All the π^2 Series and the Catalan Series

H. Vic Dannon
vic0@comcast.net
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Abstract By π^2 Series we mean a series like Euler's

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+...=\frac{1}{8}\pi^2,$$

of reciprocals of squares, that sum up to an algebraic number times π^2

Euler obtained a family of infinitely many such series:

For natural numbers l and m with no common factor, and $\frac{1}{2}l<m$, Euler had[1]

$$\frac{1}{l^2}+\frac{1}{(2m-l)^2}+\frac{1}{(2m+l)^2}+\frac{1}{(4m-l)^2}+\frac{1}{(4m+l)^2}+\frac{1}{(6m-l)^2}+...=\left(\frac{1}{2m\sin(\frac{l}{2m}\pi)}\right)^2\pi^2$$

$l=1$, and $m=2 \Rightarrow 1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+...=\frac{1}{8}\pi^2$.

The Residue Theorem yields all the four families of π^2 series.

[1] Leonardi Euleri, "Introductio in Analysin Infinitorum" p.187, Section #174.

These extend Euler's work, and show that the Catalan Series is NOT a π^2 series. Thus, resolving Euler pursuit of a π^2 series formula for the Catalan Series.

If $l < \frac{1}{2}m$, l and m are natural numbers with no common factor.

Then, applying the Residue Theorem to $\pi\cot(\pi z)\dfrac{1}{(z+\frac{l}{m})^2}$, we obtain the series

$$R_{l/m} = \frac{1}{l^2} + \frac{1}{(m-l)^2} + \frac{1}{(m+l)^2} + \frac{1}{(2m-l)^2} + \frac{1}{(2m+l)^2} + \frac{1}{(3m-l)^2} + ... = \frac{\pi^2}{m^2}\frac{1}{\sin^2(\frac{l}{m}\pi)}$$

This family of series is identical with Euler's.

Applying the Residue theorem to $\pi\dfrac{1}{\sin(\pi z)}\dfrac{1}{(z+\frac{l}{m})^2}$ yields the alternating series

$$S_{l/m} = \frac{1}{l^2} - \frac{1}{(m-l)^2} - \frac{1}{(m+l)^2} + \frac{1}{(2m-l)^2} + \frac{1}{(2m+l)^2} - \frac{1}{(3m-l)^2} - ... = \frac{\pi^2}{m^2}\frac{\cos(\frac{l}{m}\pi)}{\sin^2(\frac{l}{m}\pi)}$$

with pattern $+--++--....$

Applying the Residue Theorem to $\pi\tan(\pi z)\dfrac{1}{(z-\frac{l}{m})^2}$ yields the series

$$T_{l/m} = \frac{1}{(m-2l)^2} + \frac{1}{(m+2l)^2}$$
$$+\frac{1}{(3m-2l)^2} + \frac{1}{(3m+2l)^2}$$
$$+\frac{1}{(5m-2l)^2} + \frac{1}{(5m+2l)^2} + ... = \frac{\pi^2}{4m^2}\frac{1}{\cos^2(\frac{l}{m}\pi)}$$

Applying the Residue Theorem to $\pi\frac{1}{\cos(\pi z)}\frac{1}{(z-\frac{l}{m})^2}$, yields the alternating series

$$C_{l/m} = \frac{1}{(m-2l)^2} - \frac{1}{(m+2l)^2}$$
$$-\frac{1}{(3m-2l)^2} + \frac{1}{(3m+2l)^2}$$
$$+\frac{1}{(5m-2l)^2} - \frac{1}{(5m+2l)^2} - ... = \frac{\pi^2}{4m^2}\frac{\sin(\frac{l}{m}\pi)}{\cos^2(\frac{l}{m}\pi)}$$

with pattern $+--++--$....

The alternating Catalan's Series

$$1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}-\frac{1}{11^2}+\frac{1}{13^2}-...$$

sums up to approximately

G = 0.9159 6559 4177 2190 1505 4603 5149 3238 4110 774....

Euler sought an answer to the question whether the Catalan Series is a π^2 series.[2]

[2] Euler's letters to James Stirling in "James Stirling, This about Series and such things" Scottish Academic Press 1988, pp. 142-151.

The Catalan Series is the difference between

$$1+\tfrac{1}{5^2}+\tfrac{1}{9^2}+\tfrac{1}{13^2}+\ldots,\text{ and } \tfrac{1}{3^2}+\tfrac{1}{7^2}+\tfrac{1}{11^2}+\tfrac{1}{15^2}+\ldots.$$

The series $1+\frac{1}{5^2}+\frac{1}{9^2}+\frac{1}{13^2}+\ldots$ is obtained from

$$\frac{1}{l^2}+\frac{1}{(2m+l)^2}+\frac{1}{(4m+l)^2}+\frac{1}{(6m+l)^2}+\ldots\text{with } l=1,\ m=2$$

And the series $\frac{1}{3^2}+\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{15^2}+\ldots.$ is obtained from

$$\frac{1}{(2m-l)^2}+\frac{1}{(4m-l)^2}+\frac{1}{(6m-l)^2}+\ldots\text{with } l=1,\ m=2$$

Thus, the Catalan Series belongs to the family of series

$$\begin{aligned}&\frac{1}{l^2}-\frac{1}{(2m-l)^2}\\ &+\frac{1}{(2m+l)^2}-\frac{1}{(4m-l)^2}\\ &+\frac{1}{(4m+l)^2}-\frac{1}{(6m-l)^2}+\ldots\end{aligned}$$

which pattern of

$$+-+-+-+-\ldots.$$

is incompatible with the pattern

$$+,-,-,+,+,-,-,+,+,\ldots.$$

of the alternating π^2 series,

And which terms are different from the terms of any of the π^2 series.

Hence, by its pattern, and terms, the Catalan Series is not a π^2 series. The Catalan series does not fit the pattern of any of the possible π^2 series.

We further show that adding or subtracting alternating π^2 series does not yield the Catalan Series.

Therefore, it is not a π^2 series. That is, the Catalan series does not sum up to an algebraic number multiplying π^2.

This resolves Euler's pursuit of a π^2 series formula for the Catalan Series. Similarly,

$$1-\frac{1}{3^4}+\frac{1}{5^4}-\frac{1}{7^4}+\ldots \quad \text{is NOT a } \pi^4 \text{ series}$$

$$1-\frac{1}{3^6}+\frac{1}{5^6}-\frac{1}{7^6}+\ldots \quad \text{is NOT a } \pi^6 \text{ series}$$

....

$$1-\frac{1}{3^{2n}}+\frac{1}{5^{2n}}-\frac{1}{7^{2n}}+\ldots \quad \text{is NOT a } \pi^{2n} \text{ series.}$$

Contents

7.$_{\cos}$)
$$\pi^2 = 4m^2 \frac{\cos^2(\frac{l}{m}\pi)}{\sin(\frac{l}{m}\pi)}\left(\frac{1}{(m-2l)^2} - \frac{1}{(m+2l)^2} - \frac{1}{(3m-2l)^2} + \frac{1}{(3m+2l)^2} + \frac{1}{(5m-2l)^2} - \frac{1}{(5m+2l)^2} - ...\right)$$

7.$_{\tan}$)
$$\pi^2 = \frac{4m^2\cot(\frac{l}{m}\pi)}{1+\tan(\frac{l}{m}\pi)}\left\{\frac{1}{(m-2l)^2} + \frac{1}{(m+2l)^2} + \frac{1}{(3m-2l)^2} + \frac{1}{(3m+2l)^2} + \frac{1}{(5m-2l)^2} + \frac{1}{(5m+2l)^2} + ...\right\}$$

8) 15/38

9) $(2k+1)\ /\ 2^n$

10) 1/4

11) 1/8

12) 3/8

13) $R_{1/8} + R_{3/8} = R_{1/4}$

14) $S_{1/8} - S_{3/8}$

15) $R_{1/8} - R_{3/8} = S_{1/4}$

16) 1/16

17) 3/16

18) 5/16

19) 7/16

20) $R_{1/16} + R_{3/16} + R_{5/16} + R_{7/16}$

21) $R_{1/16} - R_{3/16}$

22) $R_{5/16} - R_{7/16}$

23) $R_{3/16} - R_{5/16}$

24) $R_{1/16} - R_{5/16}$

25) $R_{1/16} - R_{7/16}$

26) $R_{3/16} - R_{7/16}$

27) $1 \,/\, p$

28) 1/3

29) $l \,/\, p^n$

30) 1/9

31) 2/9

32) 4/9

33) 1/6

34) 1/18

35) 5/18

36) 7/18

References

1.

Residue of $f(z)$ Singular at z_0

$$\boxed{\mathrm{Res}\left\{f(z)\right\}_{z=z_0} \equiv a_{-1} = \frac{1}{2\pi i}\oint_{\zeta=z_0+\varepsilon e^{i\phi}} f(\zeta)d\zeta}$$

Proof: $f(z) = .. + \dfrac{a_{-k}}{(z-z_0)^k} + .. + \dfrac{a_{-2}}{(z-z_0)^2} +$

$$+\frac{a_{-1}}{z-z_0}+$$

$$+a_0 + ... + a_n(z-z_0)^n + ..$$

$$\Rightarrow \oint_{\zeta=z_0+\rho e^{i\phi}} f(\zeta)d\zeta =$$

$$= ... + a_{-k}\underbrace{\oint_{\zeta=z_0+\varepsilon e^{i\phi}}\frac{1}{(\zeta-z_0)^k}d\zeta}_{\frac{1}{\varepsilon^k}\varepsilon i\underbrace{\oint\frac{e^{i\phi}}{e^{ki\phi}}d\phi}_{0}} + ... + a_{-2}\underbrace{\oint_{\zeta=z_0+\varepsilon e^{i\phi}}\frac{1}{(\zeta-z_0)^2}d\zeta}_{\frac{1}{\varepsilon^2}\varepsilon i\underbrace{\oint\frac{e^{i\phi}}{e^{2i\phi}}d\phi}_{0}}$$

$$+a_{-1}\underbrace{\oint_{\zeta=z_0+\varepsilon e^{i\phi}}\frac{1}{\zeta-z_0}d\zeta}_{\frac{1}{\varepsilon}\varepsilon i\oint\frac{e^{i\phi}}{e^{i\phi}}d\phi = i\underbrace{\oint d\phi}_{2\pi}}$$

$$+a_0 \underbrace{\oint_{\zeta = z_0 + \varepsilon e^{i\phi}} d\zeta}_{\varepsilon i \underbrace{\oint e^{i\phi} d\phi}_{0}} + \ldots + a_n \underbrace{\oint_{\zeta = z_0 + \varepsilon e^{i\phi}} (\zeta - z_0)^n d\zeta}_{\varepsilon^{n+1} i \underbrace{\oint e^{in\phi} e^{i\phi} d\phi}_{0}} + \ldots$$

$$\Rightarrow \operatorname{Res}\{f(z)\}_{z=z_0} \equiv a_{-1} = \frac{1}{2\pi i} \oint_{\zeta = z_0 + \varepsilon e^{i\phi}} f(\zeta) d\zeta . \square$$

2.

Residue at Pole of Order k

2.1 $f(z) = \dfrac{a_{-k}}{(z-z_0)^k} + ... + \dfrac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + ...$

$$\Rightarrow \boxed{\text{Res}_{-1,k}\left\{f(z)\right\}_{z=z_0} = a_{-1} = \left[\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left\{(z-z_0)^k f(z)\right\}\right]_{z=z_0}}$$

2.2 $f(z) = \dfrac{a_{-2}}{(z-z_0)^2} + \dfrac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + ...$

$$\Rightarrow \boxed{\text{Res}_{-1,2}\left\{f(z)\right\}_{z=z_0} = a_{-1} = \left[\frac{d}{dz}\left\{(z-z_0)^2 f(z)\right\}\right]_{z=z_0}}$$

2.3 $f(z) = \dfrac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + ...$

$$\Rightarrow \boxed{\text{Res}_{-1,1}\left\{f(z)\right\}_{z=z_0} = a_{-1} = \left[(z-z_0)f(z)\right]_{z=z_0}}$$

3.

Residue at Pole of Infinite order

3.1 $e^{-\frac{1}{z}} = 1 - \frac{1}{z} + \frac{1}{2!}\left(-\frac{1}{z}\right)^2 + \frac{1}{3!}\left(-\frac{1}{z}\right)^3 + \frac{1}{4!}\left(-\frac{1}{z}\right)^4 + ... \Rightarrow$

$$\Rightarrow \text{Res}_{-2}\{e^{-\frac{1}{z}}\}_{z=0} = \frac{1}{2}$$

$$\Rightarrow \text{Res}_{-1}\{e^{-\frac{1}{z}}\}_{z=0} = -1$$

$$\Rightarrow \text{Res}_{0}\{e^{-\frac{1}{z}}\}_{z=0} = 1$$

4.

$$\operatorname{Res}\left\{\frac{\cot z\coth z}{z^3}\right\}_{z=0} = -\frac{7}{45}$$

Proof: divide the series,

$$\frac{1}{z^3}\frac{\cos z}{\sin z}\frac{\cosh z}{\sinh z} = \frac{1}{z^3}\frac{1-\frac{1}{2!}z^2+\frac{1}{4!}z^4+..}{z-\frac{1}{3!}z^3+\frac{1}{5!}z^5+...}\times\frac{1+\frac{1}{2!}z^2+\frac{1}{4!}z^4+..}{z+\frac{1}{3!}z^3+\frac{1}{5!}z^5+...}$$

$$= \frac{1}{z^3}\frac{1+\frac{1}{4!}z^4-\frac{1}{2!}z^2+..}{z+\frac{1}{5!}z^5-\frac{1}{3!}z^3+...}\times\frac{1+\frac{1}{4!}z^4+\frac{1}{2!}z^2+..}{z+\frac{1}{5!}z^5+\frac{1}{3!}z^3+...}$$

$$\approx \frac{1}{z^5}\frac{\left(1+\frac{1}{4!}z^4\right)^2-\left(\frac{1}{2!}z^2\right)^2}{\left(1+\frac{1}{5!}z^4\right)^2-\left(\frac{1}{3!}z^2\right)^2}$$

$$\approx \frac{1}{z^5}\frac{1-\left(\frac{1}{4}-\frac{1}{12}\right)z^4+\frac{1}{(4!)^2}z^8}{1-\left(\frac{1}{36}-\frac{1}{60}\right)z^4+\frac{1}{(5!)^2}z^8}$$

$$\approx \frac{1}{z^5}\frac{1-\frac{1}{6}z^4}{1-\frac{14}{90}z^4}$$

$$\approx \frac{1}{z^5}\left(1-\frac{1}{6}z^4\right)\left(1+\frac{1}{90}z^4\right)$$

$$= \left(1-\left[\frac{1}{6}-\frac{1}{90}\right]z^4-\frac{1}{540}z^8\right)$$

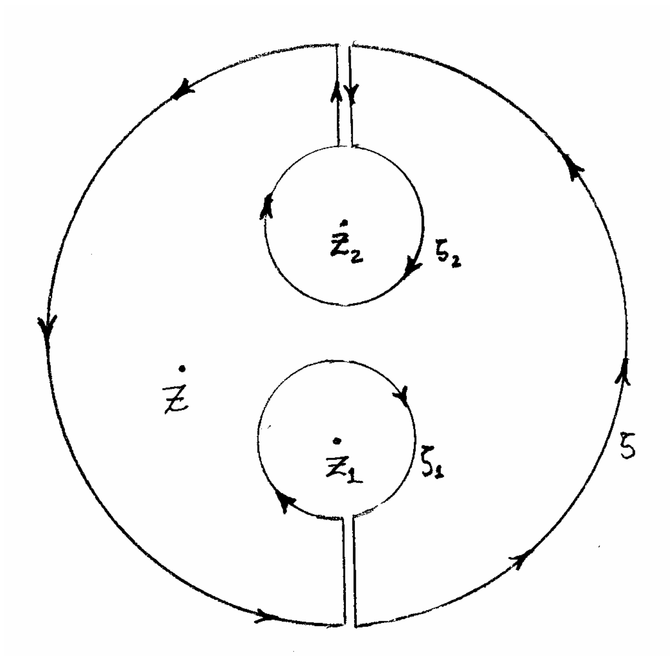
$$= \frac{1}{z^5}\left(1-\frac{14}{90}z^4-\frac{1}{540}z^8\right)$$

$$= \underbrace{1}_{a_{-5}}\frac{1}{z^5}-\underbrace{\frac{14}{90}}_{a_{-1}}\frac{1}{z}+\underbrace{\frac{1}{540}}_{a_3}z^3...$$

5.

Residue Theorem for a_{-1}

5.1



$$\boxed{\frac{1}{2\pi i}\oint_C f(\zeta)d\zeta = \text{Res}_{-1}\left\{f(z)\right\}_{z=z_1} + \text{Res}_{-1}\left\{f(z)\right\}_{z=z_2}}$$

<u>***Proof***</u>:

$$\oint_{\zeta\in C} f(\zeta)d\zeta + \oint_{\zeta_1\in c_1} f(\zeta_1)d\zeta_1 + \oint_{\zeta_2\in c_2} f(\zeta_2)d\zeta_2 = 0$$

$$\frac{1}{2\pi i}\oint_{\zeta\in C} f(\zeta)d\zeta = \frac{1}{2\pi i}\oint_{\zeta_1=z_1+\rho e^{i\phi}} f(\zeta_1)d\zeta_1 + \frac{1}{2\pi i}\oint_{\zeta_2=z_2+\rho e^{i\phi}} f(\zeta_2)d\zeta_2,$$

For $\zeta_1 \in c_1,\;\; f(\zeta_1) = \dfrac{a_{-k_1,1}}{(\zeta_1 - z_1)^{k_1}} + ... + \dfrac{a_{-1,1}}{\zeta_1 - z_1} + a_{0,1} + a_{1,1}(\zeta_1 - z_1) + ...$

$$\Rightarrow \frac{1}{2\pi i} \oint_{\zeta_1 = z_1 + \rho e^{i\phi}} f(\zeta_1) d\zeta = a_{-1,1} = \text{Res}_{-1}\left\{f(z)\right\}_{z=z_1}$$

For $\zeta_2 \in c_2$,

$$f(\zeta_2) = \frac{a_{-k_2,2}}{(\zeta_2 - z_2)^{k_2}} + ... + \frac{a_{-1,2}}{\zeta_2 - z_2} + a_{0,1} + a_{1,2}(\zeta_2 - z_2) + ...$$

$$\Rightarrow \frac{1}{2\pi i} \oint_{\zeta_2 = z_2 + \rho e^{i\phi}} f(\zeta_2) d\zeta_2 = a_{-1,2} = \text{Res}_{-1}\left\{f(z)\right\}_{z=z_2}$$

$$\frac{1}{2\pi i} \oint_C f(\zeta) d\zeta = \text{Res}_{-1}\left\{f(z)\right\}_{z=z_1} + \text{Res}_{-1}\left\{f(z)\right\}_{z=z_2} . \square$$

5.2 $f(z)$ has poles at $z_1, z_2, ..., z_N \Rightarrow$

$$\frac{1}{2\pi i} \oint_C f(\zeta) d\zeta = \text{Res}_{-1}\left\{f(z)\right\}_{z=z_1} + ... + \text{Res}_{-1}\left\{f(z)\right\}_{z=z_N}$$

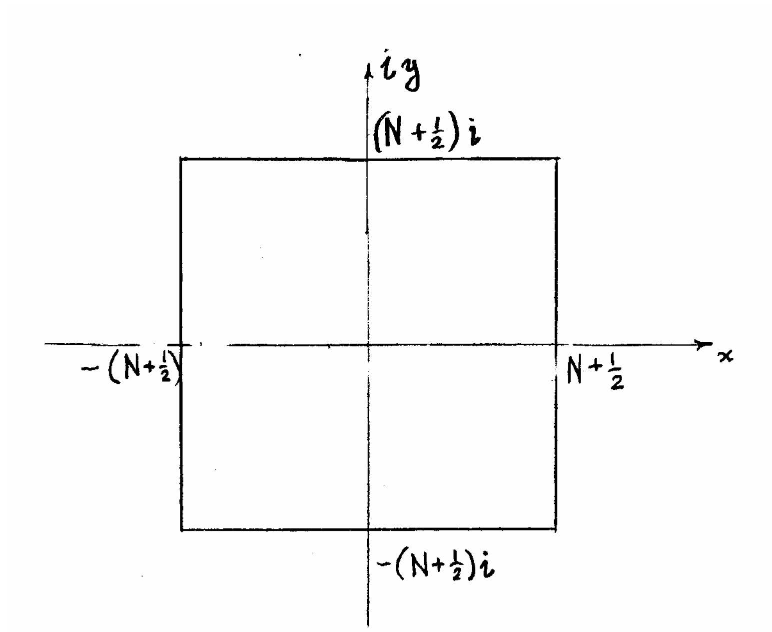
$\mathbf{6_{cot}}$.

$$\boxed{\pi\cot(\pi z)f(z)}$$

$$\boxed{\text{Res}\{\pi\cot(\pi z)f(z)\}_{z=n} = f(n)}$$

$$\boxed{\begin{array}{l} ...f(-3)+f(-2)+f(-1)+f(0)+f(1)+f(2)+f(3)+... \\ \qquad\qquad + \sum \text{Res}_{-1}\left\{\pi\cot(\pi\sigma)f(\sigma)\right\}_{\sigma=\text{pole of } f(\sigma)} = 0 \end{array}}$$

$\mathbf{6_{cot}.1}$ $\left|\cot\pi z\right| \le A$ **on** $\square_{N+\frac{1}{2}}$ **for any** N



$$\underline{\left. y = -N - \frac{1}{2}, \text{ and } -N - \frac{1}{2} \le x \le N + \frac{1}{2} \right|} \Rightarrow$$

$$\left|\cot\pi z\right| = \frac{\left|e^{i\pi z} + e^{-i\pi z}\right|}{\left|e^{i\pi z} - e^{-i\pi z}\right|}$$

$$\leq \frac{\left|e^{i\pi z}\right| + \left|e^{-i\pi z}\right|}{\left|\left|e^{i\pi z}\right| - \left|e^{-i\pi z}\right|\right|}$$

$$= \frac{\left|e^{i\pi(x+i[-N-\frac{1}{2}])}\right| + \left|e^{-i\pi(x+i[-N-\frac{1}{2}])}\right|}{\left|\left|e^{i\pi(x+i[-N-\frac{1}{2}])}\right| - \left|e^{-i\pi(x+i[-N-\frac{1}{2}])}\right|\right|}$$

$$= \frac{\left|e^{i\pi x+\pi N+\pi/2}\right| + \left|e^{-i\pi x-\pi N-\pi/2}\right|}{\left|\left|e^{i\pi x+\pi N+\pi/2}\right| - \left|e^{-i\pi x-\pi N-\pi/2}\right|\right|}$$

$$= \frac{\left(e^{\pi N+\pi/2} + e^{-\pi N-\pi/2}\right)}{\left(e^{\pi N+\pi/2} - e^{-\pi N-\pi/2}\right)} \frac{e^{-\pi N+\pi/2}}{e^{-\pi N+\pi/2}}$$

$$= \frac{e^{\pi} + e^{-2\pi N}}{e^{\pi} - e^{-2\pi N}}$$

$$= \frac{e^{\pi} + \frac{1}{e^{2\pi N}}}{e^{\pi} - \frac{1}{e^{2\pi N}}}$$

$$< \frac{e^{\pi} + 1}{e^{\pi} - 1}$$

$$= 1 + \frac{2}{e^{\pi} - 1} < 1.1. \square$$

$\underline{y = N + \frac{1}{2}, \text{ and } -N - \frac{1}{2} \leq x \leq N + \frac{1}{2}\Big|} \Rightarrow$

$$\left|\cot \pi z\right| = \left|\frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}}\right|$$

$$\leq \frac{\left|e^{i\pi z}\right| + \left|e^{-i\pi z}\right|}{\left|\left|e^{i\pi z}\right| - \left|e^{-i\pi z}\right|\right|}$$

$$= \frac{\left| e^{i\pi(x+i[N+\frac{1}{2}])} \right| + \left| e^{-i\pi(x+i[N+\frac{1}{2}])} \right|}{\left| \left| e^{i\pi(x+i[N+\frac{1}{2}])} \right| - \left| e^{-i\pi(x+i[N+\frac{1}{2}])} \right| \right|}$$

$$= \frac{\left| e^{i\pi x-\pi N-\pi/2} \right| + \left| e^{-i\pi x+\pi N+\pi/2} \right|}{\left| \left| e^{i\pi x-\pi N-\pi/2} \right| - \left| e^{-i\pi x+\pi N+\pi/2} \right| \right|}$$

$$= \frac{(e^{-\pi N-\pi/2} + e^{\pi N+\pi/2})}{(e^{\pi N+\pi/2} - e^{-\pi N-\pi/2})} \frac{e^{-\pi N+\pi/2}}{e^{-\pi N+\pi/2}}$$

$$= \frac{e^{\pi} + e^{-2\pi N}}{e^{\pi} - e^{-2\pi N}}$$

$$= \frac{e^{\pi} + \frac{1}{e^{2\pi N}}}{e^{\pi} - \frac{1}{e^{2\pi N}}}$$

$$< \frac{e^{\pi} + 1}{e^{\pi} - 1}$$

$$= 1 + \frac{2}{e^{\pi} - 1} < 1.1. \square$$

$\underline{x = N + \frac{1}{2},\ \text{and}\ -N - \frac{1}{2} \leq y \leq N + \frac{1}{2}} \Rightarrow$

For $y = 0$, $\left|\cot \pi z\right| = \left|\cot \pi(N + \tfrac{1}{2})\right|$

$$= \left|\cot(\pi N + \tfrac{\pi}{2})\right|$$

$$= \left|\cot \frac{\pi}{2}\right|$$

$$= \tan 0 = 0.$$

For $y \neq 0$, $\left|\cot \pi z\right| = \left|\cot\left(\pi N + \tfrac{\pi}{2} + i\pi y\right)\right|$

$$= \left|\cot\left(\tfrac{\pi}{2} + i\pi y\right)\right|$$

$$= \left|\tan\left(i\pi y\right)\right|$$

$$= \left|\frac{\sin i\pi y}{\cos i\pi y}\right|$$

$$= \left|\frac{\dfrac{e^{i\pi iy} - e^{-i\pi iy}}{2i}}{\dfrac{e^{i\pi iy} + e^{-i\pi iy}}{2}}\right|$$

$$= \left|\frac{e^{-\pi y} - e^{\pi y}}{e^{-\pi y} + e^{\pi y}}\right|$$

$$= \left|\frac{1 - e^{2\pi y}}{1 + e^{2\pi y}}\right| = \left|\frac{e^{2\pi y} - 1}{e^{2\pi y} + 1}\right|$$

For $y > 0$, $\quad \dfrac{e^{2\pi y} - 1}{e^{2\pi y} + 1} < 1.\square$

For $y < 0$, $\quad \dfrac{1 - e^{2\pi y}}{1 + e^{2\pi y}} = \dfrac{1 - \dfrac{1}{e^{2\pi(-y)}}}{1 + \dfrac{1}{e^{2\pi(-y)}}} < 1.\square$

$\underline{x = -N - \dfrac{1}{2}, \text{ and } -N - \dfrac{1}{2} \leq y \leq N + \dfrac{1}{2}\Big|} \Rightarrow$

For $y = 0$, $\quad \left|\cot \pi z\right| = \left|\cot \pi(-N - \tfrac{1}{2})\right|$

$$= \left|\cot(-\pi N - \tfrac{\pi}{2})\right|$$

$$= \left|\cot(-\tfrac{\pi}{2})\right|$$

$$= \tan 0 = 0.$$

For $y \neq 0$, $\quad \left|\cot \pi z\right| = \left|\cot\left(-\pi N - \tfrac{\pi}{2} + i\pi y\right)\right|$

$$= \left|\cot\left(-\tfrac{\pi}{2} + i\pi y\right)\right|$$

$$= \left|\tan\left(i\pi y\right)\right|$$

$$= \left|\frac{\sin i\pi y}{\cos i\pi y}\right|$$

$$= \left|\frac{\dfrac{e^{i\pi i y} - e^{-i\pi i y}}{2i}}{\dfrac{e^{i\pi i y} + e^{-i\pi i y}}{2}}\right|$$

$$= \left|\frac{e^{-\pi y} - e^{\pi y}}{e^{-\pi y} + e^{\pi y}}\right|$$

$$= \left|\frac{1 - e^{2\pi y}}{1 + e^{2\pi y}}\right| = \left|\frac{e^{2\pi y} - 1}{e^{2\pi y} + 1}\right|$$

For $y > 0$, $\dfrac{e^{2\pi y} - 1}{e^{2\pi y} + 1} < 1.\square$

For $y < 0$, $\dfrac{1 - e^{2\pi y}}{1 + e^{2\pi y}} = \dfrac{1 - \dfrac{1}{e^{2\pi(-y)}}}{1 + \dfrac{1}{e^{2\pi(-y)}}} < 1.\square$

6_{cot}.2 $\cot \pi z = \dfrac{\cos \pi z}{\sin \pi z}$ **has poles at** $z = n = ... -2, -1, 0, 1, 2, ...$

6_{cot}.3 $$\boxed{\operatorname{Res}\left\{\pi \cot(\pi z)\right\}_{z=n} = 1}$$

Proof: $\operatorname{Res}\left\{\pi \cot(\pi z)\right\}_{z=n} = \left[(z-n)\pi \dfrac{\cos \pi z}{\sin \pi z}\right]_{z=n}$

$$= \left[\frac{D_z(\pi z - \pi n)}{D_z \sin(\pi z)} \cos(\pi z) \right]_{z=n}$$

$$= \left[\frac{\pi}{\pi \cos(\pi z)} \cos(\pi z) \right]_{z=n} = 1. \square$$

6_{cot}.4 $$\boxed{\operatorname{Res}\left\{ \pi \cot(\pi z) f(z) \right\}_{z=n} = f(n)}$$

6_{cot}.5 $\left| f(z) \right|_{\square_{N+\frac{1}{2}}} \le \frac{M}{z^k} \Rightarrow$

$$\boxed{\begin{array}{c} ...f(-3) + f(-2) + f(-1) + f(0) + f(1) + f(2) + f(3) + ... \\ + \sum \operatorname{Res}_{-1}\left\{ \pi \cot(\pi \sigma) f(\sigma) \right\}_{\sigma = \text{pole of } f(\sigma)} = 0 \end{array}}$$

Proof:

$$\oint_{\square_{N+\frac{1}{2}}} \pi \cot(\pi\zeta) f(\zeta) d\zeta = \sum \operatorname{Res}_{-1}\left\{ \pi \cot(\pi z) f(z) \right\}_{z = \text{pole of } \cot \pi z \text{ in } \square_{N+\frac{1}{2}}}$$

$$+ \sum \operatorname{Res}_{-1}\left\{ \pi \cot(\pi \sigma) f(\sigma) \right\}_{\sigma = \text{pole of } f(\sigma) \text{ in } \square_{N+\frac{1}{2}}}$$

$$\left| \oint_{\square_{N+\frac{1}{2}}} \pi \cot(\pi\zeta) f(\zeta) d\zeta \right| \le \pi \underbrace{\left| \cot \pi\zeta \right|}_{\le A} \frac{M}{N^k} \underbrace{(\text{length} \square_{N+\frac{1}{2}})}_{8\left(N+\frac{1}{2}\right)} \xrightarrow[N \to \infty]{} 0$$

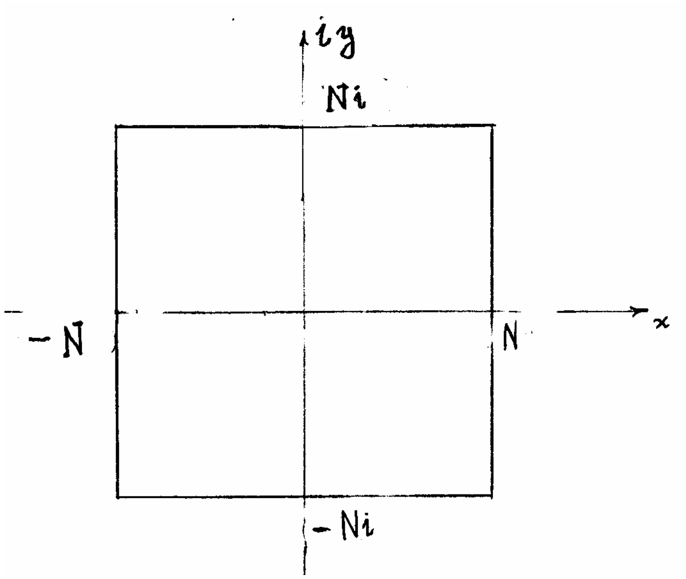
$$\operatorname{Res}\left\{ \pi \cot(\pi z) f(z) \right\}_{z=n} = f(n). \square$$

$\mathbf{6}_{\mathbf{tan}}$.

$$\boxed{\pi\tan(\pi z)f(z)}$$

$$\boxed{\begin{aligned} &..+f(-\tfrac{5}{2})+f(-\tfrac{3}{2})+f(-\tfrac{1}{2})\ +f(\tfrac{1}{2})+f(\tfrac{3}{2})+f(\tfrac{5}{2})+..= \\ &\qquad\qquad =\sum \operatorname{Res}_{-1}\left\{\pi\tan(\pi\sigma)f(\sigma)\right\}_{\sigma=pole\ of\ f(\sigma)} \end{aligned}}$$

$\mathbf{6}_{\mathbf{tan}}$.1 $\left|\tan\pi z\right| \le A$ **on** $\square_N$ **for any** N



$\underline{y=-N,\ \text{and}\ \ -N\le x\le N}\Big| \Rightarrow$

$$\left|\tan\pi z\right| = \left|\frac{e^{i\pi z}-e^{-i\pi z}}{e^{i\pi z}+e^{-i\pi z}}\right|$$

$$= \left|\frac{1-e^{-2i\pi z}}{1+e^{-2i\pi z}}\right|$$

$$= \left|\frac{1-e^{-2i\pi x-2\pi N}}{1+e^{-2i\pi x-2\pi N}}\right|$$

$$< \frac{1+e^{-2\pi N}}{1-e^{-2\pi N}}$$

$$< \frac{1+e^{-\pi}}{1-e^{-\pi}}.\square$$

$\underline{y = N,\ \text{and}\ \ -N \le x \le N}\Big| \Rightarrow$

$$\left|\tan \pi z\right| = \left|\frac{e^{i\pi z} - e^{-i\pi z}}{e^{i\pi z} + e^{-i\pi z}}\right|$$

$$= \left|\frac{e^{2i\pi z} - 1}{e^{2i\pi z} + 1}\right|$$

$$= \left|\frac{e^{-2i\pi x + 2\pi N} - 1}{e^{-2i\pi x + 2\pi N} + 1}\right|$$

$$< \frac{e^{2\pi N} + 1}{e^{2\pi N} - 1}\frac{e^{-2\pi N}}{e^{-2\pi N}}$$

$$< \frac{1 + \frac{1}{e^{2\pi N}}}{1 - \frac{1}{e^{2\pi N}}}$$

$$< \frac{1 + \frac{1}{e^{\pi}}}{1 - \frac{1}{e^{\pi}}}$$

$$< \frac{e^{\pi} + 1}{e^{\pi} - 1}$$

$$< 1 + 2\frac{1}{e^{\pi} - 1}.\square$$

$\underline{x = N,\ \text{and}\ \ -N \le y \le N}\Big| \Rightarrow$

For $y = 0$, $\left|\tan \pi z\right| = \left|\tan \pi N\right|$

$$= \tan 0 = 0.$$

For $y \neq 0$, $\left|\tan \pi z\right| = \left|\tan\left(\pi N + i\pi y\right)\right|$

$$= \left|\tan\left(i\pi y\right)\right|$$

$$= \left|\frac{\sin i\pi y}{\cos i\pi y}\right|$$

$$= \left|\frac{\dfrac{e^{i\pi i y} - e^{-i\pi i y}}{2i}}{\dfrac{e^{i\pi i y} + e^{-i\pi i y}}{2}}\right|$$

$$= \left|\frac{e^{-\pi y} - e^{\pi y}}{e^{-\pi y} + e^{\pi y}}\right|$$

$$= \left|\frac{1 - e^{2\pi y}}{1 + e^{2\pi y}}\right| = \left|\frac{e^{2\pi y} - 1}{e^{2\pi y} + 1}\right|$$

For $y > 0$, $\dfrac{e^{2\pi y} - 1}{e^{2\pi y} + 1} < 1.\square$

For $y < 0$, $\dfrac{1 - e^{2\pi y}}{1 + e^{2\pi y}} = \dfrac{1 - \dfrac{1}{e^{2\pi(-y)}}}{1 + \dfrac{1}{e^{2\pi(-y)}}} < 1.\square$

$\underline{x = -N, \text{ and } -N \leq y \leq N} \Rightarrow$

For $y = 0$, $\left|\tan \pi z\right| = \left|\tan \pi(-N)\right| = \tan 0 = 0$

For $y \neq 0$, $\left|\tan \pi z\right| = \left|\tan\left(-\pi N + i\pi y\right)\right|$

$$= \left|\tan\left(i\pi y\right)\right|$$

$$= \left| \frac{\sin i\pi y}{\cos i\pi y} \right|$$

$$= \left| \frac{\dfrac{e^{i\pi iy} - e^{-i\pi iy}}{2i}}{\dfrac{e^{i\pi iy} + e^{-i\pi iy}}{2}} \right|$$

$$= \left| \frac{e^{-\pi y} - e^{\pi y}}{e^{-\pi y} + e^{\pi y}} \right|$$

$$= \left| \frac{1 - e^{2\pi y}}{1 + e^{2\pi y}} \right| = \left| \frac{e^{2\pi y} - 1}{e^{2\pi y} + 1} \right|$$

For $y > 0$, $\dfrac{e^{2\pi y} - 1}{e^{2\pi y} + 1} < 1.\square$

For $y < 0$, $\dfrac{1 - e^{2\pi y}}{1 + e^{2\pi y}} = \dfrac{1 - \dfrac{1}{e^{2\pi(-y)}}}{1 + \dfrac{1}{e^{2\pi(-y)}}} < 1.\square$

6_{tan}.2 $\pi \tan(\pi\zeta)$ **has poles at** $z = n + \frac{1}{2}$

6_{tan}.3 $$\boxed{\operatorname{Res}\left\{\pi \tan(\pi z)\right\}_{z=n+\frac{1}{2}} = -1}$$

Proof: $\operatorname{Res}\left\{\pi \tan(\pi z)\right\}_{z=n+\frac{1}{2}} = \left[(z - [n + \tfrac{1}{2}]) \dfrac{\pi \sin(\pi z)}{\cos(\pi z)} \right]_{z=n+\frac{1}{2}}$

$$= \pi \left[\frac{D_z(z-n)}{D_z \cos(\pi z)} \sin(\pi z) \right]_{z=n+\frac{1}{2}}$$

$$= \left[\frac{1}{-\sin(\pi z)} \sin(\pi z) \right]_{z=n+\frac{1}{2}} = -1$$

$6_{\tan}$.4

$$\boxed{\text{Res}\left\{\pi\tan(\pi z)f(z)\right\}_{z=n+\frac{1}{2}} = -f(n+\tfrac{1}{2})}$$

$6_{\tan}$.5 $\left|f(z)\right|_{\square_N} \le \dfrac{M}{z^k} \Rightarrow$

$$\boxed{\begin{aligned}&..+f(-\tfrac{5}{2})+f(-\tfrac{3}{2})+f(-\tfrac{1}{2})\ +f(\tfrac{1}{2})+f(\tfrac{3}{2})+f(\tfrac{5}{2})+..=\\ &=\sum \text{Res}_{-1}\left\{\pi\tan(\pi\sigma)f(\sigma)\right\}_{\sigma=\text{pole of } f(\sigma)}\end{aligned}}$$

Proof:

$$\oint_{\square_N} \pi\tan(\pi\zeta)f(\zeta)d\zeta = \sum \text{Res}_{-1}\left\{\pi\tan(\pi\zeta)f(z)\right\}_{z=\text{pole of } \pi\tan(\pi\zeta) \text{ in } \square_N}$$

$$+\sum \text{Res}_{-1}\left\{\pi\tan(\pi\zeta)f(\sigma)\right\}_{\sigma=\text{pole of } f(\sigma) \text{ in } \square_N}$$

$$\left|\oint_{\square_N} \pi\tan(\pi\zeta)f(\zeta)d\zeta\right| \le \underbrace{\left|\pi\tan(\pi\zeta)\right|}_{\le A}\frac{M}{N^k}\underbrace{(\text{length}\square_N)}_{8N} \xrightarrow[N\to\infty]{} 0$$

$\pi\tan(\pi\zeta)$ **has poles at** $z = n+\frac{1}{2}$

$$\text{Res}_{-1}\left\{\pi\tan(\pi z)f(z)\right\}_{z=n+\frac{1}{2}} = -f(n+\tfrac{1}{2})$$

$$\Rightarrow\ ..+f(-\tfrac{5}{2})+f(-\tfrac{3}{2})+f(-\tfrac{1}{2})\ +f(\tfrac{1}{2})+f(\tfrac{3}{2})+f(\tfrac{5}{2})+...=$$

$$=\sum \text{Res}_{-1}\left\{\pi\tan(\pi\sigma)f(\sigma)\right\}_{\sigma=pole\ \ of\ \ f(\sigma)}$$

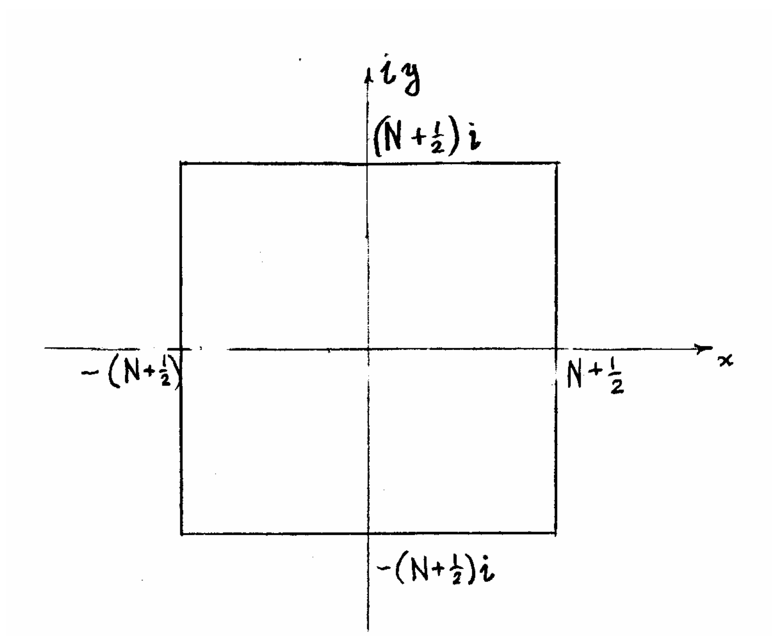
$\mathbf{6_{sin}}$.

$$\boxed{\frac{\pi}{\sin \pi z} f(z)}$$

$$\boxed{\text{Res}\left\{\pi \frac{1}{\sin \pi z} f(z)\right\}_{z=n} = (-1)^n f(n)}$$

$$\boxed{\begin{array}{l} ... - f(-3) + f(-2) - f(-1) + f(0) - f(1) + f(2) - f(3) + ... \\ \qquad + \sum \text{Res}_{-1}\left\{\frac{\pi}{\sin(\pi\sigma)} f(\sigma)\right\}_{\sigma=\text{pole of } f(\sigma)} = 0 \end{array}}$$

6_{sin}.1 $\left|\frac{1}{\sin \pi z}\right| \leq A$ **on** $\square_{N+\frac{1}{2}}$ **for any** N



$$\underline{y = -N - \frac{1}{2}, \text{ and } -N - \frac{1}{2} \leq x \leq N + \frac{1}{2}\bigg|} \Rightarrow$$

$$\left|\frac{1}{\sin \pi z}\right| = \frac{2}{\left|e^{i\pi z} - e^{-i\pi z}\right|}$$

$$= \frac{2}{\left|e^{i\pi(x+iy)} - e^{-i\pi(x+iy)}\right|}$$

$$\leq \frac{2}{\left|\left|e^{i\pi(x+iy)}\right| - \left|e^{-i\pi(x+iy)}\right|\right|}$$

$$= \frac{2}{\left|e^{-\pi y} - e^{\pi y}\right|}$$

$$= \frac{2}{e^{\pi N + \frac{\pi}{2}} - e^{-\pi N - \frac{\pi}{2}}} \frac{e^{-\pi N + \frac{\pi}{2}}}{e^{-\pi N + \frac{\pi}{2}}}$$

$$= \frac{2e^{-\pi N + \frac{\pi}{2}}}{e^{\pi} - e^{-2\pi N}}$$

$$= \frac{\frac{2e^{\frac{\pi}{2}}}{e^{\pi N}}}{e^{\pi} - \frac{1}{e^{2\pi N}}}$$

$$< \frac{2e^{\frac{\pi}{2}}}{e^{\pi} - 1} \equiv A. \square$$

$$\underline{y = N + \frac{1}{2}, \text{ and } -N - \frac{1}{2} \leq x \leq N + \frac{1}{2}} \Rightarrow$$

$$\left|\frac{1}{\sin \pi z}\right| = \frac{2}{\left|e^{i\pi z} - e^{-i\pi z}\right|}$$

$$= \frac{2}{\left|e^{i\pi(x+iy)} - e^{-i\pi(x+iy)}\right|}$$

$$\leq \frac{2}{\left|\left|e^{i\pi(x+iy)}\right| - \left|e^{-i\pi(x+iy)}\right|\right|}$$

$$= \frac{2}{\left|e^{-\pi y} - e^{\pi y}\right|}$$

$$= \frac{2}{e^{\pi N + \frac{\pi}{2}} - e^{-\pi N - \frac{\pi}{2}}} \frac{e^{-\pi N + \frac{\pi}{2}}}{e^{-\pi N + \frac{\pi}{2}}}$$

$$= \frac{2e^{-\pi N + \frac{\pi}{2}}}{e^{\pi} - e^{-2\pi N}}$$

$$= \frac{\dfrac{2e^{\frac{\pi}{2}}}{e^{\pi N}}}{e^{\pi} - \dfrac{1}{e^{2\pi N}}}$$

$$< \frac{2e^{\frac{\pi}{2}}}{e^{\pi} - 1} \equiv A. \square$$

$$\underline{x = N + \frac{1}{2}, \text{ and } -N - \frac{1}{2} \leq y \leq N + \frac{1}{2}} \Big| \Rightarrow$$

$$\left|\frac{1}{\sin \pi z}\right| = \frac{1}{\left|\sin(\pi N + \frac{\pi}{2} + i\pi y)\right|}$$

$$= \left|\csc(\pi N + \tfrac{\pi}{2} + i\pi y)\right|$$

$$= \left|\csc(\tfrac{\pi}{2} + i\pi y)\right|$$

$$= \frac{1}{\left|\sin(\frac{\pi}{2} + i\pi y)\right|}$$

$$= \frac{1}{\left|\cos(i\pi y)\right|}$$

$$= \frac{1}{\left|\dfrac{e^{i\pi(iy)} + e^{-i\pi(iy)}}{2}\right|}$$

$$= \frac{2}{e^{-\pi y} + e^{\pi y}}$$

For $y = 0$, $\quad \dfrac{2}{e^{-\pi y} + e^{\pi y}} = 1$

For $y > 0$, $\quad \dfrac{2}{e^{-\pi y} + e^{\pi y}} \dfrac{e^{\pi y}}{e^{\pi y}} = 2\dfrac{e^{\pi y}}{e^{2\pi y} + 1} < 2$

For $y < 0$, $\quad \dfrac{2}{e^{-\pi y} + e^{\pi y}} \dfrac{e^{-\pi y}}{e^{-\pi y}} = 2\dfrac{e^{\pi(-y)}}{e^{2\pi(-y)} + 1} < 2\,.\square$

$\underline{x = -N - \frac{1}{2}, \text{ and } -N - \frac{1}{2} \le y \le N + \frac{1}{2}} \Big| \Rightarrow$

$$\left|\frac{1}{\sin \pi z}\right| = \frac{1}{\left|\sin(-\pi N - \frac{\pi}{2} + i\pi y)\right|}$$

$$= \left|\csc(-\pi N - \tfrac{\pi}{2} + i\pi y)\right|$$

$$= \left|\csc(-\tfrac{\pi}{2} + i\pi y)\right|$$

$$= \frac{1}{\left|\sin(\frac{\pi}{2} + i\pi y)\right|}$$

$$= \frac{1}{\left|\cos(i\pi y)\right|}$$

$$= \frac{1}{\left|\dfrac{e^{i\pi(iy)} + e^{-i\pi(iy)}}{2}\right|}$$

$$= \frac{2}{e^{-\pi y} + e^{\pi y}}$$

For $y = 0$, $\quad \frac{2}{e^{-\pi y} + e^{\pi y}} = 1$

For $y > 0$, $\quad \frac{2}{e^{-\pi y} + e^{\pi y}} \frac{e^{\pi y}}{e^{\pi y}} = 2\frac{e^{\pi y}}{e^{2\pi y} + 1} < 2$

For $y < 0$, $\quad \frac{2}{e^{-\pi y} + e^{\pi y}} \frac{e^{-\pi y}}{e^{-\pi y}} = 2\frac{e^{\pi(-y)}}{e^{2\pi(-y)} + 1} < 2\,.\square$

$6_{\sin}$.2 $\quad \frac{1}{\sin \pi z}$ **has poles of order 1 at** $z = n = \ldots -2, -1, 0, 1, 2, \ldots$

$6_{\sin}$.3

$$\boxed{\operatorname{Res}\left\{\pi \frac{1}{\sin \pi z}\right\}_{z=n} = (-1)^n}$$

Proof: $\operatorname{Res}\left\{\pi \frac{1}{\sin \pi z}\right\}_{z=n} = \left[(z-n)\pi \frac{1}{\sin \pi z}\right]_{z=n}$

$$= \left[\frac{D_z(\pi z - \pi n)}{D_z \sin(\pi z)}\right]_{z=n}$$

$$= \left[\frac{\pi}{\pi \cos(\pi z)}\right]_{z=n} = (-1)^n.\square$$

$6_{\sin}$.4

$$\boxed{\operatorname{Res}\left\{\pi \frac{1}{\sin \pi z} f(z)\right\}_{z=n} = (-1)^n f(n)}$$

$6_{\sin}$.5 $\left|f(z)\right|_{\square_{N+\frac{1}{2}}} \le \dfrac{M}{z^k} \Rightarrow$

$$\boxed{\begin{aligned}&...-f(-3)+f(-2)-f(-1)+f(0)-f(1)+f(2)-f(3)+...\\ &\qquad\qquad +\sum \text{Res}_{-1}\left\{\frac{\pi}{\sin(\pi\sigma)}f(\sigma)\right\}_{\sigma=\text{pole of } f(\sigma)} = 0\end{aligned}}$$

Proof:

$$\oint_{\square_{N+\frac{1}{2}}} \frac{\pi}{\sin(\pi\zeta)} f(\zeta)d\zeta = \sum \text{Res}_{-1}\left\{\frac{\pi}{\sin(\pi z)}f(z)\right\}_{z=\text{pole of } \frac{\pi}{\sin(\pi\zeta)} \text{ in } \square_{N+\frac{1}{2}}}$$

$$+\sum \text{Res}_{-1}\left\{\frac{\pi}{\sin(\pi\sigma)}f(\sigma)\right\}_{\sigma=\text{pole of } f(\sigma) \text{ in } \square_{N+\frac{1}{2}}}$$

$$\left|\oint_{\square_{N+\frac{1}{2}}} \frac{\pi}{\sin(\pi\zeta)} f(\zeta)d\zeta\right| \le \pi \underbrace{\left|\frac{1}{\sin(\pi\zeta)}\right|}_{\le A} \frac{M}{N^k} \underbrace{(\text{length}\square_{N+\frac{1}{2}})}_{8\left(N+\frac{1}{2}\right)} \xrightarrow[N\to\infty]{} 0$$

$\dfrac{1}{\sin \pi z}$ **has poles at** $z = n$

$$\text{Res}_{-1}\left\{\frac{\pi}{\sin(\pi z)}f(z)\right\}_{z=\text{pole of } \frac{\pi}{\sin(\pi z)} \text{ in } \square_{N+\frac{1}{2}}} = (-1)^n f(n). \square$$

$\Rightarrow$

$$...-f(-3)+f(-2)-f(-1)+f(0)-f(1)+f(2)-f(3)+...$$

$$+\sum \text{Res}_{-1}\left\{\frac{\pi}{\sin(\pi\sigma)}f(\sigma)\right\}_{\sigma=pole\ \ of\ \ f(\sigma)} = 0$$

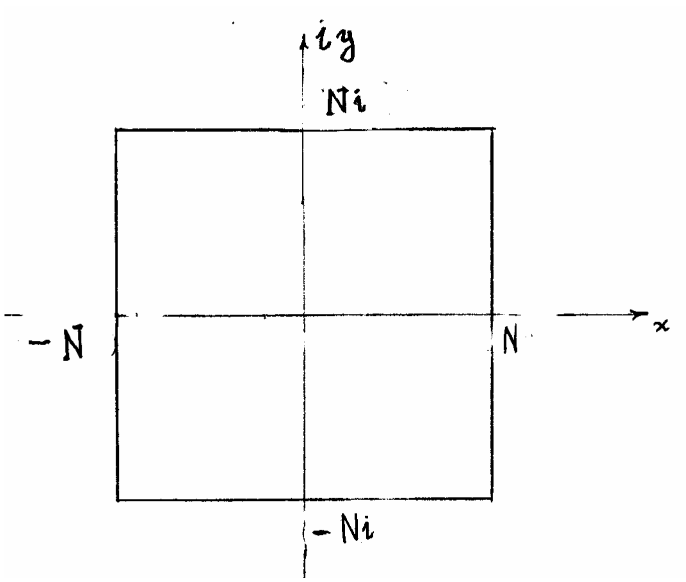
$\mathbf{6_{cos}}$.

$$\boxed{\frac{\pi}{\cos \pi z} f(z)}$$

$$\boxed{\text{Res}\left\{\frac{\pi}{\cos(\pi z)} f(z)\right\}_{z=n+\frac{1}{2}} = -(-1)^n f(n+\tfrac{1}{2})}$$

$$\boxed{\begin{aligned}&... - f(-\tfrac{5}{2}) + f(-\tfrac{3}{2}) - f(-\tfrac{1}{2}) + f(\tfrac{1}{2}) - f(\tfrac{3}{2}) + f(\tfrac{5}{2}) - ... = \\ &= \sum \text{Res}_{-1}\left\{\frac{\pi}{\cos(\pi\sigma)} f(\sigma)\right\}_{\sigma=\text{pole of } f(\sigma)}\end{aligned}}$$

$\mathbf{6_{cos}.1}$ $\left|\frac{1}{\cos \pi z}\right| \le A$ **on** $\square_N$ **for any** N



$\underline{y = -N, \text{ and } -N \le x \le N} \Rightarrow$

$$\frac{1}{\left|\cos \pi z\right|} = \frac{2}{\left|e^{i\pi z} + e^{-i\pi z}\right|}$$

$$= \frac{2}{\left|e^{i\pi(x+iy)} + e^{-i\pi(x+iy)}\right|}$$

$$\leq \frac{2}{\left|\left|e^{i\pi(x+iy)}\right| - \left|e^{-i\pi(x+iy)}\right|\right|}$$

$$= \frac{2}{\left|e^{-\pi y} - e^{\pi y}\right|}$$

$$= \frac{2}{\left|e^{-\pi(-N)} - e^{\pi(-N)}\right|}$$

$$= \frac{2}{e^{\pi N} - e^{-\pi N}}$$

$$\leq 2\frac{1}{e^{\pi} - 1}. \square$$

$\underline{y = N, \text{ and } -N \leq x \leq N} \Rightarrow$

$$\left|\frac{1}{\cos \pi z}\right| = \frac{2}{\left|e^{-i\pi z} + e^{i\pi z}\right|}$$

$$= \frac{2}{\left|e^{i\pi(x+iy)} + e^{-i\pi(x+iy)}\right|}$$

$$\leq \frac{2}{\left|\left|e^{i\pi(x+iy)}\right| - \left|e^{-i\pi(x+iy)}\right|\right|}$$

$$= \frac{2}{\left|e^{-\pi y} - e^{\pi y}\right|}$$

$$= \frac{2}{\left|e^{-\pi N} - e^{\pi N}\right|}$$

$$\leq 2\frac{1}{e^{\pi}-1}.\square$$

$\underline{x = N, \text{ and } -N \leq y \leq N|} \Rightarrow$

$$\left|\frac{1}{\cos \pi z}\right| = \frac{1}{|\cos(\pi N + i\pi y)|}$$

$$= |\sec(\pi N + i\pi y)|$$

$$= |\sec(i\pi y)|$$

$$= \frac{1}{|\cos(i\pi y)|}$$

$$= \frac{2}{|e^{i(i\pi y)} + e^{-i(i\pi y)}|}$$

$$= \frac{2}{|e^{-\pi y} + e^{\pi y}|}$$

$$= \frac{2}{e^{-\pi y} + e^{\pi y}}$$

For $y = 0$, $\quad \dfrac{2}{e^{-\pi y} + e^{\pi y}} = 1$

For $y > 0$, $\quad \dfrac{2}{e^{-\pi y} + e^{\pi y}}\dfrac{e^{\pi y}}{e^{\pi y}} = 2\dfrac{e^{\pi y}}{e^{2\pi y} + 1} < 2$

For $y < 0$, $\quad \dfrac{2}{e^{-\pi y} + e^{\pi y}}\dfrac{e^{-\pi y}}{e^{-\pi y}} = 2\dfrac{e^{\pi(-y)}}{e^{2\pi(-y)} + 1} < 2.\square$

$\underline{x = -N, \text{ and } -N \leq y \leq N|} \Rightarrow$

$$\left|\frac{1}{\cos \pi z}\right| = \frac{1}{|\cos(-\pi N + i\pi y)|}$$

$$= \left|\sec(-\pi N + i\pi y)\right|$$

$$= \left|\sec(i\pi y)\right|$$

$$= \frac{1}{\left|\cos(i\pi y)\right|}$$

$$= \frac{2}{\left|e^{i(i\pi y)} + e^{-i(i\pi y)}\right|}$$

$$= \frac{2}{\left|e^{-\pi y} + e^{\pi y}\right|}$$

$$= \frac{2}{e^{-\pi y} + e^{\pi y}}$$

For $y = 0$, $\quad \dfrac{2}{e^{-\pi y} + e^{\pi y}} = 1$

For $y > 0$, $\quad \dfrac{2}{e^{-\pi y} + e^{\pi y}} \dfrac{e^{\pi y}}{e^{\pi y}} = 2\dfrac{e^{\pi y}}{e^{2\pi y} + 1} < 2$

For $y < 0$, $\quad \dfrac{2}{e^{-\pi y} + e^{\pi y}} \dfrac{e^{-\pi y}}{e^{-\pi y}} = 2\dfrac{e^{\pi(-y)}}{e^{2\pi(-y)} + 1} < 2\,.\square$

$6_{\cos}$.2 $\quad \dfrac{1}{\cos \pi z}$ **has poles at** $z = n + \frac{1}{2} = ..., -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ...$

$6_{\cos}$.3

$$\boxed{\operatorname{Res}\left\{\frac{\pi}{\cos(\pi z)}\right\}_{z=n+\frac{1}{2}} = -(-1)^n}$$

<u>Proof</u>: $\quad \operatorname{Res}\left\{\dfrac{\pi}{\cos(\pi z)}\right\}_{z=n+\frac{1}{2}} = \left[(z - [n + \tfrac{1}{2}])\dfrac{\pi}{\cos(\pi z)}\right]_{z=n+\frac{1}{2}}$

$$= \pi \left[\frac{D_z(z - n - \frac{1}{2})}{D_z \cos(\pi z)} \right]_{z=n+\frac{1}{2}}$$

$$= \left[\frac{1}{-\sin(\pi z)} \right]_{z=n+\frac{1}{2}}$$

$$= \frac{1}{-\sin(\pi n + \frac{\pi}{2})}$$

$$= -(-1)^n . \square$$

$6_{\cos}$.4

$$\boxed{\operatorname{Res}\left\{ \frac{\pi}{\cos(\pi z)} f(z) \right\}_{z=n+\frac{1}{2}} = -(-1)^n f(n + \tfrac{1}{2})}$$

$6_{\cos}$.5 $\left| f(z) \right|_{\square_N} \le \frac{M}{z^k} \Rightarrow$

$$\boxed{\begin{aligned} &... - f(-\tfrac{5}{2}) + f(-\tfrac{3}{2}) - f(-\tfrac{1}{2}) + f(\tfrac{1}{2}) - f(\tfrac{3}{2}) + f(\tfrac{5}{2}) - ... = \\ &= \sum \operatorname{Res}_{-1} \left\{ \frac{\pi}{\cos(\pi\sigma)} f(\sigma) \right\}_{\sigma=\text{pole of } f(\sigma)} \end{aligned}}$$

Proof:

$$\oint_{\square_N} \frac{\pi}{\cos(\pi\zeta)} f(\zeta) d\zeta = \sum \operatorname{Res}_{-1} \left\{ \frac{\pi}{\cos(\pi z)} f(z) \right\}_{z=\text{pole of } \frac{\pi}{\cos(\pi z)} \text{ in } \square_N}$$

$$+ \sum \operatorname{Res}_{-1} \left\{ \frac{\pi}{\cos(\pi\sigma)} f(\sigma) \right\}_{\sigma=\text{pole of } f(\sigma) \text{ in } \square_N}$$

$$\left| \oint_{\square_N} \frac{\pi}{\cos(\pi\zeta)} f(\zeta) d\zeta \right| \le \underbrace{\left| \frac{\pi}{\cos(\pi\zeta)} \right|}_{\le A} \frac{M}{N^k} \underbrace{(\text{length} \square_N)}_{8(N)} \xrightarrow[N\to\infty]{} 0$$

$\dfrac{1}{\cos \pi z}$ **has poles at** $z = n + \frac{1}{2}$

$$\text{Res}_{-1}\left\{\frac{\pi}{\cos(\pi z)} f(z)\right\}_{z=\text{pole of } \frac{\pi}{\cos(\pi z)} \text{ in } \square_N} = \text{Res}_{-1}\left\{\frac{\pi}{\cos(\pi z)} f(z)\right\}_{z=n+\frac{1}{2}}$$

$$= -(-1)^n f(n+\tfrac{1}{2})$$

$$\Rightarrow \quad \sum (-1)^n f(n+\tfrac{1}{2}) = \sum \text{Res}_{-1}\left\{\frac{\pi}{\cos(\pi\sigma)} f(\sigma)\right\}_{\sigma=\text{pole of } f(z)}$$

$$\Rightarrow \quad ... + f(-\tfrac{7}{2}) - f(-\tfrac{5}{2}) + f(-\tfrac{3}{2}) - f(-\tfrac{1}{2}) + f(\tfrac{1}{2}) - f(\tfrac{3}{2}) + f(\tfrac{5}{2}) + ... =$$

$$= \sum \text{Re}\, s_{-1}\left\{\frac{\pi}{\cos(\pi\sigma)} f(\sigma)\right\}_{\sigma=pole\ of\ f(z)} . \square$$

7_{cot}.

$$\pi^2 = m^2 \sin^2(\tfrac{l}{m}\pi)\left(\frac{1}{l^2} + \frac{1}{(m-l)^2} + \frac{1}{(m+l)^2} + \frac{1}{(2m-l)^2} + \frac{1}{(2m+l)^2} + \frac{1}{(3m-l)^2} + \dots\right)$$

$l < \frac{1}{2}m$, l and m have no common factor

$$R_{l/m} = \frac{1}{l^2} + \frac{1}{(m-l)^2} + \frac{1}{(m+l)^2} + \frac{1}{(2m-l)^2} + \frac{1}{(2m+l)^2} + \frac{1}{(3m-l)^2} + \dots = \frac{\pi^2}{m^2 \sin^2(\frac{l}{m}\pi)}$$

Proof: $\pi\cot(\pi z)\dfrac{1}{(z-\frac{l}{m})^2}$ has poles of order 1 at $z = n$,

and a pole of order 2 at $z = \frac{l}{m}$

$$\oint_{\square_{N+\frac{1}{2}}} \pi\cot(\pi\zeta)\frac{1}{(\zeta-\frac{l}{m})^2}d\zeta = \sum \text{Res}_{-1}\left\{\pi\cot(\pi z)\frac{1}{(z-\frac{l}{m})^2}\right\}_{z=n}$$

$$+ \text{Res}_{-1}\left\{\pi\cot(\pi z)\frac{1}{(z-\frac{l}{m})^2}\right\}_{z=\frac{l}{m}}$$

$$\left| \oint_{\square_{N+\frac{1}{2}}} \pi\cot(\pi\zeta)\frac{1}{(\zeta-\frac{l}{m})^2}d\zeta \right| \le \underbrace{\left|\pi\cot\pi\zeta\right|}_{\le A} \underbrace{\oint_{\square_{N+\frac{1}{2}}} \frac{1}{(\zeta-\frac{l}{m})^2}d\zeta}_{\le\left[\frac{1}{\zeta}\right]_{\square_{N+\frac{1}{2}}}=0} = 0.\square$$

$$\mathrm{Res}_{-1}\left\{\pi\cot(\pi z)\frac{1}{(z-\frac{l}{m})^2}\right\}_{z=n} = \left[(z-n)\pi\frac{\cos\pi z}{\sin\pi z}\frac{1}{(z-\frac{l}{m})^2}\right]_{z=n}$$

$$= \left[\frac{\pi D_z(z-n)}{D_z\sin(\pi z)}\right]_{z=n}\left(\cos(\pi n)\frac{1}{(n-\frac{l}{m})^2}\right)$$

$$= \left[\frac{\pi}{\pi\cos(\pi z)}\right]_{z=n}\left(\cos(\pi n)\frac{1}{(n-\frac{l}{m})^2}\right)$$

$$= \frac{1}{(n-\frac{l}{m})^2}.\square$$

To find $\mathrm{Res}_{-1}\left\{\pi\cot(\pi z)\frac{1}{(z-\frac{l}{m})^2}\right\}_{z=\frac{l}{m}}$ **, divide the series**

$$\pi\frac{\cos(\pi z)}{\sin(\pi z)}\frac{1}{(z-\frac{l}{m})^2} = \pi\frac{\cos(\pi[z-\frac{l}{m}]+\frac{l}{m}\pi)}{\sin(\pi[z-\frac{l}{m}]+\frac{l}{m}\pi)}\frac{1}{(z-\frac{l}{m})^2}$$

$$= \pi\frac{\cos(\pi u+\frac{l}{m}\pi)}{\sin(\pi u+\frac{l}{m}\pi)}\frac{1}{u^2}$$

$$= \pi\frac{\cos(\pi u)\cos(\frac{l}{m}\pi)-\sin(\pi u)\sin(\frac{l}{m}\pi)}{\sin(\pi u)\cos(\frac{l}{m}\pi)+\cos(\pi u)\sin(\frac{l}{m}\pi)}\frac{1}{u^2}$$

$$\approx \pi \frac{(1-\frac{1}{2}u^2)\cos(\frac{l}{m}\pi)-(\pi u)\sin(\frac{l}{m}\pi)}{(\pi u)\cos(\frac{l}{m}\pi)+(1-\frac{1}{2}u^2)\sin(\frac{l}{m}\pi)}\frac{1}{u^2}$$

$$\approx \pi \frac{\cos(\frac{l}{m}\pi)-(\pi u)\sin(\frac{l}{m}\pi)}{(\pi u)\cos(\frac{l}{m}\pi)+\sin(\frac{l}{m}\pi)}\frac{1}{u^2}$$

$$= \pi \frac{\cos(\frac{l}{m}\pi)}{\sin(\frac{l}{m}\pi)}\left(\frac{1-\pi u\tan(\frac{l}{m}\pi)}{\pi u\cot(\frac{l}{m}\pi)+1}\right)\frac{1}{u^2}$$

$$= \pi \cot(\frac{l}{m}\pi)\left(\frac{1-\pi u\tan(\frac{l}{m}\pi)}{1+\pi u\cot(\frac{l}{m}\pi)}\right)\frac{1}{u^2}$$

$$\begin{array}{l|l}
1\boxed{-\pi u\{\tan(\frac{l}{m}\pi)+\cot(\frac{l}{m}\pi)\}}+\pi^2u^2\{\tan(\frac{l}{m}\pi)+\cot(\frac{l}{m}\pi)\}\cot(\frac{l}{m}\pi) & \\ \hline
1-\pi u\tan(\frac{l}{m}\pi) & 1+\pi u\cot(\frac{l}{m}\pi) \\
\underline{1+\pi u\cot(\frac{l}{m}\pi)} & \\
-\pi u\{\tan(\frac{l}{m}\pi)+\cot(\frac{l}{m}\pi)\} & \\
\underline{-\pi u\{\tan(\frac{l}{m}\pi)+\cot(\frac{l}{m}\pi)\}-\pi^2u^2\{\tan(\frac{l}{m}\pi)+\cot(\frac{l}{m}\pi)\}\cot(\frac{l}{m}\pi)} & \\
\qquad\qquad\pi^2u^2\{\tan(\frac{l}{m}\pi)+\cot(\frac{l}{m}\pi)\}\cot(\frac{l}{m}\pi)\} &
\end{array}$$

$$\approx \pi\cot(\frac{l}{m}\pi)\left[1-\pi u\{\tan(\frac{l}{m}\pi)+\cot(\frac{l}{m}\pi)\}+\pi^2u^2\{\tan(\frac{l}{m}\pi)+\cot(\frac{l}{m}\pi)\}\cot(\frac{l}{m}\pi)\right]\frac{1}{u^2}$$

$$= \pi\cot(\frac{l}{m}\pi)\frac{1}{u^2}$$

$$-\pi^2\cot(\frac{l}{m}\pi)\{\tan(\frac{l}{m}\pi)+\cot(\frac{l}{m}\pi)\}\frac{1}{u}$$

$$+\pi^3\cot(\frac{l}{m}\pi)\Big(\{\tan(\frac{l}{m}\pi)+\cot(\frac{l}{m}\pi)\}\cot(\frac{l}{m}\pi)\Big)$$

$$\text{Res}_{-1}\left\{\pi\cot(\pi z)\frac{1}{(z-\frac{l}{m})^2}\right\}_{z=\frac{l}{m}} = -\pi^2\cot(\frac{l}{m}\pi)\{\tan(\frac{l}{m}\pi)+\cot(\frac{l}{m}\pi)\}$$

$$= -\pi^2 \{1 + \cot^2(\tfrac{l}{m}\pi)\}$$

$$= -\pi^2 \frac{1}{\sin^2(\tfrac{l}{m}\pi)}$$

Therefore,

$$... + \frac{1}{(-3 - \frac{l}{m})^2} + \frac{1}{(-2 - \frac{l}{m})^2} + \frac{1}{(-1 - \frac{l}{m})^2} + \frac{1}{(0 - \frac{l}{m})^2} +$$

$$+ \frac{1}{(1 - \frac{l}{m})^2} + \frac{1}{(2 - \frac{l}{m})^2} + \frac{1}{(3 - \frac{l}{m})^2} + ... - \pi^2 \frac{1}{\sin^2(\frac{l}{m}\pi)} = 0$$

$$\pi^2 = m^2 \sin^2(\tfrac{l}{m}\pi) \left\{ \frac{1}{l^2} + \frac{1}{(m-l)^2} + \frac{1}{(m+l)^2} + \frac{1}{(2m-l)^2} + \frac{1}{(2m+l)^2} + \frac{1}{(3m-l)^2} + ... \right\}$$

$$R_{l/m} = \frac{1}{l^2} + \frac{1}{(m-l)^2} + \frac{1}{(m+l)^2} + \frac{1}{(2m-l)^2} + \frac{1}{(2m+l)^2} + \frac{1}{(3m-l)^2} + ... = \frac{\pi^2}{m^2 \sin^2(\frac{l}{m}\pi)}$$

$7_{\sin}$.

$$\pi^2 = m^2 \frac{\sin^2(\frac{l}{m}\pi)}{\cos(\frac{l}{m}\pi)} = \left(\frac{1}{l^2} - \frac{1}{(m-l)^2} - \frac{1}{(m+l)^2} + \frac{1}{(2m-l)^2} + \frac{1}{(2m+l)^2} - \frac{1}{(3m-l)^2} - \ldots \right)$$

$l < \frac{1}{2}m$, l and m have no common factor.

$$S_{l/m} = \frac{1}{l^2} - \frac{1}{(m-l)^2} - \frac{1}{(m+l)^2} + \frac{1}{(2m-l)^2} + \frac{1}{(2m+l)^2} - \frac{1}{(3m-l)^2} - \ldots = \frac{\pi^2}{m^2}\frac{\cos(\frac{l}{m}\pi)}{\sin^2(\frac{l}{m}\pi)}$$

Proof: $\pi \frac{1}{\sin(\pi z)} \frac{1}{(z-\frac{l}{m})^2}$ has poles of order 1 at $z = n$,

and a pole of order 2 at $z = \frac{l}{m}$

$$\oint_{\square_{N+\frac{1}{2}}} \pi \frac{1}{\sin(\pi\zeta)} \frac{1}{(\zeta-\frac{l}{m})^2} d\zeta = \sum \operatorname{Res}_{-1}\left\{ \pi \frac{1}{\sin(\pi z)} \frac{1}{(z-\frac{l}{m})^2} \right\}_{z=n}$$

$$+\operatorname{Res}_{-1}\left\{ \pi \frac{1}{\sin(\pi z)} \frac{1}{(z-\frac{l}{m})^2} \right\}_{z=\frac{l}{m}}$$

$$\left| \oint_{\square_{N+\frac{1}{2}}} \frac{\pi}{\sin(\pi\zeta)} \frac{1}{(\zeta - \frac{l}{m})^2} d\zeta \right| \le \underbrace{\left| \frac{\pi}{\sin(\pi\zeta)} \right|}_{\le A} \underbrace{\oint_{\square_{N+\frac{1}{2}}} \frac{1}{(\zeta - \frac{l}{m})^2} d\zeta}_{\left[\frac{1}{\zeta}\right]_{\square_{N+\frac{1}{2}}} = 0} = 0 . \square$$

$$\operatorname{Res}_{-1}\left\{ \pi \frac{1}{\sin(\pi z)} \frac{1}{(z - \frac{l}{m})^2} \right\}_{z=n} = \left[(z-n)\pi \frac{1}{\sin \pi z} \frac{1}{(z - \frac{l}{m})^2} \right]_{z=n}$$

$$= \left[\frac{\pi D_z (z-n)}{D_z \sin(\pi z)} \right]_{z=n} \frac{1}{(n - \frac{l}{m})^2}$$

$$= \left[\frac{\pi}{\pi \cos(\pi z)} \right]_{z=n} \frac{1}{(n - \frac{l}{m})^2}$$

$$= \frac{(-1)^n}{(n - \frac{l}{m})^2} . \square$$

To find $\operatorname{Res}_{-1}\left\{ \pi \frac{1}{\sin(\pi z)} \frac{1}{(z - \frac{l}{m})^2} \right\}_{z=\frac{l}{m}}$ **, divide the series**

$$\pi \frac{1}{\sin(\pi z)} \frac{1}{(z - \frac{l}{m})^2} = \pi \frac{1}{\sin[\pi(z - \frac{l}{m}) + \frac{l}{m}\pi)]} \frac{1}{(z - \frac{l}{m})^2}$$

$$= \pi \frac{1}{\sin(\pi u + \frac{l}{m}\pi)} \frac{1}{u^2}$$

$$= \pi \frac{1}{\sin(\pi u)\cos(\frac{l}{m}\pi) + \cos(\pi u)\sin(\frac{l}{m}\pi)} \frac{1}{u^2}$$

$$\approx \pi \frac{1}{(\pi u)\cos(\frac{l}{m}\pi) + (1 - \frac{1}{2}u^2)\sin(\frac{l}{m}\pi)} \frac{1}{u^2}$$

$$\approx \pi \frac{1}{(\pi u)\cos(\frac{l}{m}\pi) + \sin(\frac{l}{m}\pi)} \frac{1}{u^2}$$

$$\approx \pi \frac{1}{\sin(\frac{l}{m}\pi)} \left(\frac{1}{1 + \pi u \cot(\frac{l}{m}\pi)} \right) \frac{1}{u^2}$$

$$\begin{array}{l|l}
1 \boxed{-\pi u \cot(\tfrac{l}{m}\pi)} + \pi^2 u^2 \cot^2(\tfrac{l}{m}\pi) & \\
\hline
1 & 1 + \pi u \cot(\tfrac{l}{m}\pi) \\
\underline{1 + \pi u \cot(\tfrac{l}{m}\pi)} & \\
\quad -\pi u \cot(\tfrac{l}{m}\pi) & \\
\quad \underline{-\pi u \cot(\tfrac{l}{m}\pi) - \pi^2 u^2 \cot^2(\tfrac{l}{m}\pi)} & \\
\qquad\qquad\qquad \pi^2 u^2 \cot^2(\tfrac{l}{m}\pi) &
\end{array}$$

$$\approx \pi \frac{1}{\sin(\frac{l}{m}\pi)} \left\{ 1 - \pi u \cot(\tfrac{l}{m}\pi) + \pi^2 u^2 \cot^2(\tfrac{l}{m}\pi) \right\} \frac{1}{u^2}$$

$$= \pi \frac{1}{\sin(\frac{l}{m}\pi)} \frac{1}{u^2} - \pi^2 \frac{\cot(\frac{l}{m}\pi)}{\sin(\frac{l}{m}\pi)} \frac{1}{u} + \pi^3 \frac{\cot^2(\frac{l}{m}\pi)}{\sin(\frac{l}{m}\pi)}$$

$$\operatorname{Res}_{-1} \left\{ \pi \frac{1}{\sin(\pi z)} \frac{1}{(z - \frac{l}{m})^2} \right\}_{z = \frac{l}{m}} = -\pi^2 \frac{\cot(\frac{l}{m}\pi)}{\sin(\frac{l}{m}\pi)}$$

$$= -\pi^2 \frac{\cos(\frac{l}{m}\pi)}{\sin^2(\frac{l}{m}\pi)}$$

Therefore,

$$\dots + \frac{(-1)^{-3}}{(-3 - \frac{l}{m})^2} + \frac{(-1)^{-2}}{(-2 - \frac{l}{m})^2} + \frac{(-1)^{-1}}{(-1 - \frac{l}{m})^2} + \frac{(-1)^{0}}{(0 - \frac{l}{m})^2} +$$

$$+\frac{(-1)^1}{(1-\frac{l}{m})^2}+\frac{(-1)^2}{(2-\frac{l}{m})^2}+\frac{(-1)^3}{(3-\frac{l}{m})^2}+...-\pi^2\frac{\cos(\frac{l}{m}\pi)}{\sin^2(\frac{l}{m}\pi)}=0$$

$$\pi^2 = m^2\frac{\sin^2(\frac{l}{m}\pi)}{\cos(\frac{l}{m}\pi)}\left(\frac{1}{l^2}-\frac{1}{(m-l)^2}-\frac{1}{(m+l)^2}+\frac{1}{(2m-l)^2}+\frac{1}{(2m+l)^2}-\frac{1}{(3m-l)^2}-...\right)$$

And the Associated Series is

$$S_{l/m} = \frac{1}{l^2}-\frac{1}{(m-l)^2}-\frac{1}{(m+l)^2}+\frac{1}{(2m-l)^2}+\frac{1}{(2m+l)^2}-\frac{1}{(3m-l)^2}-...=\frac{\pi^2}{m^2}\frac{\cos(\frac{l}{m}\pi)}{\sin^2(\frac{l}{m}\pi)}.\square$$

$7_{\cos}$.

$$\boxed{\pi^2 = 4m^2 \frac{\cos^2(\frac{l}{m}\pi)}{\sin(\frac{l}{m}\pi)}\left(\frac{1}{(m-2l)^2} - \frac{1}{(m+2l)^2} - \frac{1}{(3m-2l)^2} + \frac{1}{(3m+2l)^2} + \frac{1}{(5m-2l)^2} - \frac{1}{(5m+2l)^2} - \dots\right)}$$

$l < \frac{1}{2}m$, l **and** m **have no common factor.**

$$C_{l/m} = \frac{1}{(m-2l)^2} - \frac{1}{(m+2l)^2} - \frac{1}{(3m-2l)^2} + \frac{1}{(3m+2l)^2} + \frac{1}{(5m-2l)^2} - \frac{1}{(5m+2l)^2} - \dots = \frac{\pi^2 \sin(\frac{l}{m}\pi)}{4m^2\cos^2(\frac{l}{m}\pi)}$$

<u>Proof</u>: $\pi \frac{1}{\cos(\pi z)}\frac{1}{(z-\frac{l}{m})^2}$ **has poles of order 1 at** $z = n + \frac{1}{2}$,

and a pole of order 2 at $z = \frac{l}{m}$

$$\oint_{\square_N} \pi \frac{1}{\cos(\pi\zeta)}\frac{1}{(\zeta-\frac{l}{m})^2}d\zeta = \sum \operatorname{Res}_{-1}\left\{\pi\frac{1}{\cos(\pi z)}\frac{1}{(z-\frac{l}{m})^2}\right\}_{z=n+\frac{1}{2}} + \operatorname{Res}_{-1}\left\{\pi\frac{1}{\cos(\pi z)}\frac{1}{(z-\frac{l}{m})^2}\right\}_{z=\frac{l}{m}}$$

$$\left|\oint_{\square_N} \frac{\pi}{\cos(\pi\zeta)}\frac{1}{(\zeta-\frac{l}{m})^2}d\zeta\right| \le \underbrace{\left|\frac{\pi}{\cos(\pi\zeta)}\right|}_{\le A}\underbrace{\oint_{\square_N}\frac{1}{(\zeta-\frac{l}{m})^2}d\zeta}_{\le\left[\frac{1}{|\zeta|}\right]_{\square_N}=0} = 0\,.\square$$

$$\operatorname{Res}_{-1}\left\{\pi\frac{1}{\cos(\pi z)}\frac{1}{(z-\frac{l}{m})^2}\right\}_{z=n+\frac{1}{2}}=\left[(z-n-\tfrac{1}{2})\pi\frac{1}{\cos(\pi z)}\frac{1}{(z-\frac{l}{m})^2}\right]_{z=n+\frac{1}{2}}$$

$$=\left[\frac{\pi D_z(z-n-\frac{1}{2})}{D_z\cos(\pi z)}\right]_{z=n+\frac{1}{2}}\frac{1}{(n+\frac{1}{2}-\frac{l}{m})^2}$$

$$=\left[\frac{\pi}{-\pi\sin(\pi z)}\right]_{z=n+\frac{1}{2}}\frac{1}{(n+\frac{1}{2}-\frac{l}{m})^2}$$

$$=\frac{1}{-\sin\pi(n+\frac{1}{2})}\frac{1}{(n+\frac{1}{2}-\frac{l}{m})^2}$$

$$=-(-1)^n\frac{1}{(n+\frac{1}{2}-\frac{l}{m})^2}$$

$$=(-1)^{n+1}\frac{1}{(n+\frac{1}{2}-\frac{l}{m})^2}.\square$$

To find $\operatorname{Res}_{-1}\left\{\pi\frac{1}{\cos(\pi z)}\frac{1}{(z-\frac{l}{m})^2}\right\}_{z=\frac{l}{m}}$**, divide the series**

$$\pi\frac{1}{\cos(\pi z)}\frac{1}{(z-\frac{l}{m})^2}=\pi\frac{1}{\cos(\pi[z-\frac{l}{m}]+\frac{l}{m}\pi)}\frac{1}{(z-\frac{l}{m})^2}$$

$$=\pi\frac{1}{\cos(\pi u+\frac{l}{m}\pi)}\frac{1}{u^2}$$

$$=\pi\frac{1}{\cos(\pi u)\cos(\frac{l}{m}\pi)-\sin(\pi u)\sin(\frac{l}{m}\pi)}\frac{1}{u^2}$$

$$\approx\pi\frac{1}{(1-\frac{1}{2}\pi^2u^2)\cos(\frac{l}{m}\pi)-(\pi u)\sin(\frac{l}{m}\pi)}\frac{1}{u^2}$$

$$\approx \pi \frac{1}{\cos(\frac{l}{m}\pi) - (\pi u)\sin(\frac{l}{m}\pi)} \frac{1}{u^2}$$

$$= \pi \frac{1}{\cos(\frac{l}{m}\pi)} \left(\frac{1}{1 - \pi u \tan(\frac{l}{m}\pi)} \right) \frac{1}{u^2}$$

$$\begin{array}{l|l}
\multicolumn{1}{l}{1 + \pi u \tan(\frac{l}{m}\pi) + \pi^2 u^2 \tan^2(\frac{l}{m}\pi)} & \\
\hline
1 & 1 - \pi u \tan(\frac{l}{m}\pi) \\
\underline{1 - \pi u \tan(\frac{l}{m}\pi)} & \\
\quad \pi u \tan(\frac{l}{m}\pi) & \\
\quad \underline{\pi u \tan(\frac{l}{m}\pi) - \pi^2 u^2 \tan^2(\frac{l}{m}\pi)} & \\
\qquad \qquad \pi^2 u^2 \tan^2(\frac{l}{m}\pi) &
\end{array}$$

$$= \pi \frac{1}{\cos(\frac{l}{m}\pi)} \left\{ 1 + \pi u \tan(\frac{l}{m}\pi) + \pi^2 u^2 \tan^2(\frac{l}{m}\pi) \right\} \frac{1}{u^2}$$

$$= \pi \frac{1}{\cos(\frac{l}{m}\pi)} \frac{1}{u^2} + \pi^2 \frac{\tan(\frac{l}{m}\pi)}{\cos(\frac{l}{m}\pi)} \frac{1}{u} + \pi^3 \frac{\tan^2(\frac{l}{m}\pi)}{\cos(\frac{l}{m}\pi)}$$

$$\text{Res}_{-1} \left\{ \pi \frac{1}{\cos(\pi z)} \frac{1}{(z - \frac{l}{m})^3} \right\}_{z = \frac{l}{m}} = \boxed{\pi^2 \frac{\tan(\frac{l}{m}\pi)}{\cos(\frac{l}{m}\pi)}}$$

$$= \pi^2 \frac{\sin(\frac{l}{m}\pi)}{\cos^2(\frac{l}{m}\pi)}$$

Therefore,

$$... + \frac{(-1)^{-3+1}}{(-3 + \frac{1}{2} - \frac{l}{m})^2} + \frac{(-1)^{-2+1}}{(-2 + \frac{1}{2} - \frac{l}{m})^2} + \frac{(-1)^{-1+1}}{(-1 + \frac{1}{2} - \frac{l}{m})^2} + \frac{(-1)^{0+1}}{(0 + \frac{1}{2} - \frac{l}{m})^2} +$$

$$+\frac{(-1)^{1+1}}{(1+\frac{1}{2}-\frac{l}{m})^2}+\frac{(-1)^{2+1}}{(2+\frac{1}{2}-\frac{l}{m})^2}+\frac{(-1)^{3+1}}{(3+\frac{1}{2}-\frac{l}{m})^2}+...+\frac{\pi^2\sin(\frac{l}{m}\pi)}{\cos^2(\frac{l}{m}\pi)}=0$$

$$\pi^2 = 4m^2\frac{\cos^2(\frac{l}{m}\pi)}{\sin(\frac{l}{m}\pi)}\Bigg(\frac{1}{(m-2l)^2}-\frac{1}{(m+2l)^2}-\frac{1}{(3m-2l)^2}+\frac{1}{(3m+2l)^2}+\frac{1}{(5m-2l)^2}-\frac{1}{(5m+2l)^2}-...\Bigg)$$

The Associated Series is

$$C_{l/m} = \frac{1}{(m-2l)^2}-\frac{1}{(m+2l)^2}-\frac{1}{(3m-2l)^2}+\frac{1}{(3m+2l)^2}+\frac{1}{(5m-2l)^2}-\frac{1}{(5m+2l)^2}-...=\frac{\pi^2\sin(\frac{l}{m}\pi)}{4m^2\cos^2(\frac{l}{m}\pi)}.\square$$

$7_{\tan}$.

$$\boxed{\pi^2 = 4m^2\cos^2(\tfrac{l}{m}\pi)\left\{\frac{1}{(m-2l)^2}+\frac{1}{(m+2l)^2}+\frac{1}{(3m-2l)^2}+\frac{1}{(3m+2l)^2}+\frac{1}{(5m-2l)^2}+\frac{1}{(5m+2l)^2}+\ldots\right\}}$$

$l < \frac{1}{2}m$, l and m have no common factor.

$$T_{l/m} = \frac{1}{(m-2l)^2}+\frac{1}{(m+2l)^2}+\frac{1}{(3m-2l)^2}+\frac{1}{(3m+2l)^2}+\frac{1}{(5m-2l)^2}+\frac{1}{(5m+2l)^2}+\ldots = \frac{\pi^2}{4m^2}\frac{1}{\cos^2(\frac{l}{m}\pi)}$$

Proof: $\pi\tan(\pi z)\dfrac{1}{(z-\frac{l}{m})^2}$ has poles of order 1 at $z = n+\frac{1}{2}$,

and a pole of order 2 at $z = \frac{l}{m}$

$$\oint_{\Box_N}\pi\tan(\pi\zeta)\frac{1}{(\zeta-\frac{l}{m})^2}d\zeta = \sum \text{Res}_{-1}\left\{\pi\tan(\pi z)\frac{1}{(z-\frac{l}{m})^2}\right\}_{z=n+\frac{1}{2}} + \text{Res}_{-1}\left\{\pi\tan(\pi z)\frac{1}{(z-\frac{l}{m})^2}\right\}_{z=\frac{l}{m}}$$

$$\left|\oint_{\Box_N}\pi\tan(\pi\zeta)\frac{1}{(\zeta-\frac{l}{m})^2}d\zeta\right| \le \underbrace{\left|\pi\tan(\pi\zeta)\right|}_{\le A}\underbrace{\oint_{\Box_N}\frac{1}{(\zeta-\frac{l}{m})^2}d\zeta}_{\le\left[\frac{1}{|\zeta|}\right]_{\Box_N}=0} = 0.\Box$$

$$\text{Res}_{-1}\left\{\pi\tan(\pi z)\frac{1}{(z-\frac{l}{m})^2}\right\}_{z=n+\frac{1}{2}} = \left[(z-n-\tfrac{1}{2})\pi\frac{\sin(\pi z)}{\cos(\pi z)}\frac{1}{(z-\frac{l}{m})^2}\right]_{z=n+\frac{1}{2}}$$

$$=\left[\frac{\pi D_z(z-n-\frac{1}{2})}{D_z\cos(\pi z)}\right]_{z=n+\frac{1}{2}}\sin\pi(n+\tfrac{1}{2})\frac{1}{(n+\frac{1}{2}-\frac{l}{m})^2}$$

$$=\left[\frac{\pi}{-\pi\sin(\pi z)}\right]_{z=n+\frac{1}{2}}\sin\pi(n+\tfrac{1}{2})\frac{1}{(n+\frac{1}{2}-\frac{l}{m})^2}$$

$$=\frac{-1}{(n+\frac{1}{2}-\frac{l}{m})^2}.\square$$

To find $\text{Res}_{-1}\left\{\pi\tan(\pi z)\frac{1}{(z-\frac{l}{m})^2}\right\}_{z=\frac{l}{m}}$**, divide the series**

$$\pi\tan(\pi z)\frac{1}{[z-\frac{l}{m}]^2}=\pi\frac{\sin(\pi[z-\frac{l}{m}]+\pi\frac{l}{m})}{\cos(\pi[z-\frac{l}{m}]+\pi\frac{l}{m})}\frac{1}{[z-\frac{l}{m}]^2}$$

$$=\pi\frac{\sin(\pi u+\pi\frac{l}{m})}{\cos(\pi u+\pi\frac{l}{m})}\frac{1}{u^2}$$

$$=\pi\frac{\sin(\pi u)\cos(\pi\frac{l}{m})+\cos(\pi u)\sin(\pi\frac{l}{m})}{\cos(\pi u)\cos(\pi\frac{l}{m})-\sin(\pi u)\sin(\pi\frac{l}{m})}\frac{1}{u^3}$$

$$\approx\pi\frac{(\pi u)\cos(\pi\frac{l}{m})+(1-\frac{1}{2}\pi^2u^2)\sin(\pi\frac{l}{m})}{(1-\frac{1}{2}\pi^2u^2)\cos(\pi\frac{l}{m})-(\pi u)\sin(\pi\frac{l}{m})}\frac{1}{u^3}$$

$$=\pi\frac{\sin(\pi\frac{l}{m})}{\cos(\pi\frac{l}{m})}\left(\frac{(\pi u)\cot(\pi\frac{l}{m})+(1-\frac{1}{2}\pi^2u^2)}{(1-\frac{1}{2}\pi^2u^2)-(\pi u)\tan(\pi\frac{l}{m})}\right)\frac{1}{u^3}$$

$$\approx \pi\tan(\tfrac{l}{m}\pi)\left(\frac{1+\pi u\cot(\tfrac{l}{m}\pi)}{1-\pi u\tan(\tfrac{l}{m}\pi)}\right)\frac{1}{u^2}$$

$$\begin{array}{l|l}
\underline{1+\pi u\{\cot(\tfrac{l}{m}\pi)+\tan(\tfrac{l}{m}\pi)\}+\pi^2u^2\{\cot(\tfrac{l}{m}\pi)+\tan(\tfrac{l}{m}\pi)\}\tan(\tfrac{l}{m}\pi)} & \\
1+\pi u\cot(\tfrac{l}{m}\pi) & 1-\pi u\tan(\tfrac{l}{m}\pi) \\
\underline{1-\pi u\tan(\tfrac{l}{m}\pi)} & \\
\quad \pi u\{\cot(\tfrac{l}{m}\pi)+\tan(\tfrac{l}{m}\pi)\} & \\
\quad \underline{\pi u\{\cot(\tfrac{l}{m}\pi)+\tan(\tfrac{l}{m}\pi)\}-\pi^2u^2\{\cot(\tfrac{l}{m}\pi)+\tan(\tfrac{l}{m}\pi)\}\tan(\tfrac{l}{m}\pi)} & \\
\qquad\qquad \pi^2u^2\{\cot(\tfrac{l}{m}\pi)+\tan(\tfrac{l}{m}\pi)\}\tan(\tfrac{l}{m}\pi) &
\end{array}$$

$$= \pi\tan(\tfrac{l}{m}\pi)\frac{1}{u^2}+\pi^2\tan(\tfrac{l}{m}\pi)\Big\{\cot(\tfrac{l}{m}\pi)+\tan(\tfrac{l}{m}\pi)\Big\}\frac{1}{u}$$

$$\operatorname{Res}_{-1}\left\{\pi\tan(\tfrac{l}{m}\pi)\frac{1}{(z-\tfrac{l}{m})^2}\right\}_{z=\frac{l}{m}} = \pi^2\tan(\tfrac{l}{m}\pi)\{\cot(\tfrac{l}{m}\pi)+\tan(\tfrac{l}{m}\pi)\}$$

$$= \pi^2\{1+\tan^2(\tfrac{l}{m}\pi)\}$$

$$= \pi^2\frac{1}{\cos^2(\tfrac{l}{m}\pi)}$$

Therefore,

$$\ldots-\frac{1}{(-3+\frac{1}{2}-\frac{l}{m})^2}-\frac{1}{(-2+\frac{1}{2}-\frac{l}{m})^2}-\frac{1}{(-1+\frac{1}{2}-\frac{l}{m})^2}-\frac{1}{(0+\frac{1}{2}-\frac{l}{m})^2}-$$

$$-\frac{1}{(1+\frac{1}{2}-\frac{l}{m})^2}-\frac{1}{(2+\frac{1}{2}-\frac{l}{m})^2}-\frac{1}{(3+\frac{1}{2}-\frac{l}{m})^2}-\ldots+\frac{\pi^2}{\cos^2(\frac{l}{m}\pi)}=0$$

$$\pi^2 = 4m^2\cos^2(\tfrac{l}{m}\pi)\left\{\frac{1}{(m-2l)^2}+\frac{1}{(m+2l)^2}+\frac{1}{(3m-2l)^2}+\frac{1}{(3m+2l)^2}+\frac{1}{(5m-2l)^2}+\frac{1}{(5m+2l)^2}+...\right\}$$

The Associated Series is

$$T_{l/m} = \frac{1}{(m-2l)^2}+\frac{1}{(m+2l)^2}+\frac{1}{(3m-2l)^2}+\frac{1}{(3m+2l)^2}+\frac{1}{(5m-2l)^2}+\frac{1}{(5m+2l)^2}+... = \frac{\pi^2}{4m^2}\frac{1}{\cos^2(\frac{l}{m}\pi)}. \square$$

8.

8.1

$$\pi^2 = 38^2 \sin^2(\tfrac{15}{38}\pi)\left\{\frac{1}{15^2} + \frac{1}{(38-15)^2} + \frac{1}{(38+15)^2} + \frac{1}{[2(38)-15]^2} + \frac{1}{[2(38)+15]^2} + \frac{1}{[3(38)-15]^2} + ...\right\}$$

$$R_{15/38} = \left\{\frac{1}{15^2} + \frac{1}{(38-15)^2} + \frac{1}{(38+15)^2} + \frac{1}{[2(38)-15]^2} + \frac{1}{[2(38)+15]^2} + \frac{1}{[3(38)-15]^2} + ...\right\} = \frac{\pi^2}{38^2 \sin^2(\tfrac{15}{38}\pi)}$$

Proof: By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\dfrac{1}{(z-\tfrac{15}{38})^2}$.□

8.2

$$\pi^2 = 38^2 \frac{\sin^2(\tfrac{15}{38}\pi)}{\cos(\tfrac{15}{38}\pi)}\left\{\frac{1}{15^2} - \frac{1}{(38-15)^2} - \frac{1}{(38+15)^2} + \frac{1}{[2(38)-15]^2} + \frac{1}{[2(38)+15]^2} - \frac{1}{[3(38)-15]^2} - \frac{1}{[3(38)+15]^2} + ...\right\}$$

$$S_{15/38} = \left\{\frac{1}{15^2} - \frac{1}{(38-15)^2} - \frac{1}{(38+15)^2} + \frac{1}{[2(38)-15]^2} + \frac{1}{[2(38)+15]^2} - \frac{1}{[3(38)-15]^2} - \frac{1}{[3(38)+15]^2} + ...\right\} = \frac{\pi^2\cos(\tfrac{15}{38}\pi)}{38^2 \sin^2(\tfrac{15}{38}\pi)}$$

Proof: By $\mathbf{7_{sin}}$. based on $\pi \dfrac{1}{\sin(\pi z)} \dfrac{1}{(z+\frac{15}{38})^2}$. □

8.3

$$\begin{aligned}\pi^2 = 4(38)^2 \frac{\cos^2(\frac{15}{38}\pi)}{\sin(\frac{15}{38}\pi)} \bigg(&\frac{1}{(38-2(15))^2} - \frac{1}{(38+2(15))^2} \\ &- \frac{1}{(3(38)-2(15))^2} + \frac{1}{(3(38)+2(15))^2} \\ &+ \frac{1}{(5(38)-2(15))^2} - \frac{1}{(5(38)+2(15))^2} - \ldots \bigg)\end{aligned}$$

$$\begin{aligned}C_{15/38} = &\frac{1}{(38-2(15))^2} - \frac{1}{(38+2(15))^2} \\ &- \frac{1}{(3(38)-2(15))^2} + \frac{1}{(3(38)+2(15))^2} \\ &+ \frac{1}{(5(38)-2(15))^2} - \frac{1}{(5(38)+2(15))^2} - \ldots = \frac{\pi^2 \sin(\frac{15}{38}\pi)}{4(38)^2\cos^2(\frac{15}{38}\pi)}\end{aligned}$$

Proof: By $\mathbf{7_{cos}}$. based on $\pi \dfrac{1}{\cos(\pi z)} \dfrac{1}{(z-\frac{15}{38})^2}$. □

8.4

$$\begin{aligned}\pi^2 = 4(38^2)\cos^2(\tfrac{15}{38}\pi) \bigg\{ &\frac{1}{(38-2(15))^2} + \frac{1}{(38+2(15))^2} \\ &+ \frac{1}{(3(38-2(15))^2} + \frac{1}{(3(38)+2(15))^2} \\ &+ \frac{1}{(5(38)-2(15))^2} + \frac{1}{(5(38)+2(15))^2} + \ldots \bigg\}\end{aligned}$$

$$T_{15/38} = \frac{1}{(38-2(15))^2} + \frac{1}{(38+2(15))^2}$$

$$+\frac{1}{(3(38)-2(15))^2}+\frac{1}{(3(38)+2(15))^2}$$

$$+\frac{1}{(5(38)-2(15))^2}+\frac{1}{(5(38)+2(15))^2}+...=\frac{\pi^2}{4(38)^2}\frac{1}{\cos^2(\frac{15}{38}\pi)}$$

Proof: By $\mathbf{7_{tan}}$. based on $\pi\tan(\pi z)\dfrac{1}{(z-\frac{15}{38})^2}$.□

9.

9.1
$$\pi^2 = 2^{2n}\sin^2(\tfrac{2k+1}{2^n}\pi)\left\{\frac{1}{(2k+1)^2}+\frac{1}{(2^n-2k-1)^2}+\frac{1}{(2^n+2k+1)^2}+\frac{1}{(2\cdot 2^n-2k-1)^2}+\frac{1}{(2\cdot 2^n+2k+1)^2}+\frac{1}{(3\cdot 2^n-2k-1)^2}+\dots\right\}$$

$$R_{(2k+1)/2^n} = \frac{1}{(2k+1)^2}+\frac{1}{(2^n-2k-1)^2}+\frac{1}{(2^n+2k+1)^2}+\frac{1}{(2\cdot 2^n-2k-1)^2}+\frac{1}{(2\cdot 2^n+2k+1)^2}+\frac{1}{(3\cdot 2^n-2k-1)^2}+\dots = \frac{\pi^2}{2^{2n}\sin^2(\frac{2k+1}{2^n}\pi)}$$

$$2k+1 < 2^{n-1}$$

$$\sin^2(\tfrac{2k+1}{2^n}\pi) = \tfrac{1}{2}[1-\cos(\tfrac{2k+1}{2^{n-1}}\pi)]$$

$$= 1-\tfrac{1}{\sqrt{2}}\sqrt{1+\cos(\tfrac{2k+1}{2^{n-2}}\pi)} = \dots.$$

$$= \text{ a number composed of } \sqrt{2}\text{'s}$$

<u>*Proof*</u>: By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\dfrac{1}{(z+\frac{2k+1}{2^n})^2}$.□

9.2
$$\pi^2 = 2^{2n}\frac{\sin^2(\frac{2k+1}{2^n}\pi)}{\cos(\frac{2k+1}{2^n}\pi)}\left\{\frac{1}{(2k+1)^2}+\frac{1}{(2^n-2k-1)^2}-\frac{1}{(2^n+2k+1)^2}-\frac{1}{(2\cdot 2^n-2k-1)^2}+\frac{1}{(2\cdot 2^n+2k+1)^2}+\frac{1}{(3\cdot 2^n-2k-1)^2}-\frac{1}{(3\cdot 2^n+2k+1)^2}+\dots\right\}$$

$$S_{(2k+1)/2^n} = \frac{1}{(2k+1)^2} - \frac{1}{(2^n - 2k - 1)^2} - \frac{1}{(2^n + 2k + 1)^2}$$
$$+ \frac{1}{(2\cdot 2^n - 2k - 1)^2} + \frac{1}{(2\cdot 2^n + 2k + 1)^2}$$
$$- \frac{1}{(3\cdot 2^n - 2k - 1)^2} - \frac{1}{(3\cdot 2^n + 2k + 1)^2} + ... = \frac{\pi^2 \cos(\frac{2k+1}{2^n}\pi)}{2^{2n}\sin^2(\frac{2k+1}{2^n}\pi)}$$

$$\cos(\tfrac{2k+1}{2^n}\pi) = \sqrt{\tfrac{1}{2}[1 + \cos(\tfrac{2k+1}{2^{n-1}}\pi)]}$$
$$= \text{ a number composed of } \sqrt{2}\text{'s}$$

Proof: By $\mathbf{7_{sin}}$. based on $\pi\dfrac{1}{\sin(\pi z)}\dfrac{1}{(z + \frac{2k+1}{2^n})^2}$.□

9.3
$$\boxed{\begin{aligned}\pi^2 = 4(2^{2n})\frac{\cos^2(\frac{2k+1}{2^n}\pi)}{\sin(\frac{2k+1}{2^n}\pi)}\Bigg(&\frac{1}{(2^n - 2(2k+1))^2} - \frac{1}{(2^n + 2(2k+1))^2}\\ &- \frac{1}{(3(2^n) - 2(2k+1))^2} + \frac{1}{(3(2^n) + 2(2k+1))^2}\\ &+ \frac{1}{(5(2^n) - 2(2k+1))^2} - \frac{1}{(5(2^n) + 2(2k+1))^2} - ...\Bigg)\end{aligned}}$$

$$C_{l/m} = \frac{1}{(2^n - 2(2k+1))^2} - \frac{1}{(2^n + 2(2k+1))^2}$$
$$- \frac{1}{(3(2^n) - 2(sk+1))^2} + \frac{1}{(3(2^n) + 2(2k+1))^2}$$
$$+ \frac{1}{(5(2^n) - 2(2k+1))^2} - \frac{1}{(5(2^n) + 2(2k+1))^2} - ... = \frac{\pi^2 \sin(\frac{l}{m}\pi)}{4m^2\cos^2(\frac{l}{m}\pi)}$$

Proof: By $\mathbf{7_{cos}}$. based on $\pi\dfrac{1}{\cos(\pi z)}\dfrac{1}{(z - \frac{2k+1}{2^n})^2}$.□

9.4

$$\pi^2 = 4(2^{2n})\cos^2(\tfrac{2k+1}{2^n}\pi)\Bigg(\frac{1}{(2^n - 2(2k+1))^2} + \frac{1}{(2^n + 2(2k+1))^2} + \frac{1}{(3(2^n) - 2(2k+1))^2} + \frac{1}{(3(2^n) + 2(2k+1))^2} + \frac{1}{(5(2^n) - 2(2k+1))^2} + \frac{1}{(5(2^n) + 2(2k+1))^2} - \dots\Bigg)$$

$$T_{l/m} = \frac{1}{(2^n - 2(2k+1))^2} + \frac{1}{(2^n + 2(2k+1))^2} + \frac{1}{(3(2^n) - 2(2k+1))^2} + \frac{1}{(3(2^n) + 2(2k+1))^2} + \frac{1}{(5(2^n) - 2(2k+1))^2} + \frac{1}{(5(2^n) + 2(2k+1))^2} - \dots =$$

$$= \frac{\pi^2}{4(2^{2n})}\frac{1}{\cos^2(\frac{2k+1}{2^n}\pi)}$$

__Proof__: By $\mathbf{7_{tan}}$. based on $\pi\tan(\pi z)\dfrac{1}{(z - \frac{2k+1}{2^n})^2}$.□

10.

10.1
$$\boxed{\pi^2 = 8\left(\frac{1}{1}+\frac{1}{3^2} + \frac{1}{5^2}+\frac{1}{7^2} + \frac{1}{9^2}+\frac{1}{11^2}+...\right)} \text{ (Euler)}$$

$$R_{1/4} = \frac{1}{1}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\frac{1}{13^2}+... = \frac{\pi^2}{8} \text{ (Euler)}$$

Proof: By $\mathbf{7}_{\mathbf{cot}}$. based on $\pi\cot(\pi z)\dfrac{1}{(z-\frac{1}{4})^2}$.

$$\pi^2 = 4^2\sin^2(\tfrac{1}{4}\pi)\left(\frac{1}{1^2}+\frac{1}{(4-1)^2} + \frac{1}{(4+1)^2}+\frac{1}{(2(4)-1)^2} + \frac{1}{(2(4)+1)^2}+\frac{1}{(3(4)-1)^2}+...\right)$$

Or, by $\mathbf{7}_{\mathbf{tan}}$. based on $\pi\tan(\pi z)\dfrac{1}{(z-\frac{1}{4})^2}$,

$$\pi^2 = 4(4)^2\cos^2(\tfrac{1}{4}\pi)\left(\frac{1}{(4-2)^2}+\frac{1}{(4+2)^2} + \frac{1}{(3(4)-2)^2}+\frac{1}{(3(4)+2)^2} + \frac{1}{(5(4)-2)^2}+\frac{1}{(5(4)+2)^2}-...\right).\square$$

10.2

$$\boxed{\pi^2 = 8\sqrt{2}\left(\frac{1}{1}+\frac{1}{3^2} - \frac{1}{5^2}-\frac{1}{7^2} + \frac{1}{9^2}+\frac{1}{11^2}-...\right)}$$

$$S_{1/4} = \frac{1}{1^2}-\frac{1}{3^2}-\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}-\frac{1}{11^2}-\frac{1}{13^2}+... = \frac{\pi^2}{8\sqrt{2}}$$

Proof: By $\mathbf{7_{sin}}$. based on $\pi\frac{1}{\sin(\pi z)}\frac{1}{(z+\frac{1}{4})^2}$.□

$$\pi^2 = 4^2\frac{\sin^2(\frac{1}{4}\pi)}{\cos(\frac{1}{4}\pi)}\left(\frac{1}{1^2}+\frac{1}{(4-1)^2} - \frac{1}{(4+1)^2}-\frac{1}{(2(4)-1)^2} + \frac{1}{(2(4)+1)^2}+\frac{1}{(3(4)-1)^2}-...\right).\square$$

Or, by $\mathbf{7_{cos}}$. based on $\pi\frac{1}{\cos(\pi z)}\frac{1}{(z-\frac{1}{4})^2}$,

$$\pi^2 = 4(4)^2\frac{\cos^2(\frac{1}{4}\pi)}{\sin(\frac{1}{4}\pi)}\left(\frac{1}{(4-2)^2}-\frac{1}{(4+2)^2} - \frac{1}{(3(4)-2)^2}+\frac{1}{(3(4)+2)^2} + \frac{1}{(5(4)-2)^2}-\frac{1}{(5(4)+2)^2}-...\right).\square$$

10.3

$$\boxed{\sqrt{2} = \frac{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \dots}{1 - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} - \frac{1}{13^2} + \dots}}$$

11.

11.1

$$\boxed{\pi^2 = 8^2 \sin^2(\tfrac{1}{8}\pi)\left(\frac{1}{1} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{17^2} + \frac{1}{23^2} + \dots\right)}$$

$$R_{1/8} = \frac{1}{1} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{17^2} + \frac{1}{23^2} + \dots = \frac{\pi^2}{8^2 \sin^2(\frac{1}{8}\pi)}$$

Proof: By **$7_{\mathbf{cot}}$.** based on $\pi\cot(\pi z)\dfrac{1}{(z+\frac{1}{8})^2}$.□

$$\pi^2 = 8^2 \sin^2(\tfrac{1}{8}\pi)\left(\frac{1}{1^2} + \frac{1}{(8-1)^2} + \frac{1}{(8+1)^2} + \frac{1}{(2(8)-1)^2} + \frac{1}{(2(8)+1)^2} + \frac{1}{(3(8)-1)^2} + \dots\right).\square$$

11.2

$$\boxed{\pi^2 = 8^2 \frac{\sin^2(\frac{1}{8}\pi)}{\cos(\frac{1}{8}\pi)}\left(\frac{1}{1} - \frac{1}{7^2} - \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{17^2} - \frac{1}{23^2} - \dots\right)}$$

$$S_{1/8} = \frac{1}{1} - \frac{1}{7^2} - \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{17^2} - \frac{1}{23^2} - \dots = \frac{\pi^2\cos(\frac{1}{8}\pi)}{8^2 \sin^2(\frac{1}{8}\pi)}$$

$$\sin^2(\tfrac{1}{8}\pi) = \frac{2-\sqrt{2}}{4}$$

$$\cos(\tfrac{1}{8}\pi) = \frac{\sqrt{2+\sqrt{2}}}{2}$$

Proof: By $\mathbf{7_{sin}}$. based on $\pi \frac{1}{\sin(\pi z)} \frac{1}{(z+\frac{1}{8})^2}$.□

$$\pi^2 = 8^2 \frac{\sin^2(\frac{1}{8}\pi)}{\cos(\frac{1}{8}\pi)} \Bigg(\frac{1}{1^2} - \frac{1}{(8-1)^2} - \frac{1}{(8+1)^2} + \frac{1}{(2(8)-1)^2} + \frac{1}{(2(8)+1)^2} - \frac{1}{(3(8)-1)^2} - \dots \Bigg).\square$$

$$\sin^2(\tfrac{1}{8}\pi) = \tfrac{1}{2}\Big[1-\cos(\tfrac{1}{4}\pi)\Big] = \frac{2-\sqrt{2}}{4}.\square$$

$$\cos(\tfrac{1}{8}\pi) = \sqrt{\tfrac{1}{2}\Big[1+\cos(\tfrac{1}{4}\pi)\Big]} = \sqrt{\tfrac{1}{2}\Big[1+\tfrac{\sqrt{2}}{2}\Big]} = \frac{\sqrt{2+\sqrt{2}}}{2}.\square$$

11.3

$$\boxed{\pi^2 = 8^2 \frac{\cos^2(\frac{1}{8}\pi)}{\sin(\frac{1}{8}\pi)} \Bigg(\frac{1}{3^2} - \frac{1}{5^2} - \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{19^2} - \frac{1}{21^2} - \dots \Bigg)}$$

$$C_{1/8}=\frac{1}{3^2}-\frac{1}{5^2}-\frac{1}{11^2}+\frac{1}{13^2}+\frac{1}{19^2}-\frac{1}{21^2}-...=\frac{\pi^2\sin(\frac{1}{8}\pi)}{8^2\cos^2(\frac{1}{8}\pi)}$$

$$\cos^2(\tfrac{1}{8}\pi)=\frac{2+\sqrt{2}}{4}$$

$$\sin(\tfrac{1}{8}\pi)=\frac{\sqrt{2-\sqrt{2}}}{2}$$

Proof: By $\mathbf{7_{cos}}$. based on $\pi\frac{1}{\cos(\pi z)}\frac{1}{(z-\frac{1}{8})^2}$.□

$$\pi^2=4(8^2)\frac{\cos^2(\frac{1}{8}\pi)}{\sin(\frac{1}{8}\pi)}\left(\frac{1}{(8-2)^2}-\frac{1}{(8+2)^2}-\frac{1}{(3(8)-2)^2}+\frac{1}{(3(8)+2)^2}+\frac{1}{(5(8)-2)^2}-\frac{1}{(5(8)+2)^2}-...\right).\square$$

$$\sin(\tfrac{1}{8}\pi)=\sqrt{\tfrac{1}{2}\left[1-\cos(\tfrac{1}{4}\pi)\right]}=\sqrt{\tfrac{1}{2}\left[1-\tfrac{\sqrt{2}}{2}\right]}=\frac{\sqrt{2-\sqrt{2}}}{2}.\square$$

11.4

$$\boxed{\pi^2=8^2\cos^2(\tfrac{1}{8}\pi)\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{11^2}+\frac{1}{13^2}+\frac{1}{19^2}+\frac{1}{21^2}-...\right)}$$

$$T_{1/8} = \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{21^2} - ... = \frac{\pi^2}{8^2 \cos^2(\frac{1}{8}\pi)}$$

$\cos^2(\frac{1}{8}\pi) = \frac{1}{2}(1 + \frac{\sqrt{2}}{2})$ **depends on** $\sqrt{2}$**'s**

Proof: By $\mathbf{7_{tan}}$. based on $\pi \tan(\pi z)\frac{1}{(z - \frac{1}{8})^2}$.□

$$\pi^2 = 4(8^2)\cos^2(\tfrac{1}{8}\pi)\left(\frac{1}{(8-2)^2} + \frac{1}{(8+2)^2} + \frac{1}{(3(8)-2)^2} + \frac{1}{(3(8)+2)^2} + \frac{1}{(5(8)-2)^2} + \frac{1}{(5(8)+2)^2} - ...\right).\square$$

$$\cos^2(\tfrac{1}{8}\pi) = \tfrac{1}{2}(1 + \cos\tfrac{1}{4}\pi)$$

$$= \tfrac{1}{2}(1 + \tfrac{\sqrt{2}}{2}).\square$$

12.

12.1

$$\pi^2 = 8^2 \sin^2(\tfrac{3}{8}\pi)\left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{21^2} + \dots\right)$$

$$R_{3/8} = \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{21^2} + \frac{1}{27^2} + \dots = \frac{\pi^2}{8^2 \sin^2(\frac{3}{8}\pi)}$$

$$\sin^2(\tfrac{3}{8}\pi) = \frac{2+\sqrt{2}}{4} = \textbf{depends on } \sqrt{2}\text{'s}$$

Proof: By **7$_{\text{cot}}$.** based on $\pi\cot(\pi z)\dfrac{1}{(z+\frac{3}{8})^2}$,

$$\pi^2 = 8^2 \sin^2(\tfrac{3}{8}\pi)\left(\frac{1}{3^2} + \frac{1}{(8-3)^2} + \frac{1}{(8+3)^2} + \frac{1}{(2(8)-3)^2} + \frac{1}{(2(8)+3)^2} + \frac{1}{(3(8)-3)^2} + \dots\right) .\square$$

12.2

$$\pi^2 = 8^2 \frac{\sin^2(\frac{3}{8}\pi)}{\cos(\frac{3}{8}\pi)}\left(\frac{1}{3^2} - \frac{1}{5^2} - \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{19^2} - \frac{1}{21^2} + \dots\right)$$

$$S_{3/8} = \frac{1}{3^2} - \frac{1}{5^2} - \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{19^2} - \frac{1}{21^2} - \frac{1}{27^2} + \dots = \frac{\pi^2 \cos(\frac{3}{8}\pi)}{8^2 \sin^2(\frac{3}{8}\pi)}$$

$$\sin^2(\tfrac{3}{8}\pi) = \frac{2+\sqrt{2}}{4} = \text{depends on } \sqrt{2}\text{'s}$$

$$\cos(\tfrac{3}{8}\pi) = \frac{\sqrt{2-\sqrt{2}}}{2} = \text{ depends on } \sqrt{2}\text{'s}$$

Proof By $\mathbf{7_{sin}}$. based on $\pi\frac{1}{\sin(\pi z)}\frac{1}{(z+\frac{3}{8})^2}$,

$$\pi^2 = 8^2\frac{\sin^2(\frac{3}{8}\pi)}{\cos(\frac{3}{8}\pi)}\Bigg(\frac{1}{3^2} - \frac{1}{(8-3)^2} - \frac{1}{(8+3)^2} + \frac{1}{(2(8)-3)^2} + \frac{1}{(2(8)+3)^2} - \frac{1}{(3(8)-3)^2} - ...\Bigg) . \square$$

$$\sin^2(\tfrac{3}{8}\pi) = \tfrac{1}{2}\Big[1-\cos(\tfrac{3}{4}\pi)\Big] = \frac{2+\sqrt{2}}{4} . \square$$

$$\cos(\tfrac{3}{8}\pi) = \sqrt{\tfrac{1}{2}\Big[1+\cos(\tfrac{3}{4}\pi)\Big]} = \frac{\sqrt{2-\sqrt{2}}}{2} . \square$$

12.3

$$\boxed{\pi^2 = (8^2)\frac{\cos^2(\frac{3}{8}\pi)}{\sin(\frac{3}{8}\pi)}\Bigg(\frac{1}{1^2} - \frac{1}{7^2} - \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{17^2} - \frac{1}{23^2} - ...\Bigg)}$$

$$C_{3/8} = \frac{1}{1^2} - \frac{1}{7^2} - \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{17^2} - \frac{1}{23^2} - \frac{1}{25^2} + ... = \frac{\pi^2\sin(\frac{3}{8}\pi)}{8^2\cos^2(\frac{3}{8}\pi)}$$

$$\cos^2(\tfrac{3}{8}\pi) = \frac{2-\sqrt{2}}{4} = \text{depends on } \sqrt{2}\text{'s}$$

$$\sin(\tfrac{3}{8}\pi) = \frac{\sqrt{2+\sqrt{2}}}{2} = \textbf{ depends on } \sqrt{2}\text{'s}$$

Proof By $\mathbf{7_{cos}}$. based on $\pi \frac{1}{\cos(\pi z)} \frac{1}{(z-\frac{3}{8})^2}$,

$$\pi^2 = 4(8^2)\frac{\cos^2(\frac{3}{8}\pi)}{\sin(\frac{3}{8}\pi)}\left(\frac{1}{(8-2(3))^2} - \frac{1}{(8+2(3))^2} - \frac{1}{(3(8)-2(3))^2} + \frac{1}{(3(8)+2(3))^2} + \frac{1}{(5(8)-2(3))^2} - \frac{1}{(5(8)+2(3))^2} - \dots\right).\square$$

$$\cos^2(\tfrac{3}{8}\pi) = \tfrac{1}{2}\left[1+\cos(\tfrac{3}{4}\pi)\right] = \frac{2-\sqrt{2}}{4}.\square$$

$$\sin(\tfrac{3}{8}\pi) = \sqrt{\tfrac{1}{2}\left[1+\cos(\tfrac{3}{4}\pi)\right]} = \frac{\sqrt{2+\sqrt{2}}}{2}.\square$$

12.4

$$\boxed{\pi^2 = (8^2)\cos^2(\tfrac{3}{8}\pi)\left(\frac{1}{1^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{17^2} + \frac{1}{23^2} - \dots\right)}$$

$$T_{3/8} = \frac{1}{1^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{17^2} + \frac{1}{23^2} + \frac{1}{25^2} + \dots = \frac{\pi^2}{8^2}\frac{1+\tan(\frac{3}{8}\pi)}{\cot(\frac{3}{8}\pi)}$$

$$\cos^2(\tfrac{3}{8}\pi) = \frac{2-\sqrt{2}}{4} = \textbf{depends on } \sqrt{2}\text{'s}$$

$$\sin(\tfrac{3}{8}\pi) = \frac{\sqrt{2+\sqrt{2}}}{2} = \textbf{ depends on } \sqrt{2}\text{'s}$$

Proof **By $\mathbf{7_{tan}}$. based on** $\pi\tan(\pi z)\dfrac{1}{(z-\frac{3}{8})^2}$,

$$\pi^2 = 4(8^2)\cos^2(\tfrac{3}{8}\pi)\left(\frac{1}{(8-2(3))^2}+\frac{1}{(8+2(3))^2} + \frac{1}{(3(8)-2(3))^2}+\frac{1}{(3(8)+2(3))^2} + \frac{1}{(5(8)-2(3))^2}+\frac{1}{(5(8)+2(3))^2}-...\right).\square$$

$$\cos^2(\tfrac{3}{8}\pi) = \tfrac{1}{2}\left[1+\cos(\tfrac{3}{4}\pi)\right] = \frac{2-\sqrt{2}}{4}.\square$$

$$\sin(\tfrac{3}{8}\pi) = \sqrt{\tfrac{1}{2}\left[1+\cos(\tfrac{3}{4}\pi)\right]} = \frac{\sqrt{2+\sqrt{2}}}{2}.\square$$

13.

$R_{1/8} + R_{3/8} = R_{1/4}$

$$\boxed{R_{1/8} + R_{3/8} = \frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ... = \frac{\pi^2}{8} = R_{1/4}}$$

Proof: By $\mathbf{7_{cot}}$.

$$R_{1/8} = \frac{1}{1} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{17^2} + \frac{1}{23^2} + \frac{1}{25^2} - ... = \frac{\pi^2}{8^2 \sin^2(\frac{1}{8}\pi)}$$

$$R_{3/8} = \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{21^2} + \frac{1}{27^2} + ... = \frac{\pi^2}{8^2 \sin^2(\frac{3}{8}\pi)}$$

$$\Rightarrow \quad R_{1/8} + R_{3/8} = \frac{1}{1} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ... = \frac{\pi^2}{8} = R_{1/4}$$

Or,

$$R_{1/8} + R_{3/8} = \frac{\pi^2}{8^2}\left\{\frac{1}{\sin^2(\frac{1}{8}\pi)} + \frac{1}{\sin^2(\frac{3}{8}\pi)}\right\}$$

$$= \frac{\pi^2}{8^2}\left\{\frac{1}{\frac{1}{2}\{1-\cos(\frac{1}{4}\pi)\}} + \frac{1}{\frac{1}{2}\{1-\cos(\frac{3}{4}\pi)\}}\right\}$$

$$= \frac{\pi^2}{8^2} 2\left\{\frac{1}{1-\frac{\sqrt{2}}{2}} + \frac{1}{1+\frac{\sqrt{2}}{2}}\right\}$$

$$= \frac{\pi^2}{8 \cdot 4}\frac{2}{1-\frac{2}{4}} = \frac{\pi^2}{8} = R_{1/4}$$

14.

$S_{1/8} - S_{3/8}$

$$1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}-\frac{1}{9^2}+\frac{1}{11^2}-\frac{1}{13^2}+\frac{1}{15^2}+\frac{1}{17^2}-\frac{1}{19^2}+\frac{1}{21^2}-\frac{1}{23^2}-..=$$
$$=\frac{\pi^2}{8^2}\left\{\frac{\cos(\frac{1}{8}\pi)}{\sin^2(\frac{1}{8}\pi)}-\frac{\cos(\frac{3}{8}\pi)}{\sin^2(\frac{3}{8}\pi)}\right\}$$

Proof: By $\mathbf{7_{sin}}$.

$$S_{1/8}=\frac{1}{1}-\frac{1}{7^2}-\frac{1}{9^2}+\frac{1}{15^2}+\frac{1}{17^2}-\frac{1}{23^2}-\frac{1}{25^2}+...=\frac{\pi^2\cos(\frac{1}{8}\pi)}{8^2\sin^2(\frac{1}{8}\pi)}$$

$$S_{3/8}=\frac{1}{3^2}-\frac{1}{5^2}-\frac{1}{11^2}+\frac{1}{13^2}+\frac{1}{19^2}-\frac{1}{21^2}-\frac{1}{27^2}+...=\frac{\pi^2\cos(\frac{3}{8}\pi)}{8^2\sin^2(\frac{3}{8}\pi)}$$

$S_{1/8} - S_{3/8} =$

$$1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}-\frac{1}{9^2}+\frac{1}{11^2}-\frac{1}{13^2}+\frac{1}{15^2}+\frac{1}{17^2}-\frac{1}{19^2}+\frac{1}{21^2}-\frac{1}{23^2}-..=$$

$$=\frac{\pi^2}{8^2}\left\{\frac{\cos(\frac{1}{8}\pi)}{\sin^2(\frac{1}{8}\pi)}-\frac{\cos(\frac{3}{8}\pi)}{\sin^2(\frac{3}{8}\pi)}\right\}$$

$$=\frac{\pi^2}{8^2}\left\{\frac{\frac{\sqrt{2+\sqrt{2}}}{2}}{\frac{2-\sqrt{2}}{2^2}}-\frac{\frac{\sqrt{2-\sqrt{2}}}{2}}{\frac{2+\sqrt{2}}{2^2}}\right\}$$

$$=\frac{\pi^2}{8^2}\left\{\left(\sqrt{2+\sqrt{2}}\right)^3-\left(\sqrt{2-\sqrt{2}}\right)^3\right\}$$

15.

$$R_{1/8} - R_{3/8} = S_{1/4}$$

$$= \frac{1}{1} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} - \frac{1}{13^2} + \frac{1}{15^2} + \frac{1}{17^2} - \frac{1}{19^2} - \frac{1}{21^2}$$

$$+ \frac{1}{23^2} + \frac{1}{25^2} - \frac{1}{27^2} - \frac{1}{29^2} ... = S_{1/4} = \frac{\pi^2}{8\sqrt{2}}$$

Proof:

By $\mathbf{7_{cot}}$.

$$R_{1/8} = \frac{1}{1} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{17^2} + \frac{1}{23^2} + \frac{1}{25^2} - ... = \frac{\pi^2}{8^2 \sin^2(\frac{1}{8}\pi)}$$

$$R_{3/8} = \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{21^2} + \frac{1}{27^2} + ... = \frac{\pi^2}{8^2 \sin^2(\frac{3}{8}\pi)}$$

$$R_{1/8} - R_{3/8} =$$

$$= \frac{1}{1} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} - \frac{1}{13^2} + \frac{1}{15^2} + \frac{1}{17^2} - \frac{1}{19^2} - \frac{1}{21^2}$$

$$+ \frac{1}{23^2} + \frac{1}{25^2} - \frac{1}{27^2} - \frac{1}{29^2} ... = S_{1/4} = \frac{\pi^2}{8\sqrt{2}}$$

16.

16.1

$$\boxed{\pi^2 = 16^2\sin^2(\tfrac{1}{16}\pi)\left(\frac{1}{1}+\frac{1}{15^2}+\frac{1}{17^2}+\frac{1}{31^2}+\frac{1}{33^2}+\frac{1}{47^2}+\ldots\right)}$$

$$R_{1/16} = \frac{1}{1}+\frac{1}{15^2}+\frac{1}{17^2}+\frac{1}{31^2}+\frac{1}{33^2}+\frac{1}{47^2}+\frac{1}{49^2}+\ldots=\frac{\pi^2}{16^2\sin^2(\frac{1}{16}\pi)}$$

$$\sin^2(\tfrac{1}{16}\pi) = \tfrac{1}{2}(\sqrt{2-\sqrt{2+\sqrt{2}}}) = \text{algebraic number}$$

Proof: By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\dfrac{1}{(z+\frac{1}{16})^2}$,

$$\pi^2 = 16^2\sin^2(\tfrac{1}{16}\pi)\left(\frac{1}{1^2}+\frac{1}{(16-1)^2}+\frac{1}{(16+1)^2}+\frac{1}{(2(16)-1)^2}+\frac{1}{(2(16)+1)^2}+\frac{1}{(3(16)-1)^2}+\ldots\right).\square$$

$$\sin^2(\tfrac{1}{16}\pi) = \tfrac{1}{2}\left[1-\cos(\tfrac{1}{8}\pi)\right]$$

$$= \tfrac{1}{2}\left(1-\sqrt{\tfrac{1}{2}[1+\cos\tfrac{1}{4}\pi]}\right)$$

$$= \tfrac{1}{2}\left(1-\sqrt{\tfrac{1}{2}[1+\tfrac{\sqrt{2}}{2}]}\right) = \tfrac{2-\sqrt{2+\sqrt{2}}}{4} = \text{ algebraic number.}\square$$

16.2

$$\boxed{\pi^2 = 16^2 \frac{\sin^2(\frac{1}{16}\pi)}{\cos(\frac{1}{16}\pi)}\left(\frac{1}{1} - \frac{1}{15^2} - \frac{1}{17^2} + \frac{1}{31^2} + \frac{1}{33^2} - \frac{1}{47^2} - \ldots\right)}$$

$$S_{1/16} = \frac{1}{1} - \frac{1}{15^2} - \frac{1}{17^2} + \frac{1}{31^2} + \frac{1}{33^2} - \frac{1}{47^2} - \frac{1}{49^2} + \ldots = \frac{\pi^2 \cos(\frac{1}{16}\pi)}{16^2 \sin^2(\frac{1}{16}\pi)}$$

$$\frac{\sin^2(\frac{1}{16}\pi)}{\cos(\frac{1}{16}\pi)} = \frac{\frac{1}{4}(2 - \sqrt{2+\sqrt{2}})}{\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}} = \text{algebraic number}$$

Proof: By $\mathbf{7_{sin}}$. based on $\pi \frac{1}{\sin(\pi z)} \frac{1}{(z+\frac{1}{16})^2}$,

$$\pi^2 = 16^2 \frac{\sin^2(\frac{1}{16}\pi)}{\cos(\frac{1}{16}\pi)}\left(\frac{1}{1^2} - \frac{1}{(16-1)^2} - \frac{1}{(16+1)^2} + \frac{1}{(2(16)-1)^2} + \frac{1}{(2(16)+1)^2} - \frac{1}{(3(16)-1)^2} + \ldots\right). \square$$

$$\frac{\sin^2(\frac{1}{16}\pi)}{\cos(\frac{1}{16}\pi)} = \frac{\frac{1}{4}(2 - \sqrt{2+\sqrt{2}})}{\sqrt{\frac{1}{2}\left[1+\cos(\frac{1}{8}\pi)\right]}}$$

$$= \frac{\frac{1}{4}(2 - \sqrt{2+\sqrt{2}})}{\sqrt{\frac{1}{2}\left[1+\sqrt{\frac{1}{2}\left[1+\cos(\frac{1}{4}\pi)\right]}\right]}}$$

$$= \frac{\frac{1}{4}(2 - \sqrt{2+\sqrt{2}})}{\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}} = \text{algebraic number}. \square$$

16.3

$$\boxed{\pi^2 = 16^2 \frac{\cos^2(\frac{1}{16}\pi)}{\sin(\frac{1}{16}\pi)}\left(\frac{1}{7^2} - \frac{1}{9^2} - \frac{1}{23^2} + \frac{1}{25^2} + \frac{1}{39^2} - \frac{1}{41^2} - ...\right)}$$

$$C_{1/16} = \frac{1}{7^2} - \frac{1}{9^2} - \frac{1}{23^2} + \frac{1}{25^2} + \frac{1}{39^2} - \frac{1}{41^2} - ... = \frac{\pi^2 \sin(\frac{1}{16}\pi)}{16^2 \cos^2(\frac{1}{16}\pi)}$$

Proof: By **$7_{\cos}$.** based on $\pi \frac{1}{\cos(\pi z)} \frac{1}{(z - \frac{1}{16})^2}$,

$$\pi^2 = 4(16^2) \frac{\cos^2(\frac{1}{16}\pi)}{\sin(\frac{1}{16}\pi)}\left(\frac{1}{(16-2)^2} - \frac{1}{(16+2)^2} - \frac{1}{(3(16)-2)^2} + \frac{1}{(3(16)+2)^2} + \frac{1}{(5(16)-2)^2} - \frac{1}{(5(16)+2)^2} - ...\right)$$

16.4

$$\boxed{\pi^2 = 16^2 \cos^2(\tfrac{1}{16}\pi)\left(\frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{23^2} + \frac{1}{25^2} + \frac{1}{39^2} + \frac{1}{41^2} + ...\right)}$$

$$T_{1/16} = \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{23^2} + \frac{1}{25^2} + \frac{1}{39^2} + \frac{1}{41^2} + ... = \frac{\pi^2}{16^2} \frac{1}{\cos^2(\frac{1}{16}\pi)}$$

Proof: By **$7_{\tan}$.** based on $\pi \tan(\pi z) \frac{1}{(z - \frac{1}{16})^2}$,

$$\pi^2 = 4(16^2)\cos^2(\tfrac{1}{16}\pi)\left(\frac{1}{(16-2)^2} + \frac{1}{(16+2)^2} + \frac{1}{(3(16)-2)^2} + \frac{1}{(3(16)+2)^2} + \frac{1}{(5(16)-2)^2} + \frac{1}{(5(16)+2)^2} - \ldots\right)$$

17.

17.1

$$\pi^2 = 16^2 \sin^2\left(\tfrac{3}{16}\pi\right)\left(\frac{1}{3^2} + \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{29^2} + \frac{1}{35^2} + \frac{1}{45^2} + ...\right)$$

$$R_{3/16} = \frac{1}{3^2} + \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{29^2} + \frac{1}{35^2} + \frac{1}{45^2} + \frac{1}{51^2} - ... = \frac{\pi^2}{16^2 \sin^2\left(\tfrac{3}{16}\pi\right)}$$

$$\sin^2(\tfrac{3}{16}\pi) = \tfrac{1}{4}(2 - \sqrt{2 - \sqrt{2}}) = \text{algebraic number}$$

Proof: By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\dfrac{1}{(z + \frac{3}{16})^2}$,

$$\pi^2 = 16^2 \sin^2(\tfrac{3}{16}\pi)\left(\frac{1}{3^2} + \frac{1}{(16-3)^2} + \frac{1}{(16+3)^2} + \frac{1}{(2(16)-3)^2} + \frac{1}{(2(16)+3)^2} + \frac{1}{(3(16)-3)^2} + ...\right).\square$$

$$\sin^2(\tfrac{3}{16}\pi) = \tfrac{1}{2}\left[1 - \cos(\tfrac{3}{8}\pi)\right]$$

$$= \tfrac{1}{2}\left(1 - \sqrt{\tfrac{1}{2}[1 + \cos\tfrac{3}{4}\pi]}\right)$$

$$= \tfrac{1}{2}\left(1 - \sqrt{\tfrac{1}{2}[1 - \tfrac{\sqrt{2}}{2}]}\right)$$

$$= \tfrac{1}{4}(2 - \sqrt{2 - \sqrt{2}}) = \text{ algebraic number.}\square$$

17.2
$$\pi^2 = 16^2 \frac{\sin^2\left(\frac{3}{16}\pi\right)}{\cos\left(\frac{3}{16}\pi\right)}\left(\frac{1}{3^2} - \frac{1}{13^2} - \frac{1}{19^2} + \frac{1}{29^2} + \frac{1}{35^2} - \frac{1}{45^2} - \ldots\right)$$

$$S_{3/16} = \frac{1}{3^2} - \frac{1}{13^2} - \frac{1}{19^2} + \frac{1}{29^2} + \frac{1}{35^2} - \frac{1}{45^2} - \frac{1}{51^2} + \ldots = \frac{\pi^2 \cos\left(\frac{3}{16}\pi\right)}{16^2 \sin^2\left(\frac{3}{16}\pi\right)}$$

$$\frac{\sin^2(\frac{3}{16}\pi)}{\cos(\frac{3}{16}\pi)} = \frac{2 - \sqrt{2 - \sqrt{2}}}{2\sqrt{2 + \sqrt{2 - \sqrt{2}}}} = \text{algebraic number}$$

Proof: By $\mathbf{7_{sin}}$. based on $\pi \frac{1}{\sin(\pi z)} \frac{1}{(z + \frac{3}{16})^2}$,

$$\pi^2 = 16^2 \sin^2(\tfrac{3}{16}\pi)\left(\frac{1}{3^2} - \frac{1}{(16-3)^2} - \frac{1}{(16+3)^2} + \frac{1}{(2(16)-3)^2} + \frac{1}{(2(16)+3)^2} - \frac{1}{(3(16)-3)^2} + \ldots\right).\square$$

$$\frac{\sin^2(\frac{3}{16}\pi)}{\cos(\frac{3}{16}\pi)} = \frac{\frac{1}{4}(2 - \sqrt{2 - \sqrt{2}})}{\sqrt{\frac{1}{2}\left[1 + \cos(\frac{3}{8}\pi)\right]}}$$

$$= \frac{\frac{1}{4}(2 - \sqrt{2 - \sqrt{2}})}{\sqrt{\frac{1}{2}\left[1 + \sqrt{\frac{1}{2}\left[1 + \cos(\frac{3}{4}\pi)\right]}\right]}}$$

$$= \frac{\frac{1}{4}(2 - \sqrt{2 - \sqrt{2}})}{\frac{1}{2}\sqrt{2 + \sqrt{2 - \sqrt{2}}}} = \text{algebraic number.}\square$$

17.3

$$\pi^2 = (16^2)\frac{\cos^2(\frac{3}{16}\pi)}{\sin(\frac{3}{16}\pi)}\left(\frac{1}{5^2} - \frac{1}{11^2} - \frac{1}{21^2} + \frac{1}{27^2} + \frac{1}{37^2} - \frac{1}{43^2} - \dots\right)$$

$$C_{3/16} = \frac{1}{5^2} - \frac{1}{11^2} - \frac{1}{21^2} + \frac{1}{27^2} + \frac{1}{37^2} - \frac{1}{43^2} - \dots = \frac{\pi^2 \sin\left(\frac{3}{16}\pi\right)}{16^2 \cos^2\left(\frac{3}{16}\pi\right)}$$

Proof: By $\mathbf{7_{cos}}$. based on $\pi\frac{1}{\cos(\pi z)}\frac{1}{(z-\frac{3}{16})^2}$,

$$\pi^2 = 4(16^2)\frac{\cos^2(\frac{3}{16}\pi)}{\sin(\frac{3}{16}\pi)}\left(\frac{1}{(16-2(3))^2} - \frac{1}{(16+2(3))^2} - \frac{1}{(3(16)-2(3))^2} + \frac{1}{(3(16)+2(3))^2} + \frac{1}{(5(16)-2(3))^2} - \frac{1}{(5(16)+2(3))^2} - \dots\right).\square$$

17.4

$$\pi^2 = (16^2)\cos^2(\tfrac{3}{16}\pi)\left(\frac{1}{5^2} + \frac{1}{11^2} + \frac{1}{21^2} + \frac{1}{27^2} + \frac{1}{37^2} + \frac{1}{43^2} - \dots\right)$$

$$T_{3/16} = \frac{1}{5^2} + \frac{1}{11^2} + \frac{1}{21^2} + \frac{1}{27^2} + \frac{1}{37^2} + \frac{1}{43^2} + \dots = \frac{\pi^2}{16^2}\frac{1}{\cos^2(\frac{3}{16}\pi)}$$

$$\cos^2(\tfrac{3}{16}\pi) = \frac{2+\sqrt{2-\sqrt{2}}}{2} = \text{depends on } \sqrt{2}\text{'s}$$

Proof: By $\mathbf{7_{tan}}$. based on $\pi\tan(\pi z)\dfrac{1}{(z-\frac{3}{16})^2}$,

$$\pi^2 = 4(16^2)\cos^2(\tfrac{3}{16}\pi)\left(\frac{1}{(16-2(3))^2}+\frac{1}{(16+2(3))^2}\right.$$
$$+\frac{1}{(3(16)-2(3))^2}+\frac{1}{(3(16)+2(3))^2}$$
$$\left.+\frac{1}{(5(16)-2(3))^2}+\frac{1}{(5(16)+2(3))^2}-...\right).\square$$

$$\cos^2(\tfrac{3}{16}\pi) = \tfrac{1}{2}\left[1+\cos(\tfrac{3}{8}\pi)\right]$$

$$= 1+\sqrt{\tfrac{1}{2}[1+\cos\tfrac{3}{4}\pi]}$$

$$= 1+\sqrt{\tfrac{1}{2}[1-\tfrac{\sqrt{2}}{2}]}$$

$$= \frac{2+\sqrt{2-\sqrt{2}}}{2}.\square$$

18.

18.1
$$\boxed{\pi^2 = 16^2\sin^2(\tfrac{5}{16})\left(\frac{1}{5^2}+\frac{1}{11^2}+\frac{1}{21^2}+\frac{1}{27^2}+\frac{1}{37^2}+\frac{1}{43^2}+...\right)}$$

$$R_{5/16} = \frac{1}{5^2}+\frac{1}{11^2}+\frac{1}{21^2}+\frac{1}{27^2}+\frac{1}{37^2}+\frac{1}{43^2}+\frac{1}{53^2}-...=\frac{\pi^2}{16^2\sin^2(\frac{5}{16})}$$

$$\sin^2(\tfrac{5}{16}\pi) = \tfrac{1}{4}(2-\sqrt{2-\sqrt{2}}) = \text{ algebraic number}$$

Proof: By $\mathbf{7_{cot}}$. based on $\cot(\pi z)\dfrac{1}{(z+\frac{5}{16})^2}$,

$$\pi^2 = 16^2\sin^2(\tfrac{5}{16}\pi)\left(\frac{1}{5^2}+\frac{1}{(16-5)^2}+\frac{1}{(16+5)^2}+\frac{1}{(2(16)-5)^2}+\frac{1}{(2(16)+5)^2}+\frac{1}{(3(16)-5)^2}+...\right).\square$$

$$\sin^2(\tfrac{5}{16}\pi) = \tfrac{1}{2}\left[1-\cos(\tfrac{5}{8}\pi)\right]$$

$$= \tfrac{1}{2}\left(1-\sqrt{\tfrac{1}{2}[1+\cos\tfrac{5}{4}\pi]}\right)$$

$$= \tfrac{1}{2}\left(1-\sqrt{\tfrac{1}{2}[1-\tfrac{\sqrt{2}}{2}]}\right)$$

$$= \tfrac{1}{4}(2-\sqrt{2-\sqrt{2}}) = \text{ algebraic number}.\square$$

18.2

$$\boxed{\pi^2 = 16^2 \frac{\sin^2(\frac{5}{16})}{\cos(\frac{5}{16})}\left(\frac{1}{5^2} - \frac{1}{11^2} - \frac{1}{21^2} + \frac{1}{27^2} + \frac{1}{37^2} - \frac{1}{43^2} - ....\right)}$$

$$S_{5/16} = \frac{1}{5^2} - \frac{1}{11^2} - \frac{1}{21^2} + \frac{1}{27^2} + \frac{1}{37^2} - \frac{1}{43^2} - \frac{1}{53^2} + ... = \frac{\pi^2 \cos(\frac{5}{16})}{16^2 \sin^2(\frac{5}{16})}$$

$$\frac{\sin^2(\frac{5}{16}\pi)}{\cos(\frac{5}{16}\pi)} = \frac{2 - \sqrt{2 - \sqrt{2}}}{2\sqrt{(2 + \sqrt{2 - \sqrt{2}})}} = \text{algebraic number}$$

Proof: By $\mathbf{7_{sin}}$. based on $\pi \dfrac{1}{\sin(\pi z)} \dfrac{1}{(z + \frac{5}{16})^2}$,

$$\pi^2 = 16^2 \frac{\sin^2(\frac{5}{16})}{\cos(\frac{5}{16})}\left(\frac{1}{5^2} - \frac{1}{(16-5)^2} - \frac{1}{(16+5)^2} + \frac{1}{(2(16)-5)^2} + \frac{1}{(2(16)+5)^2} - \frac{1}{(3(16)-5)^2} + ...\right). \square$$

$$\frac{\sin^2(\frac{5}{16}\pi)}{\cos(\frac{5}{16}\pi)} = \frac{\frac{1}{4}(2 - \sqrt{2 - \sqrt{2}})}{\sqrt{\frac{1}{2}\left[1 + \cos(\frac{5}{8}\pi)\right]}}$$

$$= \frac{\frac{1}{4}(2 - \sqrt{2 - \sqrt{2}})}{\sqrt{\frac{1}{2}\left[1 + \sqrt{\frac{1}{2}\left[1 + \cos(\frac{5}{4}\pi)\right]}\right]}}$$

$$= \frac{\frac{1}{4}(2 - \sqrt{2 - \sqrt{2}})}{\frac{1}{2}\sqrt{(2 + \sqrt{2 - \sqrt{2}})}} = \text{algebraic number.} \square$$

18.3
$$\pi^2 = (16)^2 \frac{\cos^2(\frac{5}{16}\pi)}{\sin(\frac{5}{16}\pi)}\left(\frac{1}{3^2} - \frac{1}{13^2} - \frac{1}{19^2} + \frac{1}{29^2} + \frac{1}{35^2} - \frac{1}{45^2} - \dots\right)$$

$$C_{5/16} = \frac{1}{3^2} - \frac{1}{13^2} - \frac{1}{19^2} + \frac{1}{29^2} + \frac{1}{35^2} - \frac{1}{45^2} - \dots = \frac{\pi^2 \sin(\frac{5}{16})}{16^2 \cos^2(\frac{5}{16})}$$

Proof: By $\mathbf{7_{cos}}$. based on $\pi \frac{1}{\cos(\pi z)} \frac{1}{(z - \frac{5}{16})^2}$,

$$\pi^2 = 4(16)^2 \frac{\cos^2(\frac{5}{16}\pi)}{\sin(\frac{5}{16}\pi)}\left(\frac{1}{(16-2(5))^2} - \frac{1}{(16+2(5))^2} - \frac{1}{(3(16)-2(5))^2} + \frac{1}{(3(16)+2(5))^2} + \frac{1}{(5(16)-2(5))^2} - \frac{1}{(5(16)+2(5))^2} - \dots\right). \square$$

18.4
$$\pi^2 = (16)^2 \cos^2(\tfrac{5}{16}\pi)\left(\frac{1}{3^2} + \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{29^2} + \frac{1}{35^2} + \frac{1}{45^2} - \dots\right)$$

$$T_{5/16} = \frac{1}{3^2} + \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{29^2} + \frac{1}{35^2} + \frac{1}{45^2} + \dots = \frac{\pi^2}{16^2} \frac{1}{\cos^2(\frac{5}{16})}$$

Proof: By $\mathbf{7_{tan}}$. based on $\pi \tan(\pi z) \frac{1}{(z - \frac{5}{16})^2}$,

$$\pi^2 = 4(16)^2 \cos^2(\tfrac{5}{16}) \Bigg(\frac{1}{(16-2(5))^2} + \frac{1}{(16+2(5))^2} + \frac{1}{(3(16)-2(5))^2} + \frac{1}{(3(16)+2(5))^2} + \frac{1}{(5(16)-2(5))^2} + \frac{1}{(5(16)+2(5))^2} + \dots \Bigg) . \square$$

19.

19.1

$$\boxed{\pi^2 = 16^2 \sin^2(\tfrac{7}{16}\pi)\left(\frac{1}{7^2}+\frac{1}{9^2} + \frac{1}{23^2}+\frac{1}{25^2} + \frac{1}{39^2}+\frac{1}{41^2}+\dots\right)}$$

$$R_{7/16} = \frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{23^2}+\frac{1}{25^2}+\frac{1}{39^2}+\frac{1}{41^2}+\frac{1}{55^2}-.. = \frac{\pi^2}{16^2\sin^2(\frac{7}{16}\pi)}$$

$$\sin^2(\tfrac{7}{16}\pi) = \tfrac{1}{4}(2-\sqrt{2+\sqrt{2}}) = \text{algebraic number}$$

Proof: By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\dfrac{1}{(z+\frac{7}{16})^2}$,

$$\pi^2 = 16^2\sin^2(\tfrac{7}{16}\pi)\left(\frac{1}{7^2}+\frac{1}{(16-7)^2} + \frac{1}{(16+7)^2}+\frac{1}{(2(16)-7)^2} + \frac{1}{(2(16)+7)^2}+\frac{1}{(3(16)-7)^2}+\dots\right).\square$$

$$\sin^2(\tfrac{7}{16}\pi) = \tfrac{1}{2}\left[1-\cos(\tfrac{7}{8}\pi)\right]$$

$$= \tfrac{1}{2}\left(1-\sqrt{\tfrac{1}{2}[1+\cos\tfrac{7}{4}\pi]}\right)$$

$$= \tfrac{1}{2}\left(1-\sqrt{\tfrac{1}{2}[1+\tfrac{\sqrt{2}}{2}]}\right)$$

$$= \tfrac{1}{4}(2-\sqrt{2+\sqrt{2}}) = \text{algebraic number}.\square$$

19.2

$$\boxed{\pi^2 = 16^2 \frac{\sin^2(\frac{7}{16}\pi)}{\cos(\frac{7}{16}\pi)}\left(\frac{1}{7^2} - \frac{1}{9^2} - \frac{1}{23^2} + \frac{1}{25^2} + \frac{1}{39^2} - \frac{1}{41^2} - ...\right)}$$

$$S_{7/16} = \frac{1}{7^2} - \frac{1}{9^2} - \frac{1}{23^2} + \frac{1}{25^2} + \frac{1}{39^2} - \frac{1}{41^2} - .. = \frac{\pi^2}{16^2}\frac{\cos(\frac{7}{16}\pi)}{\sin^2(\frac{7}{16}\pi)}$$

$$\frac{\sin^2(\frac{7}{16}\pi)}{\cos(\frac{7}{16}\pi)} = \frac{2-\sqrt{2+\sqrt{2}}}{2\sqrt{2+\sqrt{2+\sqrt{2}}}} = \text{algebraic number}$$

Proof: By $\mathbf{7_{sin}}$. based on $\pi\frac{1}{\sin(\pi z)}\frac{1}{(z+\frac{7}{16})^2}$,

$$\pi^2 = 16^2 \frac{\sin^2(\frac{7}{16}\pi)}{\cos(\frac{7}{16}\pi)}\left(\frac{1}{7^2} - \frac{1}{(16-7)^2} - \frac{1}{(16+7)^2} + \frac{1}{(2(16)-7)^2} + \frac{1}{(2(16)+7)^2} - \frac{1}{(3(16)-7)^2} - ...\right).\square$$

$$\frac{\sin^2(\frac{7}{16}\pi)}{\cos(\frac{7}{16}\pi)} = \frac{\frac{1}{4}(2-\sqrt{2+\sqrt{2}})}{\sqrt{\frac{1}{2}\left[1+\cos(\frac{7}{8}\pi)\right]}}$$

$$= \frac{\frac{1}{4}(2-\sqrt{2+\sqrt{2}})}{\sqrt{\frac{1}{2}\left[1+\sqrt{\frac{1}{2}\left[1+\cos(\frac{7}{4}\pi)\right]}\right]}}$$

$$= \frac{\frac{1}{4}(2-\sqrt{2+\sqrt{2}})}{\sqrt{\frac{1}{2}\left[1+\sqrt{\frac{1}{2}\left[1+\frac{\sqrt{2}}{2}\right]}\right]}} =$$

$$= \frac{2-\sqrt{2+\sqrt{2}}}{2\sqrt{2+\sqrt{2+\sqrt{2}}}} = \text{algebraic number.}\square$$

19.3

$$\boxed{\pi^2 = (16)^2 \frac{\cos^2(\frac{7}{16}\pi)}{\sin(\frac{7}{16}\pi)}\left(\frac{1}{1^2} - \frac{1}{15^2} - \frac{1}{17^2} + \frac{1}{31^2} + \frac{1}{33^2} - \frac{1}{47^2} - \ldots\right)}$$

$$C_{7/16} = \frac{1}{1^2} - \frac{1}{15^2} - \frac{1}{17^2} + \frac{1}{31^2} + \frac{1}{33^2} - \frac{1}{47^2} - \ldots = \frac{\pi^2}{16^2}\frac{\sin(\frac{7}{16}\pi)}{\cos^2(\frac{7}{16}\pi)}$$

<u>*Proof*</u>: By $\mathbf{7_{cos}}$. based on $\pi\frac{1}{\cos(\pi z)}\frac{1}{(z-\frac{7}{16})^2}$,

$$\pi^2 = 4(16)^2 \frac{\cos^2(\frac{7}{16}\pi)}{\sin(\frac{7}{16}\pi)}\left(\frac{1}{(16-2(7))^2} - \frac{1}{(16+2(7))^2} - \frac{1}{(3(16)-2(7))^2} + \frac{1}{(3(16)+2(7))^2} + \frac{1}{(5(16)-2(7))^2} - \frac{1}{(5(16)+2(7))^2} - \ldots\right).\square$$

$$\sin(\tfrac{7}{16}\pi) = \sqrt{\tfrac{1}{2}\left[1-\cos(\tfrac{7}{8}\pi)\right]}$$

$$= \sqrt{\tfrac{1}{2}\left[1-\sqrt{\tfrac{1}{2}\left[1+\cos(\tfrac{7}{4}\pi)\right]}\right]}$$

$$= \sqrt{\tfrac{1}{2}\left[1-\sqrt{\tfrac{1}{2}\left[1+\tfrac{\sqrt{2}}{2}\right]}\right]}$$

$$= \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}.\square$$

19.4

$$\boxed{\pi^2 = (16)^2 \cos^2(\tfrac{7}{16}\pi)\left(\frac{1}{1^2}+\frac{1}{15^2}+\frac{1}{17^2}+\frac{1}{31^2}+\frac{1}{33^2}+\frac{1}{47^2}-...\right)}$$

$$T_{7/16} = \frac{1}{1^2}+\frac{1}{15^2}+\frac{1}{17^2}+\frac{1}{31^2}+\frac{1}{33^2}+\frac{1}{47^2}+... = \frac{\pi^2}{16^2}\frac{1}{\cos^2(\tfrac{7}{16}\pi)}$$

Proof: By $\mathbf{7_{tan}}$. based on $\pi\tan(\pi z)\dfrac{1}{(z-\tfrac{7}{16})^2}$,

$$\pi^2 = 4(16)^2\cos^2(\tfrac{7}{16}\pi)\left(\frac{1}{(16-2(7))^2}+\frac{1}{(16+2(7))^2}+\frac{1}{(3(16)-2(7))^2}+\frac{1}{(3(16)+2(7))^2}+\frac{1}{(5(16)-2(7))^2}+\frac{1}{(5(16)+2(7))^2}+...\right).\square$$

$$\cos^2(\tfrac{7}{16}\pi) = \sin^2(\tfrac{7}{16}\pi)$$

$$= \tfrac{1}{2}\Big[1-\cos(\tfrac{1}{8}\pi)\Big]$$

$$= 1-\sqrt{\tfrac{1}{2}[1+\cos(\tfrac{1}{4}\pi)]}$$

$$= 1-\sqrt{\tfrac{1}{2}[1+\tfrac{\sqrt{2}}{2}]}$$

$$= \frac{2-\sqrt{2+\sqrt{2}}}{2}.\square$$

20.

$R_{1/16}+R_{3/16}+R_{5/16}+R_{7/16}$

$$R_{1/16}=\frac{1}{1}+\frac{1}{15^2}+\frac{1}{17^2}+\frac{1}{31^2}+\frac{1}{33^2}+\frac{1}{47^2}+\frac{1}{49^2}+...=\frac{\pi^2}{16^2\sin^2(\frac{1}{16}\pi)}$$

$$R_{3/16}=\frac{1}{3^2}+\frac{1}{13^2}+\frac{1}{19^2}+\frac{1}{29^2}+\frac{1}{35^2}+\frac{1}{45^2}+\frac{1}{51^2}-...=\frac{\pi^2}{16^2\sin^2\left(\frac{3}{16}\pi\right)}$$

$$R_{5/16}=\frac{1}{5^2}+\frac{1}{11^2}+\frac{1}{21^2}+\frac{1}{27^2}+\frac{1}{37^2}+\frac{1}{43^2}+\frac{1}{53^2}-...=\frac{\pi^2}{16^2\sin^2(\frac{5}{16})}$$

$$R_{7/16}=\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{23^2}+\frac{1}{25^2}+\frac{1}{39^2}+\frac{1}{41^2}+\frac{1}{55^2}-..=\frac{\pi^2}{16^2\sin^2(\frac{7}{16}\pi)}$$

$$\Rightarrow R_{1/16}+R_{3/16}+R_{5/16}+R_{7/16}=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+...=\frac{\pi^2}{8}$$

21.

$R_{1/16} - R_{3/16}$

$$R_{1/16} = \frac{1}{1} + \frac{1}{15^2} + \frac{1}{17^2} + \frac{1}{31^2} + \frac{1}{33^2} + \frac{1}{47^2} + \frac{1}{49^2} + \ldots = \frac{\pi^2}{16^2 \sin^2(\frac{1}{16}\pi)}$$

$$R_{3/16} = \frac{1}{3^2} + \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{29^2} + \frac{1}{35^2} + \frac{1}{45^2} + \frac{1}{51^2} - \ldots = \frac{\pi^2}{16^2 \sin^2\left(\frac{3}{16}\pi\right)}$$

$\Rightarrow R_{1/16} - R_{3/16} =$

$$\frac{1}{1} - \frac{1}{3^2} - \frac{1}{13^2} + \frac{1}{15^2} + \frac{1}{17^2} - \frac{1}{19^2} - \frac{1}{29^2} + \frac{1}{31^2} + \frac{1}{33^2} +$$

$$-\frac{1}{35^2} - \frac{1}{45^2} + \frac{1}{47^2} + \frac{1}{49^2} - \frac{1}{51^2} - \frac{1}{61^2} + \ldots =$$

$$= \frac{\pi^2}{16^2}\left(\frac{1}{\sin^2(\frac{1}{16}\pi)} - \frac{1}{\sin^2(\frac{3}{16}\pi)}\right)$$

22.

$R_{5/16} - R_{7/16}$

$$R_{5/16} = \frac{1}{5^2} + \frac{1}{11^2} + \frac{1}{21^2} + \frac{1}{27^2} + \frac{1}{37^2} + \frac{1}{43^2} + \frac{1}{53^2} - ... = \frac{\pi^2}{16^2 \sin^2(\frac{5}{16})}$$

$$R_{7/16} = \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{23^2} + \frac{1}{25^2} + \frac{1}{39^2} + \frac{1}{41^2} + \frac{1}{55^2} - .. = \frac{\pi^2}{16^2 \sin^2(\frac{7}{16}\pi)}$$

$$\Rightarrow R_{5/16} - R_{7/16} =$$

$$\frac{1}{5^2} - \frac{1}{7^2} - \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{21^2} - \frac{1}{23^2} - \frac{1}{25^2} + \frac{1}{27^2} + \frac{1}{37^2} -$$

$$-\frac{1}{39^2} - \frac{1}{41^2} + \frac{1}{43^2} + \frac{1}{53^2} - \frac{1}{55^2} - \frac{1}{57^2} + ...$$

$$= \frac{\pi^2}{16^2}\left(\frac{1}{\sin^2(\frac{5}{16}\pi)} - \frac{1}{\sin^2(\frac{7}{16}\pi)} \right)$$

23.

$R_{3/16} - R_{5/16}$

$$R_{3/16} = \frac{1}{3^2} + \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{29^2} + \frac{1}{35^2} + \frac{1}{45^2} + \frac{1}{51^2} - ... = \frac{\pi^2}{16^2 \sin^2\left(\frac{3}{16}\pi\right)}$$

$$R_{5/16} = \frac{1}{5^2} + \frac{1}{11^2} + \frac{1}{21^2} + \frac{1}{27^2} + \frac{1}{37^2} + \frac{1}{43^2} + \frac{1}{53^2} - ... = \frac{\pi^2}{16^2 \sin^2(\frac{5}{16})}$$

$\Rightarrow R_{3/16} - R_{5/16}$

$$\frac{1}{3^2} - \frac{1}{5^2} - \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{19^2} - \frac{1}{21^2} - \frac{1}{27^2} + \frac{1}{29^2} + \frac{1}{35^2} -$$

$$-\frac{1}{37^2} - \frac{1}{43^2} + \frac{1}{45^2} + \frac{1}{51^2} - \frac{1}{53^2} - \frac{1}{59^2} - ... =$$

$$= \frac{\pi^2}{16^2}\left(\frac{1}{\sin^2(\frac{3}{16}\pi)} - \frac{1}{\sin^2(\frac{5}{16}\pi)}\right)$$

24.

$R_{1/16} - R_{5/16}$

$$R_{1/16} = \frac{1}{1} + \frac{1}{15^2} + \frac{1}{17^2} + \frac{1}{31^2} + \frac{1}{33^2} + \frac{1}{47^2} + \frac{1}{49^2} + ... = \frac{\pi^2}{16^2 \sin^2(\frac{1}{16}\pi)}$$

$$R_{5/16} = \frac{1}{5^2} + \frac{1}{11^2} + \frac{1}{21^2} + \frac{1}{27^2} + \frac{1}{37^2} + \frac{1}{43^2} + \frac{1}{53^2} - ... = \frac{\pi^2}{16^2 \sin^2(\frac{5}{16})}$$

$\Rightarrow R_{1/16} - R_{5/16} =$

$$\frac{1}{1} - \frac{1}{5^2} - \frac{1}{11^2} + \frac{1}{15^2} + \frac{1}{17^2} - \frac{1}{21^2} - \frac{1}{27^2} + \frac{1}{31^2} + \frac{1}{33^2} -$$

$$-\frac{1}{37^2} - \frac{1}{43^2} + \frac{1}{47^2} + \frac{1}{49^2} - \frac{1}{53^2} - \frac{1}{59^2} + ... =$$

$$= \frac{\pi^2}{16^2}\left(\frac{1}{\sin^2(\frac{1}{16}\pi)} - \frac{1}{\sin^2(\frac{5}{16}\pi)}\right)$$

25.

$R_{1/16} - R_{7/16}$

$$R_{1/16} = \frac{1}{1} + \frac{1}{15^2} + \frac{1}{17^2} + \frac{1}{31^2} + \frac{1}{33^2} + \frac{1}{47^2} + \frac{1}{49^2} + ... = \frac{\pi^2}{16^2 \sin^2(\frac{1}{16}\pi)}$$

$$R_{7/16} = \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{23^2} + \frac{1}{25^2} + \frac{1}{39^2} + \frac{1}{41^2} + \frac{1}{55^2} - .. = \frac{\pi^2}{16^2 \sin^2(\frac{7}{16}\pi)}$$

$\Rightarrow R_{1/16} - R_{7/16} =$

$$\frac{1}{1} - \frac{1}{7^2} - \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{17^2} - \frac{1}{23^2} - \frac{1}{25^2} + \frac{1}{31^2} + \frac{1}{33^2}$$

$$-\frac{1}{39^2} - \frac{1}{41^2} + \frac{1}{47^2} + \frac{1}{49^2} - \frac{1}{55^2} - \frac{1}{57^2} + ... =$$

$$= \frac{\pi^2}{16^2}\left(\frac{1}{\sin^2(\frac{1}{16}\pi)} - \frac{1}{\sin^2(\frac{7}{16}\pi)}\right)$$

26.

$R_{3/16} - R_{7/16}$

$$R_{3/16} = \frac{1}{3^2} + \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{29^2} + \frac{1}{35^2} + \frac{1}{45^2} + \frac{1}{51^2} - ... = \frac{\pi^2}{16^2 \sin^2\left(\frac{3}{16}\pi\right)}$$

$$R_{7/16} = \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{23^2} + \frac{1}{25^2} + \frac{1}{39^2} + \frac{1}{41^2} + \frac{1}{55^2} - .. = \frac{\pi^2}{16^2 \sin^2(\frac{7}{16}\pi)}$$

$\Rightarrow R_{3/16} - R_{7/16} =$

$$\frac{1}{3^2} - \frac{1}{7^2} - \frac{1}{9^2} + \frac{1}{13^2} + \frac{1}{19^2} - \frac{1}{23^2} - \frac{1}{25^2} + \frac{1}{29^2} + \frac{1}{35^2} -$$

$$-\frac{1}{39^2} - \frac{1}{41^2} + \frac{1}{45^2} + \frac{1}{51^2} - \frac{1}{55^2} - \frac{1}{61^2} + ... =$$

$$= \frac{\pi^2}{16^2}\left(\frac{1}{\sin^2(\frac{3}{16}\pi)} - \frac{1}{\sin^2(\frac{7}{16}\pi)}\right)$$

27.

27.1

$$\pi^2 = p^2 \sin^2(\tfrac{1}{p}\pi)\left\{1 + \frac{1}{(p-1)^2} + \frac{1}{(p+1)^2} + \frac{1}{(2p-1)^2} + \frac{1}{(2p+1)^2} + \frac{1}{(3p-1)^2} + \frac{1}{(3p+1)^2} + \dots\right\}$$

$p =$ prime.

$$R_{1/p} = 1 + \frac{1}{(p-1)^2} + \frac{1}{(p+1)^2} + \frac{1}{(2p-1)^2} + \frac{1}{(2p+1)^2} + \frac{1}{(3p-1)^2} + \frac{1}{(3p+1)^2} + \dots = \frac{\pi^2}{p^2 \sin^2(\frac{1}{p}\pi)}$$

Proof: By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\dfrac{1}{(z+\frac{1}{p})^2}$.□

27.2

$$\pi^2 = p^2 \frac{\sin^2(\frac{1}{p}\pi)}{\cos(\frac{1}{p}\pi)}\left\{1 - \frac{1}{(p-1)^2} - \frac{1}{(p+1)^2} + \frac{1}{(2p-1)^2} + \frac{1}{(2p+1)^2} - \frac{1}{(3p-1)^2} - \dots\right\}$$

$$S_{1/p} = 1 - \frac{1}{(p-1)^2} - \frac{1}{(p+1)^2} + \frac{1}{(2p-1)^2} + \frac{1}{(2p+1)^2} + \frac{1}{(3p-1)^2} + \frac{1}{(3p+1)^2} + \dots = \frac{\pi^2 \cos(\frac{1}{p}\pi)}{p^2 \sin^2(\frac{1}{p}\pi)}$$

Proof: By $\mathbf{7_{sin}}$. based on $\pi \dfrac{1}{\sin(\pi z)} \dfrac{1}{(z+\frac{1}{p})^2}$.□

27.3
$$\boxed{\begin{aligned}\pi^2 &= 4p^2 \frac{\cos^2(\frac{1}{p}\pi)}{\sin(\frac{1}{p}\pi)}\bigg(\frac{1}{(p-2)^2} - \frac{1}{(p+2)^2} \\ &\qquad - \frac{1}{(3p-2)^2} + \frac{1}{(3p+2)^2} \\ &\qquad + \frac{1}{(5p-2)^2} - \frac{1}{(5p+2)^2} - \dots \bigg)\end{aligned}}$$

$$\begin{aligned}C_{1/p} &= \frac{1}{(p-2)^2} - \frac{1}{(p+2)^2} \\ &\quad - \frac{1}{(3p-2)^2} + \frac{1}{(3p+2)^2} \\ &\quad + \frac{1}{(5p-2)^2} - \frac{1}{(5p+2p)^2} - \dots = \frac{\pi^2 \sin(\frac{1}{p}\pi)}{4p^2\cos^2(\frac{1}{p}\pi)}\end{aligned}$$

Proof: By $\mathbf{7_{cos}}$. based on $\pi \dfrac{1}{\cos(\pi z)} \dfrac{1}{(z-\frac{1}{p})^2}$.□

27.4
$$\boxed{\begin{aligned}\pi^2 &= 4p^2 \cos^2(\tfrac{1}{p}\pi)\bigg(\frac{1}{(p-2)^2} + \frac{1}{(p+2)^2} \\ &\qquad + \frac{1}{(3p-2)^2} + \frac{1}{(3p+2)^2} \\ &\qquad + \frac{1}{(5p-2)^2} + \frac{1}{(5p+2)^2} - \dots \bigg)\end{aligned}}$$

$$T_{1/p} = \frac{1}{(p-2)^2} - \frac{1}{(p+2)^2}$$
$$- \frac{1}{(3p-2)^2} + \frac{1}{(3p+2)^2}$$
$$+ \frac{1}{(5p-2)^2} - \frac{1}{(5p+2p)^2} - ... = \frac{\pi^2}{4p^2} \frac{1}{\cos^2(\frac{1}{p}\pi)}$$

Proof: By $\mathbf{7_{tan}}$. based on $\pi \tan(\pi z) \frac{1}{(z-\frac{1}{p})^2}$.□

28.

28.1

$$\pi^2 = 3^2 \underbrace{\sin^2(\tfrac{1}{3}\pi)}_{3/4}\left(\frac{1}{1}+\frac{1}{2^2} + \frac{1}{4^2}+\frac{1}{5^2} + \frac{1}{7^2}+\frac{1}{8^2}+...\right)$$

$$R_{1/3} = \frac{1}{1}+\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{8^2}... = 4\frac{\pi^2}{3^3}$$

Proof: By **$7_{\cot}$.** based on $\pi\cot(\pi z)\dfrac{1}{(z+\frac{1}{3})^2}$

$$\pi^2 = 3^2\sin^2(\tfrac{1}{3}\pi)\left(\frac{1}{1^2}+\frac{1}{(3-1)^2} + \frac{1}{(3+1)^2}+\frac{1}{(2(3)-1)^2} + \frac{1}{(2(3)+1)^2}+\frac{1}{(3(3)-1)^2}+...\right)$$

28.2

$$\pi^2 = 3^2 \underbrace{\frac{\sin^2(\tfrac{1}{3}\pi)}{\cos(\tfrac{1}{3}\pi)}}_{3/2}\left(\frac{1}{1}-\frac{1}{2^2} - \frac{1}{4^2}+\frac{1}{5^2} + \frac{1}{7^2}-\frac{1}{8^2}-...\right)$$

$$S_{1/3} = \frac{1}{1}-\frac{1}{2^2}-\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{7^2}-\frac{1}{8^2}-\frac{1}{10^2}+... = 2\frac{\pi^2}{3^3}.$$

Proof: By $\mathbf{7}_{\mathbf{sin}}$. based on $\dfrac{\pi}{\sin(\pi z)}\dfrac{1}{(z+\frac{1}{3})^2}$,

$$\pi^2 = 3^2\frac{\sin^2(\frac{1}{3}\pi)}{\cos(\frac{1}{3}\pi)}\left(\frac{1}{1^2}-\frac{1}{(3-1)^2}-\frac{1}{(3+1)^2}+\frac{1}{(2(3)-1)^2}+\frac{1}{(2(3)+1)^2}-\frac{1}{(3(3)-1)^2}-...\right).\square$$

28.3

$$\boxed{\pi^2 = 4(3^2)\underbrace{\frac{\cos^2(\frac{1}{3}\pi)}{\sin(\frac{1}{3}\pi)}}_{2/\sqrt{3}}\left(\frac{1}{1}-\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{13^2}-\frac{1}{17^2}-...\right)}$$

$$C_{1/3} = \frac{1}{1}-\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{13^2}-\frac{1}{17^2}-... = \frac{\pi^2}{8}\frac{1}{3\sqrt{3}}.$$

Proof: By $\mathbf{7}_{\mathbf{cos}}$. based on $\dfrac{\pi}{\cos(\pi z)}\dfrac{1}{(z-\frac{1}{3})^2}$,

$$\pi^2 = 4(3^2)\frac{\cos^2(\frac{1}{3}\pi)}{\sin(\frac{1}{3}\pi)}\left(\frac{1}{(3-2)^2}-\frac{1}{(3+2)^2}-\frac{1}{(3(3)-2)^2}+\frac{1}{(3(3)+2)^2}+\frac{1}{(5(3)-2)^2}-\frac{1}{(5(3)+2)^2}-...\right).\square$$

28.4
$$\boxed{\begin{aligned}\pi^2 = (3^2)\Big(\frac{1}{1} &+ \frac{1}{5^2}\\ &+ \frac{1}{7^2} + \frac{1}{11^2}\\ &+ \frac{1}{13^2} + \frac{1}{17^2} - \dots\Big)\end{aligned}}$$

$$T_{1/3} = \frac{1}{1} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \dots = \frac{\pi^2}{3^2}.$$

Proof: By $\mathbf{7_{tan}}$. based on $\pi\tan(\pi z)\dfrac{1}{(z-\frac{1}{3})^2}$,

$$\begin{aligned}\pi^2 = 4(3^2)\underbrace{\cos^2(\tfrac{1}{3}\pi)}_{1/4}\Big(&\frac{1}{(3-2)^2} + \frac{1}{(3+2)^2}\\ &+ \frac{1}{(3(3)-2)^2} + \frac{1}{(3(3)+2)^2}\\ &+ \frac{1}{(5(3)-2)^2} + \frac{1}{(5(3)+2)^2} - \dots\Big)\end{aligned} .\square$$

29.

29.1

$$\pi^2 = p^{2n}\sin^2(\tfrac{l}{p^n}\pi)\left\{\frac{1}{l^2}+\frac{1}{(p^n-l)^2}+\frac{1}{(p^n+l)^2}+\frac{1}{(2p^n-l)^2}+\frac{1}{(2p^n+l)^2}+\frac{1}{(3p^n-l)^2}+....\right\}$$

$$R_{l/p^n} = \frac{1}{l^2}+\frac{1}{(p^n-l)^2}+\frac{1}{(p^n+l)^2}+\frac{1}{(2p^n-l)^2}+\frac{1}{(2p^n+l)^2}+\frac{1}{(3p^n-l)^2}+... = \frac{\pi^2}{p^{2n}\sin^2(\frac{l}{p^n}\pi)}$$

Proof: By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\dfrac{1}{(z+\frac{l}{p^n})^2}$.□

29.2

$$\pi^2 = p^{2n}\frac{\sin^2(\frac{l}{p^n}\pi)}{\cos(\frac{l}{p^n}\pi)}\left\{\frac{1}{l^2}-\frac{1}{(p^n-l)^2}-\frac{1}{(p^n+l)^2}+\frac{1}{(2p^n-l)^2}+\frac{1}{(2p^n+l)^2}-\frac{1}{(3p^n-l)^2}-....\right\}$$

$$S_{l/p^n} = \frac{1}{l^2}-\frac{1}{(p^n-l)^2}-\frac{1}{(p^n+l)^2}+\frac{1}{(2p^n-l)^2}+\frac{1}{(2p^n+l)^2}-\frac{1}{(3p^n-l)^2}-... = \frac{\pi^2\cos(\frac{l}{p^n}\pi)}{p^{2n}\sin^2(\frac{l}{p^n}\pi)}$$

Proof: By $\mathbf{7_{sin}}$. based on $\pi \frac{1}{\sin(\pi z)} \frac{1}{(z + \frac{l}{p^n})^2}$. □

29.3

$$\boxed{\pi^2 = 4p^{2n} \frac{\cos^2(\frac{l}{p^n}\pi)}{\sin(\frac{l}{p^n}\pi)} \left(\frac{1}{(p^n - 2l)^2} - \frac{1}{(p^n + 2l)^2} - \frac{1}{(3p^n - 2l)^2} + \frac{1}{(3p^n + 2l)^2} + \frac{1}{(5p^n - 2l)^2} - \frac{1}{(5p^n + 2l)^2} - \ldots \right)}$$

$$C_{l/p^n} = \frac{1}{(p^n - 2l)^2} - \frac{1}{(p^n + 2l)^2} - \frac{1}{(3p^n - 2l)^2} + \frac{1}{(3p^n + 2l)^2} + \frac{1}{(5p^n - 2l)^2} - \frac{1}{(5p^n + 2l)^2} - \ldots = \frac{\pi^2 \sin(\frac{l}{p^n}\pi)}{4p^{2n} \cos^2(\frac{l}{p^n}\pi)}$$

Proof: By $\mathbf{7_{cos}}$. based on $\pi \frac{1}{\cos(\pi z)} \frac{1}{(z - \frac{l}{p^n})^2}$. □

29.4

$$\boxed{\pi^2 = 4p^{2n} \cos^2(\tfrac{l}{p^n}\pi) \left(\frac{1}{(p^n - 2l)^2} + \frac{1}{(p^n + 2l)^2} + \frac{1}{(3p^n - 2l)^2} + \frac{1}{(3p^n + 2l)^2} + \frac{1}{(5p^n - 2l)^2} + \frac{1}{(5p^n + 2l)^2} - \ldots \right)}$$

$$T_{l/p^n} = \frac{1}{(p^n - 2l)^2} + \frac{1}{(p^n + 2l)^2} + \frac{1}{(3p^n - 2l)^2} + \frac{1}{(3p^n + 2l)^2} + \frac{1}{(5p^n - 2l)^2} + \frac{1}{(5p^n + 2l)^2} - ... = \frac{\pi^2}{4p^{2n}} \frac{1}{\cos^2(\frac{l}{p^n}\pi)}$$

Proof: By $\mathbf{7_{tan}}$**.** based on $\pi \tan(\pi z) \frac{1}{(z - \frac{l}{p^n})^2}$.□

30.

30.1

$$\boxed{\pi^2 = 9^2\sin^2(\tfrac{1}{9}\pi)\left(\frac{1}{1}+\frac{1}{8^2} + \frac{1}{10^2}+\frac{1}{17^2} + \frac{1}{19^2}+\frac{1}{26^2}+...\right)}$$

$$R_{1/9} = \frac{1}{1}+\frac{1}{8^2}+\frac{1}{10^2}+\frac{1}{17^2}+\frac{1}{19^2}+\frac{1}{26^2}+... = \frac{\pi^2}{9^2\sin^2(\frac{1}{9}\pi)}$$

$\sin(\tfrac{1}{9}\pi)$ **solves the cubic** $4x^3 - 3x + \tfrac{1}{2}\sqrt{3} = 0$

Proof: By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\dfrac{1}{(z+\frac{1}{9})^2}$,

$$\pi^2 = 9^2\sin^2(\tfrac{1}{9}\pi)\left(\frac{1}{1^2}+\frac{1}{(9-1)^2} + \frac{1}{(9+1)^2}+\frac{1}{(2(9)-1)^2} + \frac{1}{(2(9)+1)^2}+\frac{1}{(3(9)-1)^2}+...\right).\square$$

$$\sin 3A = -4\sin^3 A + 3\sin A \Rightarrow$$

$$\Rightarrow \underbrace{\sin(\tfrac{1}{3}\pi)}_{\frac{1}{2}\sqrt{3}} = -4\sin^3(\tfrac{1}{9}\pi) + 3\sin(\tfrac{1}{9}\pi) \Rightarrow$$

$\Rightarrow \sin(\tfrac{1}{9}\pi)$ **solves the cubic** $4x^3 - 3x + \tfrac{1}{2}\sqrt{3} = 0.\square$

30.2

$$\pi^2 = 9^2 \frac{\sin^2(\frac{1}{9}\pi)}{\cos(\frac{1}{9}\pi)} \left(\frac{1}{1} - \frac{1}{8^2} - \frac{1}{10^2} + \frac{1}{17^2} + \frac{1}{19^2} - \frac{1}{26^2} + \ldots \right)$$

$$S_{1/9} = \frac{1}{1} - \frac{1}{8^2} - \frac{1}{10^2} + \frac{1}{17^2} + \frac{1}{19^2} - \frac{1}{26^2} - \frac{1}{28^2} + \ldots = \frac{\pi^2 \cos(\frac{1}{9}\pi)}{9^2 \sin^2(\frac{1}{9}\pi)}$$

Proof : By $\mathbf{7_{sin}}$. based on $\pi \frac{1}{\sin(\pi z)} \frac{1}{(z + \frac{1}{9})^2}$

$$\pi^2 = 9^2 \frac{\sin^2(\frac{1}{9}\pi)}{\cos(\frac{1}{9}\pi)} \left(\frac{1}{1^2} - \frac{1}{(9-1)^2} - \frac{1}{(9+1)^2} + \frac{1}{(2(9)-1)^2} + \frac{1}{(2(9)+1)^2} - \frac{1}{(3(9)-1)^2} + \ldots \right) . \square$$

30.3

$$\pi^2 = 4(9^2) \frac{\cos^2(\frac{1}{9}\pi)}{\sin(\frac{1}{9}\pi)} \left(\frac{1}{7^2} - \frac{1}{11^2} - \frac{1}{25^2} + \frac{1}{29^2} + \frac{1}{43^2} - \frac{1}{47^2} - \ldots \right)$$

$$C_{1/9} = \frac{1}{7^2} - \frac{1}{11^2} - \frac{1}{25^2} + \frac{1}{29^2} + \frac{1}{43^2} - \frac{1}{47^2} - \ldots = \frac{\pi^2 \sin(\frac{1}{9}\pi)}{4(9^2) \cos^2(\frac{1}{9}\pi)}$$

Proof : **By $\mathbf{7_{cos}}$. based on** $\pi\frac{1}{\cos(\pi z)}\frac{1}{(z-\frac{1}{9})^2}$

$$\pi^2 = 4(9^2)\frac{\cos^2(\frac{1}{9}\pi)}{\sin(\frac{1}{9}\pi)}\Bigg(\frac{1}{(9-2)^2}-\frac{1}{(9+2)^2} - \frac{1}{(3(9)-2)^2}+\frac{1}{(3(9)+2)^2} + \frac{1}{(5(9)-2)^2}-\frac{1}{(5(9)+2)^2}-...\Bigg) .\square$$

30.4

$$\boxed{\pi^2 = 4(9^2)\cos^2(\tfrac{1}{9}\pi)\Bigg(\frac{1}{7^2}+\frac{1}{11^2} + \frac{1}{25^2}+\frac{1}{29^2} + \frac{1}{43^2}+\frac{1}{47^2}-...\Bigg)}$$

$$T_{1/9} = \frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{25^2}+\frac{1}{29^2}+\frac{1}{43^2}+\frac{1}{47^2}-... = \frac{\pi^2}{4(9^2)}\frac{1}{\cos^2(\frac{1}{9}\pi)}$$

Proof : **By $\mathbf{7_{tan}}$. based on** $\pi\tan(\pi z)\frac{1}{(z-\frac{1}{9})^2}$

$$\pi^2 = 4(9^2)\cos^2(\tfrac{1}{9}\pi)\Bigg(\frac{1}{(9-2)^2}+\frac{1}{(9+2)^2} + \frac{1}{(3(9)-2)^2}+\frac{1}{(3(9)+2)^2} + \frac{1}{(5(9)-2)^2}+\frac{1}{(5(9)+2)^2}-...\Bigg) .\square$$

31.

31.1

$$\boxed{\pi^2 = 9^2 \sin^2(\tfrac{2}{9}\pi)\left(\frac{1}{2^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{16^2} + \frac{1}{20^2} + \frac{1}{25^2} + ...\right)}$$

$$R_{2/9} = \frac{1}{2^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{16^2} + \frac{1}{20^2} + \frac{1}{25^2} + ... = \frac{\pi^2}{9^2 \sin^2(\frac{2}{9}\pi)}$$

$\sin(\frac{2}{9}\pi)$ **solves the cubic** $4x^3 - 3x + \frac{1}{2} = 0$

<u>Proof</u>: By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\dfrac{1}{(z+\frac{2}{9})^2}$,

$$\pi^2 = 9^2 \sin^2(\tfrac{2}{9}\pi)\left(\frac{1}{1^2} + \frac{1}{(9-2)^2} + \frac{1}{(9+2)^2} + \frac{1}{(2(9)-2)^2} + \frac{1}{(2(9)+2)^2} + \frac{1}{(3(9)-2)^2} + ...\right). \square$$

$$\sin 3A = -4\sin^3 A + 3\sin A \Rightarrow$$

$$\Rightarrow \underbrace{\sin(3\tfrac{2}{9}\pi)}_{\frac{1}{2}} = -4\sin^3(\tfrac{2}{9}\pi) + 3\sin(\tfrac{2}{9}\pi) \Rightarrow$$

$\Rightarrow \sin(\frac{2}{9}\pi)$ **solves the cubic** $4x^3 - 3x + \frac{1}{2} = 0. \square$

31.2

$$\pi^2 = 9^2 \frac{\sin^2(\frac{2}{9}\pi)}{\cos(\frac{2}{9}\pi)}\left(\frac{1}{2^2} - \frac{1}{7^2} - \frac{1}{11^2} + \frac{1}{16^2} + \frac{1}{20^2} - \frac{1}{25^2} + ...\right)$$

$$S_{2/9} = \frac{1}{2^2} - \frac{1}{7^2} - \frac{1}{11^2} + \frac{1}{16^2} + \frac{1}{20^2} - \frac{1}{25^2} - ... = \frac{\pi^2 \cos(\frac{2}{9}\pi)}{9^2 \sin^2(\frac{2}{9}\pi)}$$

Proof: By $\mathbf{7_{sin}}$. based on $\pi \dfrac{1}{\sin(\pi z)} \dfrac{1}{(z + \frac{2}{9})^2}$,

$$\pi^2 = 9^2 \frac{\sin^2(\frac{2}{9}\pi)}{\cos(\frac{2}{9}\pi)}\left(\frac{1}{1^2} - \frac{1}{(9-2)^2} - \frac{1}{(9+2)^2} + \frac{1}{(2(9)-2)^2} + \frac{1}{(2(9)+2)^2} - \frac{1}{(3(9)-2)^2} + ...\right) . \square$$

31.3

$$\pi^2 = 4(9^2) \frac{\cos^2(\frac{2}{9}\pi)}{\sin(\frac{2}{9}\pi)}\left(\frac{1}{5^2} - \frac{1}{13^2} - \frac{1}{23^2} + \frac{1}{31^2} + \frac{1}{41^2} - \frac{1}{49^2} - ...\right)$$

$$C_{2/9} = \frac{1}{5^2} - \frac{1}{13^2} - \frac{1}{23^2} + \frac{1}{31^2} + \frac{1}{41^2} - \frac{1}{49^2} - ... = \frac{\pi^2}{4(9^2)} \frac{\sin(\frac{2}{9}\pi)}{\cos^2(\frac{2}{9}\pi)}$$

Proof: By $\mathbf{7_{cos}}$. based on $\pi \dfrac{1}{\cos(\pi z)} \dfrac{1}{(z - \frac{2}{9})^2}$,

$$\pi^2 = 4(9^2)\frac{\cos^2(\frac{2}{9}\pi)}{\sin(\frac{2}{9}\pi)}\left(\frac{1}{(9-2(2))^2} - \frac{1}{(9+2(2))^2} - \frac{1}{(3(9)-2(2))^2} + \frac{1}{(3(9)+2(2))^2} + \frac{1}{(5(9)-2(2))^2} - \frac{1}{(5(9)+2(2))^2} - ...\right).\square$$

31.4

$$\boxed{\pi^2 = 4(9^2)\cos^2(\tfrac{2}{9}\pi)\left(\frac{1}{5^2} + \frac{1}{13^2} + \frac{1}{23^2} + \frac{1}{31^2} + \frac{1}{41^2} + \frac{1}{49^2} - ...\right)}$$

$$T_{2/9} = \frac{1}{5^2} + \frac{1}{13^2} + \frac{1}{23^2} + \frac{1}{31^2} + \frac{1}{41^2} + \frac{1}{49^2} - ... = \frac{\pi^2}{4(9^2)}\frac{1}{\cos^2(\frac{2}{9}\pi)}$$

Proof: By $\mathbf{7_{tan}}$. based on $\pi\tan(\pi z)\dfrac{1}{(z-\frac{2}{9})^2}$,

$$\pi^2 = 4(9^2)\cos^2(\tfrac{2}{9}\pi)\left(\frac{1}{(9-2(2))^2} + \frac{1}{(9+2(2))^2} + \frac{1}{(3(9)-2(2))^2} + \frac{1}{(3(9)+2(2))^2} + \frac{1}{(5(9)-2(2))^2} + \frac{1}{(5(9)+2(2))^2} - ...\right).\square$$

32.

32.1
$$\boxed{\pi^2 = 9^2\sin^2(\tfrac{4}{9}\pi)\left(\frac{1}{4^2}+\frac{1}{5^2} + \frac{1}{13^2}+\frac{1}{14^2} + \frac{1}{22^2}+\frac{1}{23^2}+...\right)}$$

$$R_{4/9} = \frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{13^2}+\frac{1}{14^2}+\frac{1}{22^2}+\frac{1}{23^2}+... = \frac{\pi^2}{9^2\sin^2(\frac{4}{9}\pi)}$$

$\sin(\frac{4}{9}\pi)$ **solves the cubic** $4x^3 - 3x - \frac{1}{2}\sqrt{3} = 0$

Proof By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\dfrac{1}{(z+\frac{4}{9})^2}$

$$\pi^2 = 9^2\sin^2(\tfrac{4}{9}\pi)\left(\frac{1}{4^2}+\frac{1}{(9-4)^2} + \frac{1}{(9+4)^2}+\frac{1}{(2(9)-4)^2} + \frac{1}{(2(9)+4)^2}+\frac{1}{(3(9)-4)^2}+...\right)$$

$$\sin 3A = -4\sin^3 A + 3\sin A \Rightarrow$$

$$\Rightarrow \underbrace{\sin(3\tfrac{4}{9}\pi)}_{-\frac{1}{2}\sqrt{3}} = -4\sin^3(\tfrac{4}{9}\pi) + 3\sin(\tfrac{4}{9}\pi) \Rightarrow$$

$\Rightarrow \sin(\frac{4}{9}\pi)$ **solves the cubic** $4x^3 - 3x - \frac{1}{2}\sqrt{3} = 0$.□

32.2

$$\boxed{\pi^2 = 9^2 \frac{\sin^2(\frac{4}{9}\pi)}{\cos(\frac{4}{9}\pi)} \left(\frac{1}{4^2} - \frac{1}{5^2} - \frac{1}{13^2} + \frac{1}{14^2} + \frac{1}{22^2} - \frac{1}{23^2} - \dots \right)}$$

$$S_{4/9} = \frac{1}{4^2} - \frac{1}{5^2} - \frac{1}{13^2} + \frac{1}{14^2} + \frac{1}{22^2} - \frac{1}{23^2} - \dots = \frac{\pi^2 \cos(\frac{4}{9}\pi)}{9^2 \sin^2(\frac{4}{9}\pi)}$$

Proof By $\mathbf{7_{sin}}$. based on $\pi \frac{1}{\sin(\pi z)} \frac{1}{(z + \frac{4}{9})^2}$,

$$\pi^2 = 9^2 \sin^2(\tfrac{4}{9}\pi) \left(\frac{1}{4^2} - \frac{1}{(9-4)^2} - \frac{1}{(9+4)^2} + \frac{1}{(2(9)-4)^2} + \frac{1}{(2(9)+4)^2} - \frac{1}{(3(9)-4)^2} - \dots \right) . \square$$

32.3

$$\boxed{\pi^2 = 4(9)^2 \frac{\cos^2(\frac{4}{9}\pi)}{\sin(\frac{4}{9}\pi)} \left(\frac{1}{1^2} - \frac{1}{17^2} - \frac{1}{19^2} + \frac{1}{35^2} + \frac{1}{37^2} - \frac{1}{53^2} - \dots \right)}$$

$$C_{4/9} = \frac{1}{1^2} - \frac{1}{17^2} - \frac{1}{19^2} + \frac{1}{35^2} + \frac{1}{37^2} - \frac{1}{53^2} - \dots = \frac{\pi^2}{4(9^2)} \frac{\sin(\frac{4}{9}\pi)}{\cos^2(\frac{4}{9}\pi)}$$

Proof By $\mathbf{7_{cos}}$. based on $\pi \frac{1}{\cos(\pi z)} \frac{1}{(z - \frac{4}{9})^2}$,

$$\pi^2 = 4(9)^2 \frac{\cos^2(\frac{4}{9}\pi)}{\sin(\frac{4}{9}\pi)}\left(\frac{1}{(9-2(4))^2} - \frac{1}{(9+2(4))^2} - \frac{1}{(3(9)-2(4))^2} + \frac{1}{(3(9)+2(4))^2} + \frac{1}{(5(9)-2(4))^2} - \frac{1}{(5(9)+2(4))^2} - ...\right). \square$$

32.4

$$\boxed{\pi^2 = 4(9)^2 \cos^2(\tfrac{4}{9}\pi)\left(\frac{1}{1^2} + \frac{1}{17^2} + \frac{1}{19^2} + \frac{1}{35^2} + \frac{1}{37^2} + \frac{1}{53^2} - ...\right)}$$

$$T_{4/9} = \frac{1}{1^2} + \frac{1}{17^2} + \frac{1}{19^2} + \frac{1}{35^2} + \frac{1}{37^2} + \frac{1}{53^2} + ... = \frac{\pi^2}{4(9^2)} \frac{1+\tan(\frac{4}{9}\pi)}{\cot(\frac{4}{9}\pi)}$$

Proof By $\mathbf{7_{tan}}$. based on $\pi\tan(\pi z)\dfrac{1}{(z-\frac{4}{9})^2}$,

$$\pi^2 = 4(9)^2 \cos^2(\tfrac{4}{9}\pi)\left(\frac{1}{(9-2(4))^2} - \frac{1}{(9+2(4))^2} - \frac{1}{(3(9)-2(4))^2} + \frac{1}{(3(9)+2(4))^2} + \frac{1}{(5(9)-2(4))^2} - \frac{1}{(5(9)+2(4))^2} - ...\right). \square$$

$\cos^2(\frac{4}{9}\pi) = \frac{1}{2}[1+\cos(\frac{8}{9}\pi)].\square$

33.

33.1
$$\pi^2 = 3^2\left(\frac{1}{1}+\frac{1}{5^2} +\frac{1}{7^2}+\frac{1}{11^2} +\frac{1}{13^2}+\frac{1}{17^2}+...\right)$$

$$R_{1/6} = \frac{1}{1}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{13^2}+\frac{1}{17^2}+... = \frac{\pi^2}{9}$$

Proof: By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\frac{1}{(z+\frac{1}{6})^2}$

$$\pi^2 = 6^2\underbrace{\sin^2(\tfrac{1}{6}\pi)}_{1/2^2}\left(\frac{1}{1^2}+\frac{1}{(6-1)^2} +\frac{1}{(6+1)^2}+\frac{1}{(2(6)-1)^2} +\frac{1}{(2(6)+1)^2}+\frac{1}{(3(6)-1)^2}+...\right)$$

33.2
$$\pi^2 = \tfrac{2}{\sqrt{3}}3^2\left(\frac{1}{1}-\frac{1}{5^2} -\frac{1}{7^2}+\frac{1}{11^2} +\frac{1}{13^2}-\frac{1}{17^2}-...\right)$$

$$S_{1/6} = \frac{1}{1}-\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{13^2}-\frac{1}{17^2}-... = \frac{\pi^2}{9}\frac{\sqrt{3}}{2}$$

Proof: By $\mathbf{7_{sin}}$. based on $\pi\frac{1}{\sin(\pi z)}\frac{1}{(z+\frac{1}{6})^2}$,

$$\pi^2 = 6^2 \underbrace{\frac{\sin^2(\frac{1}{6}\pi)}{\cos(\frac{1}{6}\pi)}}_{(1/4)/(\sqrt{3}/2)} \left(\frac{1}{1^2} + \frac{1}{(6-1)^2} + \frac{1}{(6+1)^2} + \frac{1}{(2(6)-1)^2} + \frac{1}{(2(6)+1)^2} + \frac{1}{(3(6)-1)^2} + \dots \right) . \square$$

33.3

$$\sqrt{3} = 2 \frac{\frac{1}{1} - \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} - \frac{1}{17^2} - \dots}{\frac{1}{1} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \dots}$$

Proof: $\frac{S_{1/6}}{R_{1/6}} . \square$

33.4

$$\boxed{\pi^2 = \frac{3^3}{2} \left(\frac{1}{1} - \frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{8^2} - \dots \right)}$$

$$C_{1/6} = \frac{1}{1} - \frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{8^2} - \dots = \frac{2}{3^3}\pi^2$$

Proof: By $\mathbf{7_{cos}}$. based on $\pi \frac{1}{\cos(\pi z)} \frac{1}{(z - \frac{1}{6})^2}$,

$$\pi^2 = 4(6^2)\underbrace{\frac{\cos^2(\frac{1}{6}\pi)}{\sin(\frac{1}{6}\pi)}}_{(3/4)/(1/2)}\left(\frac{1}{(6-2)^2} - \frac{1}{(6+2)^2} - \frac{1}{(3(6)-2)^2} + \frac{1}{(3(6)+2)^2} + \frac{1}{(5(6)-2)^2} - \frac{1}{(5(6)+2)^2} - ...\right)$$

33.5

$$\boxed{\pi^2 = \frac{6^3}{2}\left(\frac{1}{1} + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} - ...\right)}$$

$$T_{1/6} = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} + ... = \frac{2}{6^3}\pi^2$$

Proof: By $\mathbf{7_{tan}}$. based on $\pi\tan(\pi z)\dfrac{1}{(z-\frac{1}{6})^2}$,

$$\pi^2 = 4(6^2)\underbrace{\cos^2(\tfrac{1}{6}\pi)}_{3/4}\left(\frac{1}{(6-2)^2} + \frac{1}{(6+2)^2} + \frac{1}{(3(6)-2)^2} + \frac{1}{(3(6)+2)^2} + \frac{1}{(5(6)-2)^2} + \frac{1}{(5(6)+2)^2} - ...\right)$$

34.

34.1

$$\pi^2 = 18^2 \sin^2(\tfrac{1}{18}\pi)\left(\frac{1}{1} + \frac{1}{17^2} + \frac{1}{19^2} + \frac{1}{35^2} + \frac{1}{37^2} + \frac{1}{53^2} + \dots\right)$$

$$R_{1/18} = \frac{1}{1} + \frac{1}{17^2} + \frac{1}{19^2} + \frac{1}{35^2} + \frac{1}{37^2} + \frac{1}{53^2} + \dots = \frac{\pi^2}{18^2 \sin^2(\frac{1}{18}\pi)}$$

Proof: By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\dfrac{1}{(z+\frac{1}{18})^2}$

$$\pi^2 = 18^2 \sin^2(\tfrac{1}{18}\pi)\left(\frac{1}{1^2} + \frac{1}{(18-1)^2} + \frac{1}{(18+1)^2} + \frac{1}{(2(18)-1)^2} + \frac{1}{(2(18)+1)^2} + \frac{1}{(3(18)-1)^2} + \dots\right)$$

34.2

$$\pi^2 = 18^2 \frac{\sin^2(\frac{1}{18}\pi)}{\cos(\frac{1}{18}\pi)}\left(\frac{1}{1} - \frac{1}{17^2} - \frac{1}{19^2} + \frac{1}{35^2} + \frac{1}{37^2} - \frac{1}{53^2} - \dots\right)$$

$$S_{1/18} = \frac{1}{1} - \frac{1}{17^2} - \frac{1}{19^2} + \frac{1}{35^2} + \frac{1}{37^2} - \frac{1}{53^2} - \dots = \frac{\pi^2 \cos(\frac{1}{18}\pi)}{18^2 \sin^2(\frac{1}{18}\pi)}$$

$$\sin(\tfrac{1}{18}\pi) = \sqrt{\tfrac{1}{2}[1-\cos(\tfrac{1}{9}\pi)]} = \sqrt{\tfrac{1}{2}[1-\sin(\tfrac{4}{9}\pi)]}$$

And $\sin(\tfrac{4}{9}\pi)$ solves the cubic $4x^3 - 3x - \tfrac{1}{2}\sqrt{3} = 0$

Proof: By $\mathbf{7_{sin}}$. based on $\pi \dfrac{1}{\sin(\pi z)} \dfrac{1}{(z+\frac{1}{18})^2}$

$$\pi^2 = 18^2 \frac{\sin^2(\frac{1}{18}\pi)}{\cos(\frac{1}{18}\pi)} \left(\frac{1}{1^2} - \frac{1}{(18-1)^2} - \frac{1}{(18+1)^2} + \frac{1}{(2(18)-1)^2} + \frac{1}{(2(18)+1)^2} - \frac{1}{(3(18)-1)^2} + \ldots \right).\square$$

$$\sin 3A = -4\sin^3 A + 3\sin A \Rightarrow$$

$$\Rightarrow \underbrace{\sin(3\tfrac{4}{9}\pi)}_{-\frac{1}{2}\sqrt{3}} = -4\sin^3(\tfrac{4}{9}\pi) + 3\sin(\tfrac{4}{9}\pi) \Rightarrow$$

$\Rightarrow \sin(\tfrac{4}{9}\pi)$ solves the cubic $4x^3 - 3x - \tfrac{1}{2}\sqrt{3} = 0.\square$

34.3

$$\boxed{\pi^2 = 9^2 \frac{\cos^2(\frac{1}{18}\pi)}{\sin(\frac{1}{18}\pi)} \left(\frac{1}{4^2} - \frac{1}{5^2} - \frac{1}{13^2} + \frac{1}{14^2} + \frac{1}{22^2} - \frac{1}{23^2} - \ldots \right)}$$

$$C_{1/18} = \frac{1}{4^2} - \frac{1}{5^2} - \frac{1}{13^2} + \frac{1}{14^2} + \frac{1}{22^2} - \frac{1}{23^2} - \ldots = \frac{\pi^2}{9^2} \frac{\sin(\frac{1}{18}\pi)}{\cos^2(\frac{1}{18}\pi)}$$

Proof: By $\mathbf{7_{cos}}$. based on $\pi \frac{1}{\cos(\pi z)} \frac{1}{(z - \frac{1}{18})^2}$

$$\pi^2 = 4(18^2)\frac{\cos^2(\frac{1}{18}\pi)}{\sin(\frac{1}{18}\pi)}\Bigg(\frac{1}{(18-2)^2} - \frac{1}{(18+2)^2} - \frac{1}{(3(18)-2)^2} + \frac{1}{(3(18)+2)^2} + \frac{1}{(5(18)-2)^2} - \frac{1}{(5(18)+2)^2} - ...\Bigg)$$

34.4
$$\boxed{\pi^2 = 9^2\cos^2(\tfrac{1}{18}\pi)\Bigg(\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{13^2} + \frac{1}{14^2} + \frac{1}{22^2} + \frac{1}{23^2} - ...\Bigg)}$$

$$T_{1/18} = \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{13^2} + \frac{1}{14^2} + \frac{1}{22^2} + \frac{1}{23^2} + ... = \frac{\pi^2}{9^2}\frac{1}{\cos^2(\frac{1}{18}\pi)}$$

Proof: By $\mathbf{7_{tan}}$. based on $\pi \tan(\pi z) \frac{1}{(z - \frac{1}{18})^2}$

$$\pi^2 = 4(18^2)\cos^2(\tfrac{1}{18}\pi)\Bigg(\frac{1}{(18-2)^2} + \frac{1}{(18+2)^2} + \frac{1}{(3(18)-2)^2} + \frac{1}{(3(18)+2)^2} + \frac{1}{(5(18)-2)^2} + \frac{1}{(5(18)+2)^2} - ...\Bigg)$$

35.

35.1

$$\pi^2 = 18^2 \sin^2(\tfrac{5}{18}\pi)\left(\frac{1}{5^2} + \frac{1}{13^2} + \frac{1}{23^2} + \frac{1}{31^2} + \frac{1}{41^2} + \frac{1}{49^2} + ...\right)$$

$$R_{5/18} = \frac{1}{5^2} + \frac{1}{13^2} + \frac{1}{23^2} + \frac{1}{31^2} + \frac{1}{41^2} + \frac{1}{49^2} + ... = \frac{\pi^2}{18^2 \sin^2(\frac{5}{18}\pi)}$$

Proof: By **7_{cot}.** based on $\pi \cot(\pi z)\dfrac{1}{(z+\frac{5}{18})^2}$,

$$\pi^2 = 18^2 \sin^2(\tfrac{5}{18}\pi)\left(\frac{1}{5^2} + \frac{1}{(18-5)^2} + \frac{1}{(18+5)^2} + \frac{1}{(2(18)-5)^2} + \frac{1}{(2(18)+5)^2} + \frac{1}{(3(18)-5)^2} + ...\right)$$

35.2

$$\pi^2 = 18^2 \frac{\sin^2(\frac{5}{18}\pi)}{\cos(\frac{5}{18}\pi)}\left(\frac{1}{5^2} - \frac{1}{13^2} - \frac{1}{23^2} + \frac{1}{31^2} + \frac{1}{41^2} - \frac{1}{49^2} - ...\right)$$

$$S_{5/18} = \frac{1}{5^2} - \frac{1}{13^2} - \frac{1}{23^2} + \frac{1}{31^2} + \frac{1}{41^2} - \frac{1}{49^2} - ... = \frac{\pi^2 \cos(\frac{5}{18}\pi)}{18^2 \sin^2(\frac{5}{18}\pi)}$$

<u>*Proof*</u>: By $\mathbf{7_{sin}}$. Based on $\pi \dfrac{1}{\sin(\pi z)} \dfrac{1}{(z+\frac{5}{18})^2}$,

$$\pi^2 = 18^2 \frac{\sin^2(\frac{5}{18}\pi)}{\cos(\frac{5}{18}\pi)} \Bigg(\frac{1}{5^2} - \frac{1}{(18-5)^2} - \frac{1}{(18+5)^2} + \frac{1}{(2(18)-5)^2} + \frac{1}{(2(18)+5)^2} - \frac{1}{(3(18)-5)^2} - \dots \Bigg) . \square$$

35.3

$$\boxed{\pi^2 = 9^2 \frac{\cos^2(\frac{5}{18}\pi)}{\sin(\frac{5}{18}\pi)} \Bigg(\frac{1}{2^2} - \frac{1}{7^2} - \frac{1}{11^2} + \frac{1}{16^2} + \frac{1}{20^2} - \frac{1}{25^2} - \dots \Bigg)}$$

$$C_{5/18} = \frac{1}{2^2} - \frac{1}{7^2} - \frac{1}{11^2} + \frac{1}{16^2} + \frac{1}{20^2} - \frac{1}{25^2} - \dots = \frac{\pi^2 \sin(\frac{5}{18}\pi)}{9^2 \cos^2(\frac{5}{18}\pi)}$$

<u>*Proof*</u>: By $\mathbf{7_{cos}}$. Based on $\pi \dfrac{1}{\cos(\pi z)} \dfrac{1}{(z-\frac{5}{18})^2}$,

$$\pi^2 = 4(18^2) \frac{\cos^2(\frac{5}{18}\pi)}{\sin(\frac{5}{18}\pi)} \Bigg(\frac{1}{(18-2(5))^2} - \frac{1}{(18+2(5))^2} - \frac{1}{(3(18)-2(5))^2} + \frac{1}{(3(18)+2(5))^2} + \frac{1}{(5(18)-2(5))^2} - \frac{1}{(5(18)+2(5))^2} - \dots \Bigg)$$

35.4

$$\boxed{\pi^2 = 9^2\cos^2(\tfrac{5}{18}\pi)\left(\frac{1}{2^2}+\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{16^2}+\frac{1}{20^2}+\frac{1}{25^2}-...\right)}$$

$$T_{5/18} = \frac{1}{2^2}+\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{16^2}+\frac{1}{20^2}+\frac{1}{25^2}+... = \frac{\pi^2}{9^2}\frac{1}{\cos^2(\tfrac{5}{18}\pi)}$$

Proof: By $\mathbf{7_{tan}}$. Based on $\pi\tan(\pi z)\dfrac{1}{(z-\frac{5}{18})^2}$,

$$\pi^2 = 4(18^2)\cos^2(\tfrac{5}{18}\pi)\left(\frac{1}{(18-2(5))^2}+\frac{1}{(18+2(5))^2}+\frac{1}{(3(18)-2(5))^2}+\frac{1}{(3(18)+2(5))^2}+\frac{1}{(5(18)-2(5))^2}+\frac{1}{(5(18)+2(5))^2}+...\right)$$

36.

36.1

$$\pi^2 = 18^2 \sin^2(\tfrac{7}{18}\pi)\left(\frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{25^2} + \frac{1}{29^2} + \frac{1}{43^2} + \frac{1}{47^2} + ...\right)$$

$$R_{7/18} = \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{25^2} + \frac{1}{29^2} + \frac{1}{43^2} + \frac{1}{47^2} + ... = \frac{\pi^2}{18^2 \sin^2(\frac{7}{18}\pi)}$$

Proof: By $\mathbf{7_{cot}}$. based on $\pi \cot(\pi z)\dfrac{1}{(z + \frac{7}{18})^2}$,

$$\pi^2 = 18^2 \sin^2(\tfrac{7}{18}\pi)\left(\frac{1}{7^2} + \frac{1}{(18-7)^2} + \frac{1}{(18+7)^2} + \frac{1}{(2(18)-7)^2} + \frac{1}{(2(18)+7)^2} + \frac{1}{(3(18)-7)^2} + ...\right).\square$$

36.2

$$\pi^2 = 18^2 \frac{\sin^2(\frac{7}{18}\pi)}{\cos(\frac{7}{18}\pi)}\left(\frac{1}{7^2} - \frac{1}{11^2} - \frac{1}{25^2} + \frac{1}{29^2} + \frac{1}{43^2} - \frac{1}{47^2} - ...\right)$$

$$S_{7/18} = \frac{1}{7^2} - \frac{1}{11^2} - \frac{1}{25^2} + \frac{1}{29^2} + \frac{1}{43^2} - \frac{1}{47^2} - ... = \frac{\pi^2 \cos(\frac{7}{18}\pi)}{18^2 \sin^2(\frac{7}{18}\pi)}$$

Proof: By $\mathbf{7_{sin}}$. Based on $\pi \dfrac{1}{\sin(\pi z)} \dfrac{1}{(z + \frac{7}{18})^2}$,

$$\pi^2 = 18^2 \frac{\sin^2(\frac{7}{18}\pi)}{\cos(\frac{7}{18}\pi)} \left(\frac{1}{7^2} - \frac{1}{(18-7)^2} - \frac{1}{(18+7)^2} + \frac{1}{(2(18)-7)^2} + \frac{1}{(2(18)+7)^2} - \frac{1}{(3(18)-7)^2} - \ldots \right) . \square$$

36.3

$$\boxed{\pi^2 = 9^2 \frac{\cos^2(\frac{7}{18}\pi)}{\sin(\frac{7}{18}\pi)} \left(\frac{1}{1} - \frac{1}{8^2} - \frac{1}{10^2} + \frac{1}{17^2} + \frac{1}{19^2} - \frac{1}{26^2} - \ldots \right)}$$

$$C_{7/18} = \frac{1}{1^2} - \frac{1}{8^2} - \frac{1}{10^2} + \frac{1}{17^2} + \frac{1}{19^2} - \frac{1}{26^2} - \ldots = \frac{\pi^2 \sin(\frac{7}{18}\pi)}{9^2 \cos^2(\frac{7}{18}\pi)}$$

Proof: By $\mathbf{7_{cos}}$. Based on $\pi \dfrac{1}{\cos(\pi z)} \dfrac{1}{(z - \frac{7}{18})^2}$,

$$\pi^2 = 4(18^2) \frac{\cos^2(\frac{7}{18}\pi)}{\sin(\frac{7}{18}\pi)} \left(\frac{1}{(18-2(7))^2} - \frac{1}{(18+2(7))^2} - \frac{1}{(3(18)-2(7))^2} + \frac{1}{(3(18)+2(7))^2} + \frac{1}{(5(18)-2(7))^2} - \frac{1}{(5(18)+2(7))^2} - \ldots \right) . \square$$

36.4
$$\boxed{\pi^2 = 9^2\cos^2(\tfrac{7}{18}\pi)\left(\frac{1}{1}+\frac{1}{8^2} + \frac{1}{10^2}+\frac{1}{17^2} + \frac{1}{19^2}+\frac{1}{26^2}-...\right)}$$

$$T_{7/18} = \frac{1}{1^2}+\frac{1}{8^2}+\frac{1}{10^2}+\frac{1}{17^2}+\frac{1}{19^2}+\frac{1}{26^2}+... = \frac{\pi^2}{9^2}\frac{1}{\cos^2(\tfrac{7}{18}\pi)}$$

Proof: By $\mathbf{7_{tan}}$. Based on $\pi\tan(\pi z)\dfrac{1}{(z-\tfrac{7}{18})^2}$,

$$\pi^2 = 4(18^2)\cos^2(\tfrac{7}{18}\pi)\left(\frac{1}{(18-2(7))^2}+\frac{1}{(18+2(7))^2} + \frac{1}{(3(18)-2(7))^2}+\frac{1}{(3(18)+2(7))^2} + \frac{1}{(5(18)-2(7))^2}+\frac{1}{(5(18)+2(7))^2}+...\right).\square$$

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