Expansions of $\theta = \tan\alpha$ in Powers of $\arctan\theta = \alpha$

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Abstract: We expand $\theta = \tan\alpha$ in powers of $\arctan\theta = \alpha$.

$$\boxed{\begin{aligned}\pi = \tfrac{1}{2}\{&(\arctan 2\pi) + \tfrac{1}{3}(\arctan 2\pi)^3 + \tfrac{2}{15}(\arctan 2\pi)^5 + \tfrac{17}{315}(\arctan 2\pi)^7\\ &+ \tfrac{62}{2835}(\arctan 2\pi)^9 + \tfrac{(819)(691)}{3^6 5^2 7(11)(91)}(\arctan 2\pi)^{11}\\ &+ \tfrac{5461}{3^5 5^2 7(11)(13)}(\arctan 2\pi)^{13}(2\pi) + ... + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!}B_n\alpha^{2n-1}(2\pi) + ...\}\end{aligned}}$$

$$\boxed{\begin{aligned}e = &\arctan e + \tfrac{1}{3}(\arctan e)^3 + \tfrac{2}{15}(\arctan e)^5 + \tfrac{17}{315}(\arctan e)^7\\ &+ \tfrac{62}{2835}(\arctan e)^9 + \tfrac{(819)(691)}{3^6 5^2 7(11)(91)}(\arctan e)^{11}(e) +\\ &+ \tfrac{5461}{3^5 5^2 7(11)(13)}(\arctan e)^{13}(e) + ... + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!}B_n(\arctan e)^{2n-1} + ...\end{aligned}}$$

$$\boxed{\begin{aligned}1 = &\tfrac{\pi}{4} + \tfrac{1}{3}(\tfrac{\pi}{4})^3 + \tfrac{2}{15}(\tfrac{\pi}{4})^5 + \tfrac{17}{315}(\tfrac{\pi}{4})^7 + \tfrac{62}{2835}(\tfrac{\pi}{4})^9 +\\ &+ \tfrac{(819)(691)}{3^6 5^2 7(11)(91)}(\tfrac{\pi}{4})^{11} + \tfrac{5461}{3^5 5^2 7(11)(13)}(\tfrac{\pi}{4})^{13} + ... + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!}B_n(\tfrac{\pi}{4})^{2n-1} + ...\end{aligned}}$$

where B_n = Bernoulli Numbers.

1.

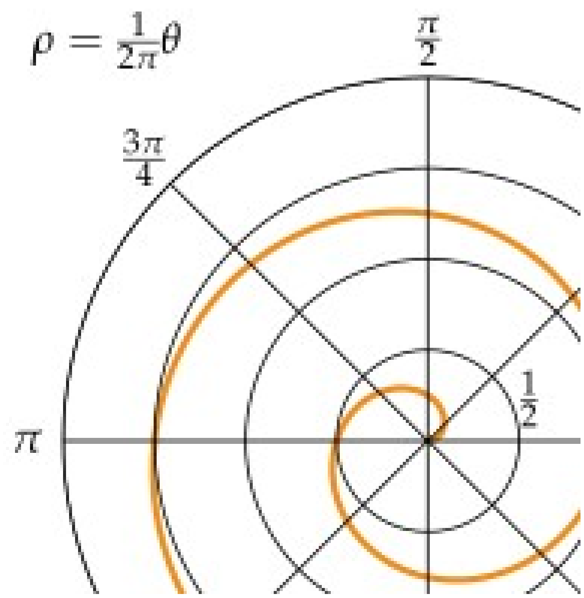
π in powers of $\arctan(2\pi)$

Archimedes Spiral[1] in polar coordinates (ρ, θ) is given by

$$\rho(\theta) = \frac{1}{2\pi}\theta .$$

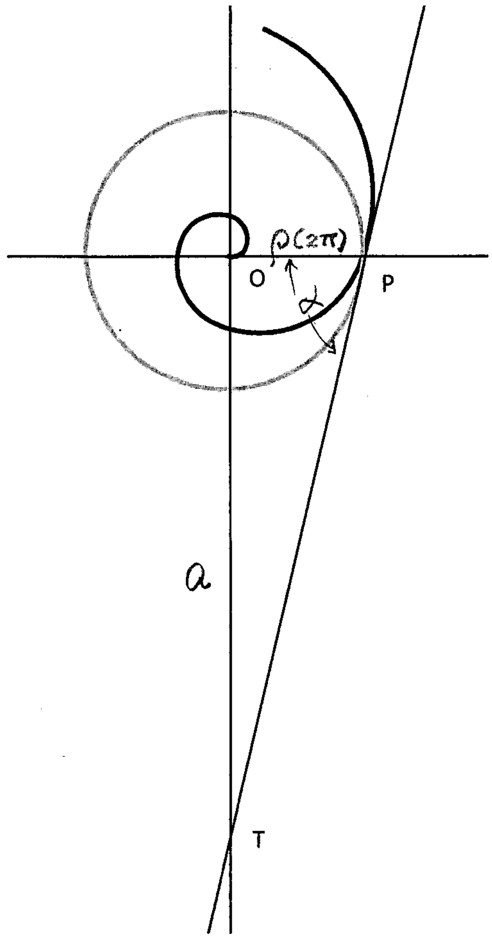
$$\Rightarrow \rho(2\pi) = 1$$

Hence, the circumference of the unit circle is 2π.



The tangent to the spiral at $\theta = 2\pi$, at the point $P(2\pi)$, intersects the y axis at the point $T(2\pi)$.

[1] https://en.wikipedia.org/wiki/Archimedean_spiral#:~:text=The%20Archimedean%20spiral%20(also%20known,rotates%20with%20constant%20angular%20velocity.



Archimedes proved[2] that

$$a = \textbf{\textit{the circumference of the circle of radius }} \rho(2\pi) = 1$$

That is,

$$\boxed{a = 2\pi \underbrace{\rho(2\pi)}_{1} = 2\pi}$$

Then,

$$\boxed{\tan\alpha = \frac{a}{\rho(2\pi)} = \frac{2\pi}{1} = 2\pi}$$

$$2\pi = \tan\alpha = \tan(\arctan 2\pi)$$

$$\pi = \frac{1}{2}\tan(\arctan 2\pi)$$

[2] Archimedes, "On Spirals", Cambridge University Press.

Since $\arctan 2\pi < \dfrac{1}{2}\pi$, we can expand $\tan(\arctan 2\pi)$ in $\arctan 2\pi$,

$$\boxed{\begin{aligned}\pi = \tfrac{1}{2}\{&(\arctan 2\pi) + \tfrac{1}{3}(\arctan 2\pi)^3 + \tfrac{2}{15}(\arctan 2\pi)^5 + \tfrac{17}{315}(\arctan 2\pi)^7 \\ &+ \tfrac{62}{2835}(\arctan 2\pi)^9 + \tfrac{(819)(691)}{3^6 5^2 7(11)(91)}(\arctan 2\pi)^{11} \\ &+ \tfrac{5461}{3^5 5^2 7(11)(13)}(\arctan 2\pi)^{13}(2\pi) + ... + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!} B_n \alpha^{2n-1}(2\pi) + ...\}\end{aligned}}$$

where $\boxed{\arctan(2\pi) \approx 1.412965137...}$

and B_n = Bernoulli Numbers

2. The Convergence Speed of the Series for π

$$B_1 = \tfrac{1}{6} \Rightarrow \quad \tfrac{1}{2}\alpha(2\pi) \approx 0.706482568$$

$$B_2 = \tfrac{1}{30} \Rightarrow \quad \tfrac{1}{6}\alpha^3(2\pi) \approx 0.470157196$$

$$B_3 = \tfrac{1}{42} \Rightarrow \quad \tfrac{1}{15}\alpha^5(2\pi) \approx 0.375461985$$

$$B_4 = \tfrac{1}{30} \Rightarrow \quad \tfrac{17}{630}\alpha^7(2\pi) \approx 0.303409025$$

$$B_5 = \tfrac{5}{66} \Rightarrow \quad \tfrac{31}{2835}\alpha^9(2\pi) \approx 0.24546617$$

$$B_6 = \tfrac{691}{2730} \Rightarrow \quad \tfrac{(819)(691)}{3^6 5^2 7(11)(91)}\alpha^{11}(2\pi) \approx 0.198613243$$

$$B_7 = \tfrac{7}{6} \Rightarrow \quad \tfrac{5461}{3^5 5^2 7(11)(13)}\alpha^{13}(2\pi) \approx 0.080352726$$

$$S_7 \approx 2.379942913$$

And the error in S_7 is

$$\left|\pi - S_7\right| \approx \left|3.141592654 - 2.379942913\right| \approx 0.76164974$$

In comparison, for the Leibniz Series of reciprocals for π,

$$S_7 = 4\{1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \tfrac{1}{13}\} \approx 3.283738484$$

And the error in S_7 is

$$\left|\pi - S_7\right| \approx \left|3.141592654 - 3.283738484\right| \approx 0.14214583$$

2.

Expansions of e, 1, $\sqrt{3}$, $\frac{1}{\sqrt{3}}$, 3,

2.1 e in Powers of $\arctan(e)$

Since $\arctan(e) \approx 1.218282905... < \frac{1}{2}\pi$

we can expand $\mathrm{e} = \tan\left(\arctan(\mathrm{e})\right)$ in powers of $\arctan(e)$

$$\boxed{\begin{aligned} e = &\arctan e + \tfrac{1}{3}(\arctan e)^3 + \tfrac{2}{15}(\arctan e)^5 + \tfrac{17}{315}(\arctan e)^7 \\ &+ \tfrac{62}{2835}(\arctan e)^9 + \tfrac{(819)(691)}{3^6 5^2 7(11)(91)}(\arctan e)^{11}(e) + \\ &+ \tfrac{5461}{3^5 5^2 7(11)(13)}(\arctan e)^{13}(e) + ... + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!}B_n(\arctan e)^{2n-1} + ... \end{aligned}}$$

where B_n = Bernoulli Numbers

2.2 1 in powers of $\arctan(1) = \dfrac{\pi}{4}$

$$\boxed{\begin{aligned} 1 = &\tfrac{\pi}{4} + \tfrac{1}{3}\left(\tfrac{\pi}{4}\right)^3 + \tfrac{2}{15}\left(\tfrac{\pi}{4}\right)^5 + \tfrac{17}{315}\left(\tfrac{\pi}{4}\right)^7 + \tfrac{62}{2835}\left(\tfrac{\pi}{4}\right)^9 + \\ &+ \tfrac{(819)(691)}{3^6 5^2 7(11)(91)}\left(\tfrac{\pi}{4}\right)^{11} + \tfrac{5461}{3^5 5^2 7(11)(13)}\left(\tfrac{\pi}{4}\right)^{13} + ... + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!}B_n\left(\tfrac{\pi}{4}\right)^{2n-1} + ... \end{aligned}}$$

2.3 $\sqrt{3}$ in powers of $\arctan\sqrt{3} = \dfrac{\pi}{3}$

$$\boxed{\begin{aligned}\sqrt{3} &= \tfrac{\pi}{3} + \tfrac{1}{3}\left(\tfrac{\pi}{3}\right)^3 + \tfrac{2}{15}\left(\tfrac{\pi}{3}\right)^5 + \tfrac{17}{315}\left(\tfrac{\pi}{3}\right)^7 + \tfrac{62}{2835}\left(\tfrac{\pi}{3}\right)^9 + \\ &+ \tfrac{(819)(691)}{3^6 5^2 7(11)(91)}\left(\tfrac{\pi}{3}\right)^{11} + \tfrac{5461}{3^5 5^2 7(11)(13)}\left(\tfrac{\pi}{3}\right)^{13} + \dots + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!} B_n \left(\tfrac{\pi}{3}\right)^{2n-1} + \dots\end{aligned}}$$

2.4 $\dfrac{1}{\sqrt{3}}$ in Powers of $\arctan\dfrac{1}{\sqrt{3}} = \dfrac{\pi}{6}$

$$\boxed{\begin{aligned}\tfrac{1}{\sqrt{3}} &= \tfrac{\pi}{6} + \tfrac{1}{3}\left(\tfrac{\pi}{6}\right)^3 + \tfrac{2}{15}\left(\tfrac{\pi}{6}\right)^5 + \tfrac{17}{315}\left(\tfrac{\pi}{6}\right)^7 + \tfrac{62}{2835}\left(\tfrac{\pi}{6}\right)^9 + \\ &+ \tfrac{(819)(691)}{3^6 5^2 7(11)(91)}\left(\tfrac{\pi}{6}\right)^{11} + \tfrac{5461}{3^5 5^2 7(11)(13)}\left(\tfrac{\pi}{6}\right)^{13} + \dots + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!} B_n \left(\tfrac{\pi}{6}\right)^{2n-1} + \dots\end{aligned}}$$

Therefore,

2.5

$$\boxed{\sqrt{3} = \frac{1}{\tfrac{\pi}{6} + \tfrac{1}{3}\left(\tfrac{\pi}{6}\right)^3 + \tfrac{2}{15}\left(\tfrac{\pi}{6}\right)^5 + \tfrac{17}{315}\left(\tfrac{\pi}{6}\right)^7 + \tfrac{62}{2835}\left(\tfrac{\pi}{6}\right)^9 + \dots}}$$

$$\boxed{3 = \frac{\tfrac{\pi}{3} + \tfrac{1}{3}\left(\tfrac{\pi}{3}\right)^3 + \tfrac{2}{15}\left(\tfrac{\pi}{3}\right)^5 + \tfrac{17}{315}\left(\tfrac{\pi}{3}\right)^7 + \tfrac{62}{2835}\left(\tfrac{\pi}{3}\right)^9 + \dots}{\tfrac{\pi}{6} + \tfrac{1}{3}\left(\tfrac{\pi}{6}\right)^3 + \tfrac{2}{15}\left(\tfrac{\pi}{6}\right)^5 + \tfrac{17}{315}\left(\tfrac{\pi}{6}\right)^7 + \tfrac{62}{2835}\left(\tfrac{\pi}{6}\right)^9 + \dots}}$$

$$\boxed{\begin{aligned}1 &= \left\{\tfrac{\pi}{6} + \tfrac{1}{3}\left(\tfrac{\pi}{6}\right)^3 + \tfrac{2}{15}\left(\tfrac{\pi}{6}\right)^5 + \tfrac{17}{315}\left(\tfrac{\pi}{6}\right)^7 + \tfrac{62}{2835}\left(\tfrac{\pi}{6}\right)^9 + \dots\right\} \times \\ &\times \left\{\tfrac{\pi}{3} + \tfrac{1}{3}\left(\tfrac{\pi}{3}\right)^3 + \tfrac{2}{15}\left(\tfrac{\pi}{3}\right)^5 + \tfrac{17}{315}\left(\tfrac{\pi}{3}\right)^7 + \tfrac{62}{2835}\left(\tfrac{\pi}{3}\right)^9 + \dots\right\}\end{aligned}}$$

References

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