# Volume of $x$ Dimensional Set

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## Abstract

The volume of the $x$-dimensional unit ball is $\boxed{V(x) = \dfrac{\pi^{\frac{1}{2}x}}{\Gamma(\frac{1}{2}x+1)}}$.

And the surface area of its Unit Sphere is $\boxed{x\dfrac{\pi^{\frac{1}{2}x}}{\Gamma(\frac{1}{2}x+1)}}$

| Fractal set | Dimension | Volume of Unit Ball | Area of Unit Sphere |
|---|---|---|---|
| Cantor Set | $\dfrac{\log 2}{\log 3} \approx 0.63$ | $\dfrac{\pi^{\frac{1}{2}\frac{\log 2}{\log 3}}}{\Gamma(\frac{1}{2}\frac{\log 2}{\log 3}+1)}$ | $\frac{\log 2}{\log 3}\dfrac{\pi^{\frac{1}{2}\frac{\log 2}{\log 3}}}{\Gamma(\frac{1}{2}\frac{\log 2}{\log 3}+1)}$ |
| Koch Snow-flake | $\dfrac{\log 4}{\log 3} \approx 1.26$ | $\dfrac{\pi^{\frac{1}{2}\frac{\log 4}{\log 3}}}{\Gamma(\frac{1}{2}\frac{\log 4}{\log 3}+1)}$ | $\frac{\log 4}{\log 3}\dfrac{\pi^{\frac{1}{2}\frac{\log 4}{\log 3}}}{\Gamma(\frac{1}{2}\frac{\log 4}{\log 3}+1)}$ |
| Measure Theory claim | 1 | $\infty$ | 0 |
| Sierpinski Triangle | $\dfrac{\log 3}{\log 2} \approx 1.585$ | $\dfrac{\pi^{\frac{1}{2}\frac{\log 3}{\log 2}}}{\Gamma(\frac{1}{2}\frac{\log 3}{\log 2}+1)}$ | $\frac{\log 3}{\log 2}\dfrac{\pi^{\frac{1}{2}\frac{\log 3}{\log 2}}}{\Gamma(\frac{1}{2}\frac{\log 3}{\log 2}+1)}$ |
| | | | |

In passing, we note that these expose more falsehoods[1] of the Lebesgue Measure theory:

In the Irreparably Flawed Lebesgue Theory,

the Cantor Set has zero dimension, and zero volume,

The Koch Snowflake has dimension 1, But infinite distance between any two points, and infinite volume.

The Sierpinski Triangle has infinite distance between any two points. Thus, infinite volume. And it has zero area. Thus, zero volume.

[1] Other falsehoods of Lebesgue Theory (a) The Un-measurable set of Rationals in (0,1) has measure zero, (b) The Un-measurable set of Irrationals in (0,1) has measure 1, (c) $\infty \cdot 0 = 0$ (d) At integration, Singularities can be ignored, (e) The Cauchy Residue Theorem is Banks.

# 1.

# Volume of $k$ Dimensional Set

In the Euclidean plane, points are tips of vectors that start at the origin, and point in any direction in the plane.

Then, each point is represented by its two coordinates $(x, y)$, and the plane has dimension 2.

The Unit Ball is the disk

$$x^2 + y^2 \leq 1, \text{ with area } \pi,$$

and the Unit Sphere is the circle

$$x^2 + y^2 = 1, \text{ with length } 2\pi.$$

In the Euclidean 3-space, points are tips of vectors that start at the origin, and point in any direction in the space.

Then, each point is represented by its three coordinates $(x, y, z)$, and the 3-space has dimension 3.

The Unit Ball is the ball

$$x^2 + y^2 + z^2 \leq 1, \text{ with volume } \tfrac{4}{3}\pi,$$

and the unit sphere is the sphere

$$x^2 + y^2 + z^2 = 1, \text{ with area } 4\pi.$$

In the Euclidean $k$-space, points are tips of vectors that start at the origin, and point in any direction in the k-space.

Then, each point is represented by its $k$ coordinates $(x_1, x_2, ..., x_k)$, and the $k$-space has dimension $k$.

The Euclidean Unit Ball is

$$x_1^2 + x_2^2 + ... + x_k^2 \leq 1,$$

with volume[2]

$$V(k) = \frac{\pi^{\frac{1}{2}k}}{\Gamma(\frac{1}{2}k + 1)}.$$

Therefore,

$$V(2l) = \frac{\pi^l}{l!}$$

$$V(2l+1) = \frac{\pi^{l+\frac{1}{2}}}{\Gamma(l+1+\frac{1}{2})}$$

$$= \frac{\pi^{l+\frac{1}{2}}}{(l+\frac{1}{2})(l-\frac{1}{2})(l-\frac{3}{2})...(\frac{1}{2})\pi^{\frac{1}{2}}}$$

$$= \frac{2^{l+1}}{(2l+1)\cdot ... \cdot 7 \cdot 5 \cdot 3}\pi^l.$$

The volume of the ball centered at the origin with radius $R$ is

$$V(k, R) = V(k)R^k$$

The Euclidean Unit Sphere is

$$x_1^2 + x_2^2 + ... + x_k^2 = 1.$$

with area

$$A(k-1)$$

The area of a spherical shell with radius $R$ is

[2] https://en.wikipedia.org/wiki/Volume_of_an_n-ball

$$A(k-1,R) = \frac{dV(k,R)}{dR} = kV(k)R^{k-1}$$

Therefore, the area of the unit $k$-sphere is

$$A(k-1) = kV(k).$$

A short Table,

| Dimension | Unit Ball Volume | Unit Sphere Area |
|---|---|---|
| $k=2$ | $\pi$ | $2\pi$ |
| $k=3$ | $\frac{4}{3}\pi$ | $4\pi$ |
| $k=4$ | $\frac{1}{2!}\pi^2$ | $2\pi^2$ |
| $k=5$ | $\frac{2^3}{5\cdot 3}\pi^2$ | $\frac{2^3}{3}\pi^2$ |
| $k=6$ | $\frac{1}{3!}\pi^3$ | $\pi^3$ |
| $k=7$ | $\frac{2^4}{7\cdot 5\cdot 3}\pi^3$ | $\frac{2^4}{5\cdot 3}\pi^3$ |
| $k=8$ | $\frac{1}{4!}\pi^4$ | $\frac{1}{3}\pi^4$ |
| $k=9$ | $\frac{2^5}{9\cdot 7\cdot 5\cdot 3}\pi^4$ | $\frac{2^5}{7\cdot 5\cdot 3}\pi^4$ |
| $k=10$ | $\frac{1}{5!}\pi^5$ | $\frac{1}{4!}\pi^5$ |
| $k=11$ | $\frac{2^6}{11\cdot 9\cdot 7\cdot 5\cdot 3}\pi^5$ | $\frac{2^6}{9\cdot 7\cdot 5\cdot 3}\pi^5$ |

The Euclidean dimension is the number of algebraically independent vectors.

The dimension of Fractal sets requires infinitesimals.

# 2.

# Fractal Dimension

To determine the dimension $d$ of a Fractal set of volume 1, we cover the set with infinitesimal cubes of side $\varepsilon$.

The volume of each cube is

$$\varepsilon^d.$$

And the number of cubes in the cover is

$$N(\varepsilon) = \frac{1}{\varepsilon^d}.$$

Hence, the Fractal Dimension is

$$\boxed{d = \frac{\log N(\varepsilon)}{\log(\frac{1}{\varepsilon})}}$$

**2.1 The fractal dimension of a Euclidean set equals its Euclidean dimension.**

*Proof*: A $k$-dimensional Euclidean set can be covered by infinitesimal cubes of volume $\varepsilon^k$.

If the set has volume 1, then the number of cubes in the cover is

$$N(\varepsilon) = \frac{1}{\varepsilon^k}.$$

And

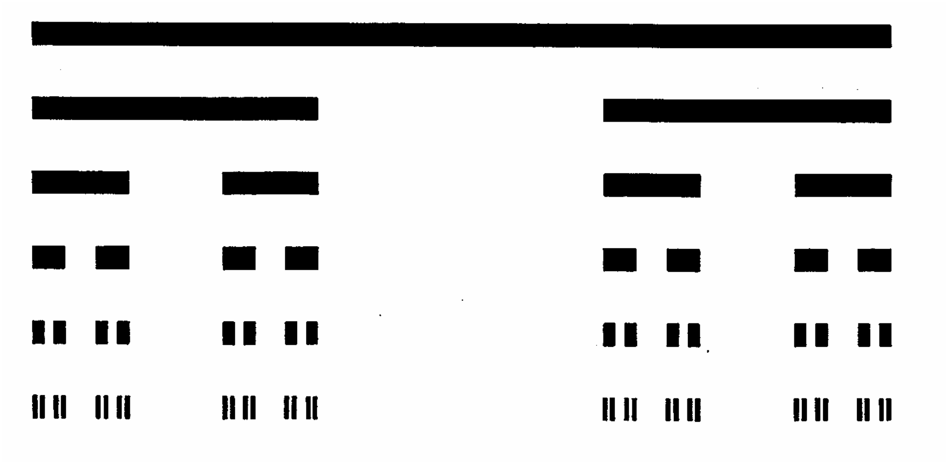
$$d = \frac{\log(\frac{1}{\varepsilon^k})}{\log(\frac{1}{\varepsilon})} = k.\square$$

Every portion of a fractal curve includes an infinite sequence of scaled identical curves. So the distance between two points on the curve is infinite. The dimension of the curve measures the extent to which it fills the area more than it fills a line.

Every portion of a fractal surface is composed of an infinite sequence of scaled identical surfaces. The dimension of the surface measures the extent to which it fills the volume more than it fills a surface.

# 3.

# The Cantor Set



In the first iteration, two intervals are left, Each has length $\frac{1}{3}$

$$N_1 = 2, \quad \varepsilon_1 = \frac{1}{3}$$

In the second iteration, four intervals are left. Each has length $\frac{1}{3^2}$

$$N_2 = 2^2, \quad \varepsilon_2 = \frac{1}{3^2}$$

In third iteration, eight intervals are left. Each as length $\frac{1}{3^3}$

$$N_3 = 2^3, \quad \varepsilon_3 = \frac{1}{3^3}$$

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Let $K$ = infinite hyper real. In the $K^{th}$ iteration,

$$N_K = 2^K, \quad \varepsilon_K = \frac{1}{3^K}.$$

$$d = \frac{\log N_K}{\log(\frac{1}{\varepsilon_K})} = \frac{\log 2^K}{\log 3^K} = \frac{\log 2}{\log 3} \approx 0.63$$

is the extent to which the Cantor set fills a line more than it fills a point.

Clearly, at dimension of 0.63, the Cantor set fills a line more than it fills a point.

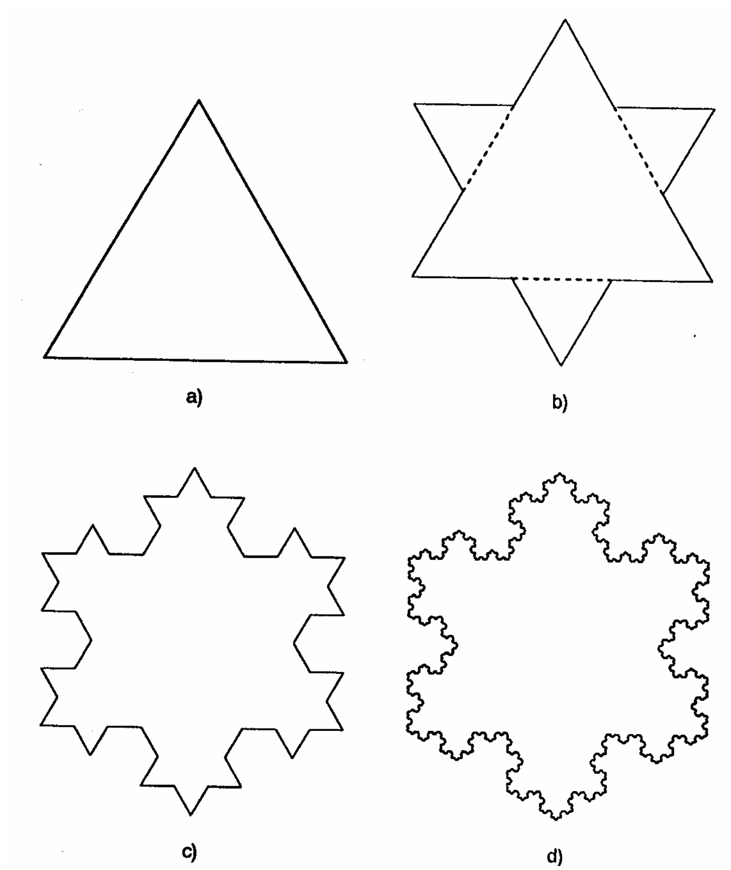
This contradicts Measure Theory conception of the Cantor Set as having zero length.

In fact, the volume of the Unit Ball here is

$$\frac{\pi^{\frac{1}{2}\frac{\log 2}{\log 3}}}{\Gamma(\frac{1}{2}\frac{\log 2}{\log 3}+1)}.$$

# 4.

# The Koch Snowflake




In the first iteration, (b), each side becomes 4 segments. Each segment has length $\frac{1}{3}$

$$N_1 = 4, \quad \varepsilon_1 = \frac{1}{3}$$

In the second iteration, (c), each side becomes 4 segments. Each segment has length $\frac{1}{3^2}$

$$N_2 = 4^2, \quad \varepsilon_2 = \frac{1}{3^2}$$

In the third iteration, (d), each side becomes 4 segments. Each segment has length $\frac{1}{3^3}$

$$N_3 = 4^3, \quad \varepsilon_3 = \frac{1}{3^3}$$

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Let $K$ = infinite hyper real. In the $K^{th}$ iteration,

$$N_K = 4^K, \quad \varepsilon_K = \frac{1}{3^K}.$$

$$d = \frac{\log N_K}{\log(\frac{1}{\varepsilon_K})} = \frac{\log 4^K}{\log 3^K} = \frac{\log 4}{\log 3} \approx 1.26$$

is the extent to which the Koch snowflake folds and fills an area more than it fills a line.

Clearly, at dimension of 1.26, the Koch snowflake fills a line more than it fills an area.

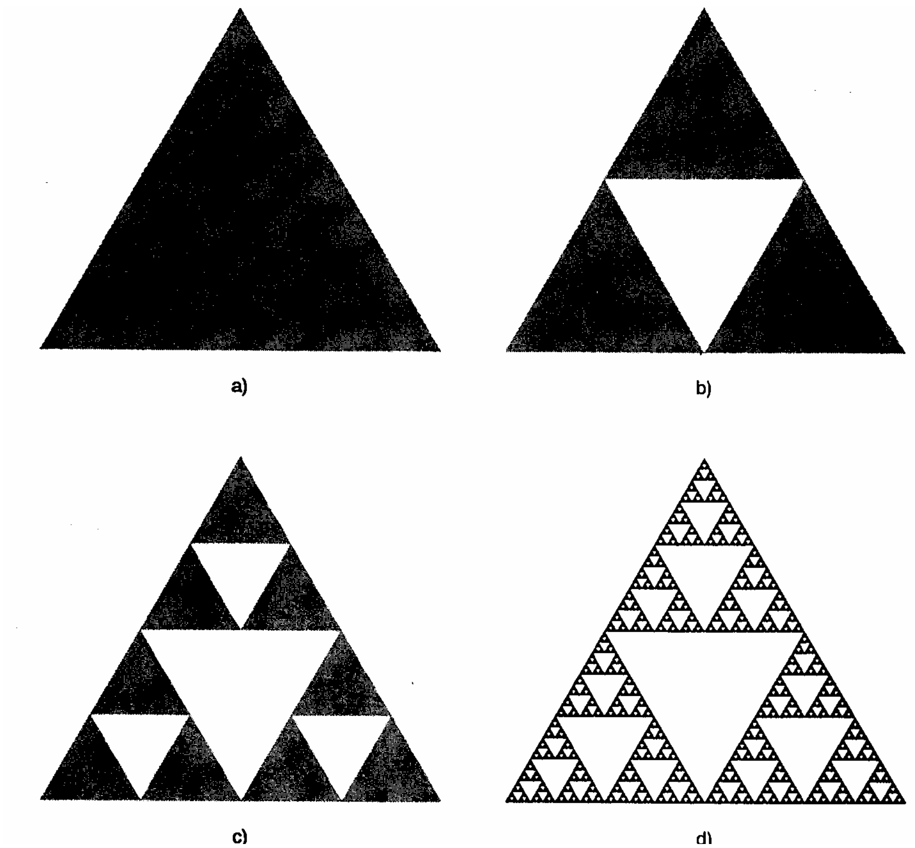
But the infinite distance between any two points says that the volume of its unit ball is infinite.

In fact, the volume of the Unit Ball here is

$$\frac{\log 4}{\log 3} \frac{\pi^{\frac{1}{2}\frac{\log 4}{\log 3}}}{\Gamma(\frac{1}{2}\frac{\log 4}{\log 3} + 1)}.$$

# 5.

# The Sierpinski Triangle




In the first iteration, (b), 3 triangles are left. Each triangle side equals $\frac{1}{2}$

$$N_1 = 3, \quad \varepsilon_1 = \frac{1}{2}$$

In the second iteration, (c), $3^2$ triangles are left. Each triangle side equals $\frac{1}{2^2}$

$$N_2 = 3^2, \quad \varepsilon_2 = \frac{1}{2^2}$$

In the third iteration, (d), $3^3$ triangles are left. Each triangle side equals $\frac{1}{2^3}$

$$N_3 = 3^3, \quad \varepsilon_3 = \frac{1}{2^3}$$

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Let $K$ = infinite hyper real. In the $K^{th}$ iteration,

$$N_K = 3^K, \quad \varepsilon_K = \frac{1}{2^K}$$

$$d = \frac{\log N_K}{\log(\frac{1}{\varepsilon_K})} = \frac{\log 3^K}{\log 3^K} = \frac{\log 3}{\log 2} \approx 1.585$$

is the extent to which the zero area Sierpinski Triangle folds and fills an area more than it fills a line.

Clearly, at dimension of 1.585, the Sierpinski Triangle fills an area more than it fills a line.

In Measure Theory, the Sierpinski Triangle is enigmatic. It has infinite distance between any two points. Thus, infinite volume. And it has zero area. Thus, zero volume.

In fact, the volume of the Unit Ball here is

$$\frac{\pi^{\frac{1}{2}\frac{\log 3}{\log 2}}}{\Gamma(\frac{1}{2}\frac{\log 3}{\log 2}+1)}.$$

# 6.

# The Unit Ball, and its Unit Sphere in $x$ Dimensions

The volume of the $x$-dimensional Unit Ball is $\boxed{V(x) = \dfrac{\pi^{\frac{1}{2}x}}{\Gamma(\frac{1}{2}x+1)}}$.

And the surface area of its Unit Sphere is $\boxed{x\dfrac{\pi^{\frac{1}{2}x}}{\Gamma(\frac{1}{2}x+1)}}$

| fractal set | dimension | volume of Unit Ball | Area of Unit Sphere |
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| Sierpinski Triangle | $\dfrac{\log 3}{\log 2} \approx 1.585$ | $\dfrac{\pi^{\frac{1}{2}\frac{\log 3}{\log 2}}}{\Gamma(\frac{1}{2}\frac{\log 3}{\log 2}+1)}$ | $\dfrac{\log 3}{\log 2}\dfrac{\pi^{\frac{1}{2}\frac{\log 3}{\log 2}}}{\Gamma(\frac{1}{2}\frac{\log 3}{\log 2}+1)}$ |

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