dz + P_t dt)dx$$

Therefore,

11.1 $P_y dy + P_z dz + P_t dt$ *is the diagonal in the box with sides* $P_y dy$, $P_z dz$, *and* $P_t dt$. *That diagonal is perpendicular to* dx

$dPdx$ *is the area of the rectangle with sides* dx, *and* $P_y dy + P_z dz + P_t dt$.

11.2 *The Infinitesimal 4-Space Circulation Theorem implies the Infinitesimal 4-Space Curl Theorem.*

Proof:

$$dPdx + dQdy + dRdz + dSdt = (P_x dx + P_y dy + P_z dz + P_t dt)dx +$$

$$+(Q_x dx + Q_y dy + Q_z dz + Q_t dt)dy +$$

$$+(R_x dx + R_y dy + R_z dz + R_t dt)dz +$$

$$+(S_x dx + S_y dy + S_z dz + S_t dt)dt$$

$$= P_x \underbrace{dxdx}_{0} + P_y \underbrace{dydx}_{-dxdy} + P_z dzdx + P_t dtdx +$$

$$+Q_x dxdy + Q_y \underbrace{dydy}_{0} + Q_z \underbrace{dzdy}_{-dydz} + Q_t \underbrace{dtdy}_{-dydt} +$$

$$+R_x \underbrace{dxdz}_{-dzdx} + R_y dydz + R_z \underbrace{dzdz}_{0} + R_t \underbrace{dtdz}_{-dzdt} +$$

$$+S_x \underbrace{dxdt}_{-dtdx} + S_y dydt + S_z dzdt + S_t \underbrace{dtdt}_{0}$$

$$= (Q_x - P_y)dxdy + (P_z - R_x)dzdx + (R_y - Q_z)dydz$$

$$+(P_t - S_x)dtdx + (S_y - Q_t)dydt + (S_z - R_t)dzdt$$

12.

Circulation Theorem

We shall state, and prove the theorem for $n = 3$. The proof extends to any $n \geq 2$

Let $P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$, be hyper-real differentiable functions, defined on a surface Σ, with closed boundary $\partial\Sigma$.

8.1 3-Space Circulation Theorem

$$\oint_{\partial\Sigma} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = \oiint_{\Sigma} dPdx + dQdy + dRdz$$

Proof: The sum of the $dPdx + dQdy + dRdz$ over the infinitesimal rectangles enclosed in the plane area Σ,

equals the sum of the circulations

$$\oint_{\substack{\text{sides of rectangles}\\ \text{bounding 3-box}}} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz.$$

That is,

$$\sum_{z=c_1}^{z=c_2} \sum_{y=b_1}^{y=b_2} \sum_{x=a_1}^{x=a_2} dPdx + dQdy + dRdz =$$

$$= \sum_{x=a_1}^{x=a_2} \oint_{\substack{\text{sides of} \\ \text{rectangles}}} P(x,y,z)dx + \sum_{y=b_1}^{y=b_2} \oint_{\substack{\text{sides of} \\ \text{rectangles}}} Q(x,y,z)dy + \sum_{z=c_1}^{z=c_2} \oint_{\substack{\text{sides of} \\ \text{rectangles}}} R(x,y,z)dz \,.$$

The sum over the areas is

$$\oiint_{\Sigma} dPdx + dQdy + dRdz \,.$$

The Interior path integrals of

$$\sum_{x=a_1}^{x=a_2} \oint_{\substack{\text{sides of} \\ \text{rectangles}}} P(x,y,z)dx + \sum_{y=b_1}^{y=b_2} \oint_{\substack{\text{sides of} \\ \text{rectangles}}} Q(x,y,z)dy + \sum_{z=c_1}^{z=c_2} \oint_{\substack{\text{sides of} \\ \text{rectangles}}} R(x,y,z)dz$$

appear in pairs of opposite signs and cancel, leaving the Integral over the boundary line,

$$\oint_{\partial\Sigma} P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = \oiint_{\Sigma} dPdx + dQdy + dRdz$$

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