All the π^3 Series and the $\zeta(3)$ Series

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Abstract By π^3 Series we mean a series like Euler's

$$1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{11^3}+...=\frac{1}{32}\pi^3,$$

of reciprocals of cubes of integers, that sum up to an algebraic number times π^3.

Euler obtained two families of infinitely many alternating series:

For natural numbers l and m, and denoting

$$k=\tan(\tfrac{l}{2m}\pi)$$

Euler had in his section #174[1], the alternating series

$$\frac{1}{l_1^3}-\frac{1}{(2m_1-l_1)^3}$$
$$+\frac{1}{(2m_1+l_1)^3}-\frac{1}{(4m_1-l_1)^3}$$
$$+\frac{1}{(4m_1+l_1)^3}-\frac{1}{(6m_1-l_1)^3}+...=\frac{\pi^3}{8m_1^3}\frac{k^2+1}{k^3}$$

$l=1,\ m=2\ \Rightarrow\ k=\tan(\tfrac{1}{4}\pi)=1\Rightarrow$

$$1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{11^3}+...=\frac{1}{32}\pi^3$$

[1] Leonardi Euleri, "Introductio in Analysin Infinitorum", Section #174

And in his section #175[2], the alternating series

$$\frac{1}{(m_1-l_1)^3}-\frac{1}{(m_1+l_1)^3}+\frac{1}{(3m_1-l_1)^3}-\frac{1}{(3m_1+l_1)^3}+\frac{1}{(5m_1-l_1)^3}-\frac{1}{(5m_1+l_1)^3}+...=\frac{\pi^3}{8m_1^3}(k^3+k)$$

$l=1,\ m=2\ \Rightarrow\ k=\tan(\frac{1}{4}\pi)=1\Rightarrow$

$$1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{11^3}+...=\frac{1}{32}\pi^3.$$

The Residue Theorem yields all the four families of π^3 series. All are alternating Series. These extend Euler's work, and show that the $\zeta(3)$ Series is NOT a π^3 series. Thus, resolving Euler pursuit of a π^3 series formula for the $\zeta(3)$ Series.

If $l<\frac{1}{2}m$, l and m are natural numbers with no common factor.

Then, applying the Residue Theorem to $\pi\cot(\pi z)\frac{1}{(z-\frac{l}{m})^3}$, we obtain the series

$$R_{l/m}=\frac{1}{l^3}-\frac{1}{(m-l)^3}+\frac{1}{(m+l)^3}-\frac{1}{(2m-l)^3}+\frac{1}{(2m+l)^3}-\frac{1}{(3m-l)^3}+...=\frac{\pi^3}{m^3}\frac{\cot(\frac{l}{m}\pi)}{\sin^2(\frac{l}{m}\pi)}$$

with pattern $+-+-+-+-....$

[2] Leonardi Euleri, "Introductio in Analysin Infinitorum", Section #175

$l = 1$, and $m = 4 \Rightarrow$

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + ... = \frac{1}{64}\pi^3.$$

This family of series is identical with Euler's #174 where

$$\frac{\cot(\frac{l}{m}\pi)}{\sin^2(\frac{l}{m}\pi)} = \frac{k^2+1}{k^3}$$

Applying the Residue Theorem to $\pi\tan(\pi z)\dfrac{1}{(z-\frac{l}{m})^3}$ yields the series

$$T_{l/m} = \frac{1}{(m-2l)^3} - \frac{1}{(m+2l)^3} + \frac{1}{(3m-2l)^3} - \frac{1}{(3m+2l)^3} + \frac{1}{(5m-2l)^3} - \frac{1}{(5m+2l)^3} + ... = \frac{\pi^3}{8m^3}\frac{\tan(\frac{l}{m}\pi)}{\cos^2(\frac{l}{m}\pi)}$$

with pattern $+-+-+-+-....$

$l = 1$, and $m = 4 \Rightarrow$

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + ... = \frac{1}{32}\pi^3.$$

This family of series is identical with Euler's #175 where

$$\frac{\tan(\frac{l}{m}\pi)}{\cos^2(\frac{l}{m}\pi)} = k^3 + k$$

Euler did not obtain two infinite families of π^3 series that follow from the Residue Theorem:

Applying the Residue theorem to $\pi\frac{1}{\sin(\pi)}\frac{1}{(z-\frac{l}{m})^3}$ yields the alternating series

$$S_{l/m}=\frac{1}{l^3}+\frac{1}{(m-l)^3}$$
$$-\frac{1}{(m+l)^3}-\frac{1}{(2m-l)^3}$$
$$+\frac{1}{(2m+l)^3}+\frac{1}{(3m-l)^3}-...=\frac{\pi^3}{m^3}\frac{1+\cos^2(\frac{l}{m}\pi)}{2\sin^3(\frac{l}{m}\pi)}$$

with pattern $++--++--....$

$l=1$, and $m=4\ \Rightarrow$

$$1+\frac{1}{3^3}-\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}+\frac{1}{11^3}+...=\frac{\pi^3}{32}\frac{3}{2\sqrt{2}}.$$

Applying the Residue Theorem to $\pi\frac{1}{\cos(\pi z)}\frac{1}{(z-\frac{l}{m})^3}$, yields the alternating series

$$C_{l/m}=\frac{1}{(m-2l)^2}+\frac{1}{(m+2l)^2}$$
$$-\frac{1}{(3m-2l)^2}-\frac{1}{(3m+2l)^2}$$
$$+\frac{1}{(5m-2l)^2}+\frac{1}{(5m+2l)^2}-...=\frac{\pi^3}{16m^3}\frac{1+\sin^2(\frac{l}{m}\pi)}{2\cos^3(\frac{l}{m}\pi)}$$

with pattern $++--++--....$

$l=1$, and $m=4\ \Rightarrow$

$$1+\frac{1}{3^3}-\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}+\frac{1}{11^3}+...=\frac{\pi^3}{32}\frac{3}{2\sqrt{2}}$$

The $\zeta(3)$ Series

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\frac{1}{6^3}+\frac{1}{7^3}+\dots$$

sums up to approximately[3]

1.202056903159594285...

Euler sought an answer to the question whether the $\zeta(3)$ Series is a π^3 series.[4]

Since

$$\zeta(3)=\left(1+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+\dots\right)+\frac{1}{2^3}\left(1+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+\dots\right)$$
$$=\left(1+\frac{1}{2^3}\right)\left(1+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+\dots\right),$$

it is sufficient to show that

$$1+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+\dots$$

is a π^3 series.

This Series is the sum of

$$1+\tfrac{1}{5^2}+\tfrac{1}{9^2}+\tfrac{1}{13^2}+\dots, \text{ and } \tfrac{1}{3^2}+\tfrac{1}{7^2}+\tfrac{1}{11^2}+\tfrac{1}{15^2}+\dots.$$

The series $1+\frac{1}{5^2}+\frac{1}{9^2}+\frac{1}{13^2}+\dots$ is obtained from

$$\frac{1}{l^2}+\frac{1}{(2m+l)^2}+\frac{1}{(4m+l)^2}+\frac{1}{(6m+l)^2}+\dots \text{with } l=1,\ m=2$$

And the series $\frac{1}{3^2}+\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{15^2}+\dots.$ is obtained from

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[3] https://en.wikipedia.org/wiki/Particular_values_of_the_Riemann_zeta_function#Odd_positive_integers

[4] Euler's letters to James Stirling in "James Stirling, This about Series and such things" Scottish Academic Press 1988, pp. 142-151.

$$\frac{1}{(2m-l)^2}+\frac{1}{(4m-l)^2}+\frac{1}{(6m-l)^2}+\ldots \text{with } l=1,\ m=2$$

Thus, the desired Series belongs to the family of series

$$\frac{1}{l^2}+\frac{1}{(2m-l)^2}$$
$$+\frac{1}{(2m+l)^2}+\frac{1}{(4m-l)^2}$$
$$+\frac{1}{(4m+l)^2}+\frac{1}{(6m-l)^2}+\ldots$$

which pattern of

$$++++++++....$$

is incompatible with the alternating pattern of any π^3 series,

And which terms are different from the terms of any of the π^3 series.

Hence, by its pattern, and terms, the $\zeta(3)$ Series is not a π^3 series.

The $\zeta(3)$ series does not fit the pattern of any of the possible π^3 series.

Adding or subtracting alternating π^3 series gives alternating series, and does not yield the $\zeta(3)$ Series.

Therefore, it is not a π^3 series. That is, the $\zeta(3)$ series does not sum up to an algebraic number multiplying π^3.

This resolves Euler's pursuit of a π^3 series formula for the $\zeta(3)$ Series. Similarly,

$$1+\frac{1}{3^5}+\frac{1}{5^5}+\frac{1}{7^5}+\ldots \quad \text{is NOT a } \pi^5 \text{ series}$$

$$1+\frac{1}{3^7}+\frac{1}{5^7}+\frac{1}{7^7}+... \quad \text{is NOT a } \pi^7 \text{ series}$$

....

$$1+\frac{1}{3^{2n+1}}+\frac{1}{5^{2n+1}}+\frac{1}{7^{2n+1}}+... \quad \text{is NOT a } \pi^{2n+1} \text{ series.}$$

Contents

7.$_{\cos}$)

$$\pi^3 = 8m^3 \frac{2\cos^3(\frac{l}{m}\pi)}{\sin^2(\frac{l}{m}\pi)+1}\left\{\frac{1}{(m-2l)^3} + \frac{1}{(m+2l)^3} - \frac{1}{(3m-2l)^3} - \frac{1}{(3m+2l)^2} + \frac{1}{(5m-2l)^2} + \frac{1}{(5m+2l)^2} - ...\right\}$$

7.$_{\tan}$)

$$\pi^3 = 8m^3 \frac{\cos^2(\frac{l}{m}\pi)}{\tan(\frac{l}{m}\pi)}\left\{\frac{1}{(m-2l)^3} - \frac{1}{(m+2l)^3} + \frac{1}{(3m-2l)^3} - \frac{1}{(3m+2l)^3} + \frac{1}{(5m-2l)^2} - \frac{1}{(5m+2l)^2} + ...\right\}$$

8) 15/38

9) $(2k+1)\ /\ 2^n$

10) 1/4

11) 1/8

12) 3/8

13) $R_{1/8} + R_{3/8} = S_{1/4}$

14) $S_{1/8} - S_{3/8}$

15) $R_{1/8} - R_{3/8} = R_{1/4}$

16) 1/16

17) 3/16

18) 5/16

19) 7/16

20) $R_{1/16} + R_{3/16} + R_{5/16} + R_{7/16}$

21) $R_{1/16} - R_{3/16}$

22) $R_{5/16} - R_{7/16}$

23) $R_{3/16} - R_{5/16}$

24) $R_{1/16} - R_{5/16}$

25) $R_{1/16} - R_{7/16}$

26) $R_{3/16} - R_{7/16}$

27) $1 / p$

28) 1/3

29) l / p^n

30) 1/9

31) 2/9

32) 4/9

33) 1/6

34) 1/18

35) 5/18

36) 7/18

References

1.

Residue of $f(z)$ Singular at z_0

$$\text{Res}\left\{f(z)\right\}_{z=z_0} \equiv a_{-1} = \frac{1}{2\pi i}\oint_{\zeta=z_0+\varepsilon e^{i\phi}} f(\zeta)d\zeta$$

Proof: $f(z) = .. + \dfrac{a_{-k}}{(z-z_0)^k} + .. + \dfrac{a_{-2}}{(z-z_0)^2} +$

$$+\frac{a_{-1}}{z-z_0}+$$

$$+a_0 + ... + a_n(z-z_0)^n + ..$$

$$\Rightarrow \oint_{\zeta=z_0+\rho e^{i\phi}} f(\zeta)d\zeta =$$

$$= ... + a_{-k}\underbrace{\oint_{\zeta=z_0+\varepsilon e^{i\phi}}\frac{1}{(\zeta-z_0)^k}d\zeta}_{\frac{1}{\varepsilon^k}\varepsilon i\underbrace{\oint\frac{e^{i\phi}}{e^{ki\phi}}d\phi}_{0}} + ... + a_{-2}\underbrace{\oint_{\zeta=z_0+\varepsilon e^{i\phi}}\frac{1}{(\zeta-z_0)^2}d\zeta}_{\frac{1}{\varepsilon^2}\varepsilon i\underbrace{\oint\frac{e^{i\phi}}{e^{2i\phi}}d\phi}_{0}}$$

$$+a_{-1}\underbrace{\oint_{\zeta=z_0+\varepsilon e^{i\phi}}\frac{1}{\zeta-z_0}d\zeta}_{\frac{1}{\varepsilon}\varepsilon i\oint\frac{e^{i\phi}}{e^{i\phi}}d\phi = i\underbrace{\oint d\phi}_{2\pi}}$$

$$+a_0 \underbrace{\oint_{\zeta = z_0 + \varepsilon e^{i\phi}} d\zeta}_{\varepsilon i \underbrace{\oint e^{i\phi} d\phi}_{0}} + \dots + a_n \underbrace{\oint_{\zeta = z_0 + \varepsilon e^{i\phi}} (\zeta - z_0)^n d\zeta}_{\varepsilon^{n+1} i \underbrace{\oint e^{in\phi} e^{i\phi} d\phi}_{0}} + \dots$$

$$\Rightarrow \operatorname{Res}\left\{ f(z) \right\}_{z=z_0} \equiv a_{-1} = \frac{1}{2\pi i} \oint_{\zeta = z_0 + \varepsilon e^{i\phi}} f(\zeta) d\zeta \,. \square$$

2.

Residue at Pole of Order k

2.1 $f(z) = \frac{a_{-k}}{(z - z_0)^k} + ... + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + ...$

$$\Rightarrow \boxed{\text{Res}_{-1,k}\left\{f(z)\right\}_{z=z_0} = a_{-1} = \left[\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left\{(z - z_0)^k f(z)\right\}\right]_{z=z_0}}$$

2.2 $f(z) = \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ...$

$$\Rightarrow \boxed{\text{Res}_{-1,2}\left\{f(z)\right\}_{z=z_0} = a_{-1} = \left[\frac{d}{dz}\left\{(z - z_0)^2 f(z)\right\}\right]_{z=z_0}}$$

2.3 $f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ...$

$$\Rightarrow \boxed{\text{Res}_{-1,1}\left\{f(z)\right\}_{z=z_0} = a_{-1} = \left[(z - z_0)f(z)\right]_{z=z_0}}$$

3.

Residue at Pole of Infinite order

3.1 $e^{-\frac{1}{z}} = 1 - \frac{1}{z} + \frac{1}{2!}\left(-\frac{1}{z}\right)^2 + \frac{1}{3!}\left(-\frac{1}{z}\right)^3 + \frac{1}{4!}\left(-\frac{1}{z}\right)^4 + ... \Rightarrow$

$$\Rightarrow \text{Res}_{-2}\{e^{-\frac{1}{z}}\}_{z=0} = \frac{1}{2}$$

$$\Rightarrow \text{Res}_{-1}\{e^{-\frac{1}{z}}\}_{z=0} = -1$$

$$\Rightarrow \text{Res}_{0}\{e^{-\frac{1}{z}}\}_{z=0} = 1$$

4.

$$\operatorname{Res}\left\{\frac{\cot z \coth z}{z^3}\right\}_{z=0} = -\frac{7}{45}$$

Proof: divide the series,

$$\frac{1}{z^3}\frac{\cos z}{\sin z}\frac{\cosh z}{\sinh z} = \frac{1}{z^3}\frac{1-\frac{1}{2!}z^2+\frac{1}{4!}z^4+..}{z-\frac{1}{3!}z^3+\frac{1}{5!}z^5+...}\times\frac{1+\frac{1}{2!}z^2+\frac{1}{4!}z^4+..}{z+\frac{1}{3!}z^3+\frac{1}{5!}z^5+...}$$

$$= \frac{1}{z^3}\frac{1+\frac{1}{4!}z^4-\frac{1}{2!}z^2+..}{z+\frac{1}{5!}z^5-\frac{1}{3!}z^3+...}\times\frac{1+\frac{1}{4!}z^4+\frac{1}{2!}z^2+..}{z+\frac{1}{5!}z^5+\frac{1}{3!}z^3+...}$$

$$\approx \frac{1}{z^5}\frac{\left(1+\frac{1}{4!}z^4\right)^2-\left(\frac{1}{2!}z^2\right)^2}{\left(1+\frac{1}{5!}z^4\right)^2-\left(\frac{1}{3!}z^2\right)^2}$$

$$\approx \frac{1}{z^5}\frac{1-\left(\frac{1}{4}-\frac{1}{12}\right)z^4+\frac{1}{(4!)^2}z^8}{1-\left(\frac{1}{36}-\frac{1}{60}\right)z^4+\frac{1}{(5!)^2}z^8}$$

$$\approx \frac{1}{z^5}\frac{1-\frac{1}{6}z^4}{1-\frac{14}{90}z^4}$$

$$\approx \frac{1}{z^5}\left(1-\frac{1}{6}z^4\right)\left(1+\frac{1}{90}z^4\right)$$

$$= \left(1-\left[\frac{1}{6}-\frac{1}{90}\right]z^4-\frac{1}{540}z^8\right)$$

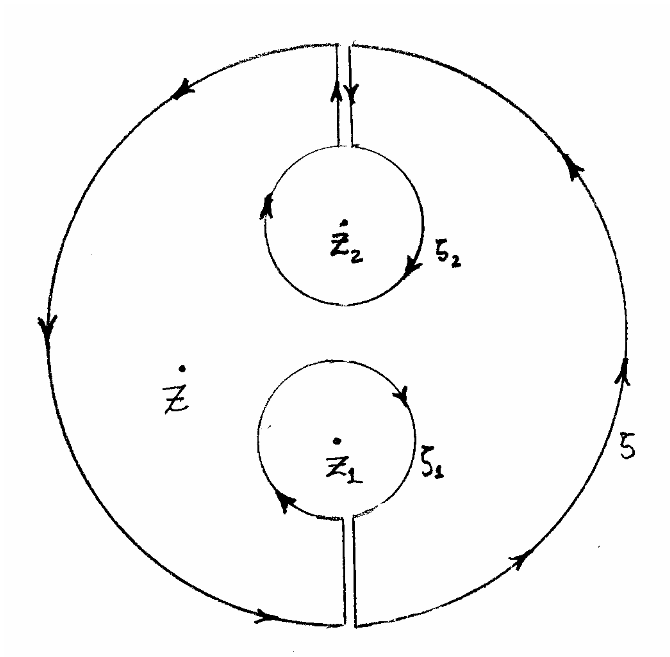
$$= \frac{1}{z^5}\left(1-\frac{14}{90}z^4-\frac{1}{540}z^8\right)$$

$$= \underbrace{1}_{a_{-5}}\frac{1}{z^5}-\underbrace{\frac{14}{90}}_{a_{-1}}\frac{1}{z}+\underbrace{\frac{1}{540}}_{a_3}z^3...$$

5.

Residue Theorem for a_{-1}

5.1



$$\frac{1}{2\pi i}\oint_C f(\zeta)d\zeta = \text{Res}_{-1}\left\{f(z)\right\}_{z=z_1} + \text{Res}_{-1}\left\{f(z)\right\}_{z=z_2}$$

Proof: $$\oint_{\zeta\in C} f(\zeta)d\zeta + \oint_{\zeta_1\in c_1} f(\zeta_1)d\zeta_1 + \oint_{\zeta_2\in c_2} f(\zeta_2)d\zeta_2 = 0$$

$$\frac{1}{2\pi i}\oint_{\zeta\in C} f(\zeta)d\zeta = \frac{1}{2\pi i}\oint_{\zeta_1=z_1+\rho e^{i\phi}} f(\zeta_1)d\zeta_1 + \frac{1}{2\pi i}\oint_{\zeta_2=z_2+\rho e^{i\phi}} f(\zeta_2)d\zeta_2,$$

For $\zeta_1 \in c_1$, $f(\zeta_1) = \dfrac{a_{-k_1,1}}{(\zeta_1 - z_1)^{k_1}} + ... + \dfrac{a_{-1,1}}{\zeta_1 - z_1} + a_{0,1} + a_{1,1}(\zeta_1 - z_1) + ...$

$$\Rightarrow \frac{1}{2\pi i} \oint_{\zeta_1 = z_1 + \rho e^{i\phi}} f(\zeta_1) d\zeta = a_{-1,1} = \text{Res}_{-1}\left\{ f(z) \right\}_{z=z_1}$$

For $\zeta_2 \in c_2$,

$$f(\zeta_2) = \frac{a_{-k_2,2}}{(\zeta_2 - z_2)^{k_2}} + ... + \frac{a_{-1,2}}{\zeta_2 - z_2} + a_{0,1} + a_{1,2}(\zeta_2 - z_2) + ...$$

$$\Rightarrow \frac{1}{2\pi i} \oint_{\zeta_2 = z_2 + \rho e^{i\phi}} f(\zeta_2) d\zeta_2 = a_{-1,2} = \text{Res}_{-1}\left\{ f(z) \right\}_{z=z_2}$$

$$\frac{1}{2\pi i} \oint_C f(\zeta) d\zeta = \text{Res}_{-1}\left\{ f(z) \right\}_{z=z_1} + \text{Res}_{-1}\left\{ f(z) \right\}_{z=z_2} . \square$$

5.2 $f(z)$ has poles at $z_1, z_2, ..., z_N \Rightarrow$

$$\frac{1}{2\pi i} \oint_C f(\zeta) d\zeta = \text{Res}_{-1}\left\{ f(z) \right\}_{z=z_1} + ... + \text{Res}_{-1}\left\{ f(z) \right\}_{z=z_N}$$

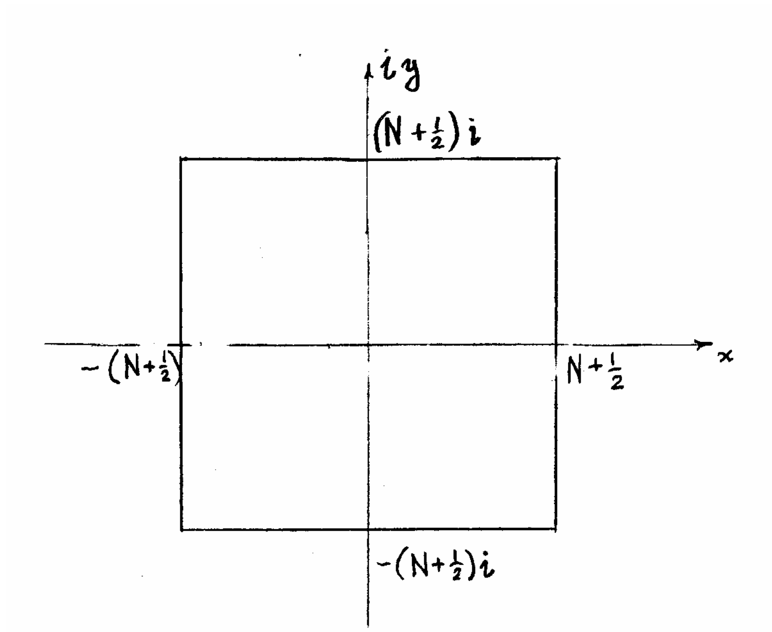
$\mathbf{6_{cot}}$.

$$\boxed{\pi\cot(\pi z)f(z)}$$

$$\boxed{\text{Res}\{\pi\cot(\pi z)f(z)\}_{z=n} = f(n)}$$

$$\boxed{\begin{aligned}&...f(-3)+f(-2)+f(-1)+f(0)+f(1)+f(2)+f(3)+...\\&\qquad+\sum \text{Res}_{-1}\left\{\pi\cot(\pi\sigma)f(\sigma)\right\}_{\sigma=\text{pole of } f(\sigma)} = 0\end{aligned}}$$

$\mathbf{6_{cot}.1}$ $\left|\cot\pi z\right| \le A$ **on** $\square_{N+\frac{1}{2}}$ **for any** N



$$\underline{y = -N - \frac{1}{2},\ \text{and}\ \ -N - \frac{1}{2} \le x \le N + \frac{1}{2}\Big|} \Rightarrow$$

$$\left|\cot\pi z\right| = \frac{\left|e^{i\pi z} + e^{-i\pi z}\right|}{\left|e^{i\pi z} - e^{-i\pi z}\right|}$$

$$\leq \frac{\left|e^{i\pi z}\right| + \left|e^{-i\pi z}\right|}{\left|\left|e^{i\pi z}\right| - \left|e^{-i\pi z}\right|\right|}$$

$$= \frac{\left|e^{i\pi(x+i[-N-\frac{1}{2}])}\right| + \left|e^{-i\pi(x+i[-N-\frac{1}{2}])}\right|}{\left|\left|e^{i\pi(x+i[-N-\frac{1}{2}])}\right| - \left|e^{-i\pi(x+i[-N-\frac{1}{2}])}\right|\right|}$$

$$= \frac{\left|e^{i\pi x+\pi N+\pi/2}\right| + \left|e^{-i\pi x-\pi N-\pi/2}\right|}{\left|\left|e^{i\pi x+\pi N+\pi/2}\right| - \left|e^{-i\pi x-\pi N-\pi/2}\right|\right|}$$

$$= \frac{\left(e^{\pi N+\pi/2} + e^{-\pi N-\pi/2}\right)}{\left(e^{\pi N+\pi/2} - e^{-\pi N-\pi/2}\right)} \frac{e^{-\pi N+\pi/2}}{e^{-\pi N+\pi/2}}$$

$$= \frac{e^{\pi} + e^{-2\pi N}}{e^{\pi} - e^{-2\pi N}}$$

$$= \frac{e^{\pi} + \frac{1}{e^{2\pi N}}}{e^{\pi} - \frac{1}{e^{2\pi N}}}$$

$$< \frac{e^{\pi} + 1}{e^{\pi} - 1}$$

$$= 1 + \frac{2}{e^{\pi} - 1} < 1.1. \square$$

$$\underline{y = N + \frac{1}{2}, \text{ and } -N - \frac{1}{2} \leq x \leq N + \frac{1}{2}} \Big| \Rightarrow$$

$$\left|\cot \pi z\right| = \left|\frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}}\right|$$

$$\leq \frac{\left|e^{i\pi z}\right| + \left|e^{-i\pi z}\right|}{\left|\left|e^{i\pi z}\right| - \left|e^{-i\pi z}\right|\right|}$$

$$= \frac{\left|e^{i\pi(x+i[N+\frac{1}{2}])}\right| + \left|e^{-i\pi(x+i[N+\frac{1}{2}])}\right|}{\left|\left|e^{i\pi(x+i[N+\frac{1}{2}])}\right| - \left|e^{-i\pi(x+i[N+\frac{1}{2}])}\right|\right|}$$

$$= \frac{\left|e^{i\pi x-\pi N-\pi/2}\right| + \left|e^{-i\pi x+\pi N+\pi/2}\right|}{\left|\left|e^{i\pi x-\pi N-\pi/2}\right| - \left|e^{-i\pi x+\pi N+\pi/2}\right|\right|}$$

$$= \frac{(e^{-\pi N-\pi/2} + e^{\pi N+\pi/2})}{(e^{\pi N+\pi/2} - e^{-\pi N-\pi/2})}\frac{e^{-\pi N+\pi/2}}{e^{-\pi N+\pi/2}}$$

$$= \frac{e^{\pi} + e^{-2\pi N}}{e^{\pi} - e^{-2\pi N}}$$

$$= \frac{e^{\pi} + \frac{1}{e^{2\pi N}}}{e^{\pi} - \frac{1}{e^{2\pi N}}}$$

$$< \frac{e^{\pi} + 1}{e^{\pi} - 1}$$

$$= 1 + \frac{2}{e^{\pi} - 1} < 1.1. \square$$

$\underline{x = N + \frac{1}{2},\ \text{and}\ -N - \frac{1}{2} \le y \le N + \frac{1}{2}} \Rightarrow$

For $y = 0$, $\left|\cot \pi z\right| = \left|\cot \pi(N + \frac{1}{2})\right|$

$$= \left|\cot(\pi N + \tfrac{\pi}{2})\right|$$

$$= \left|\cot\frac{\pi}{2}\right|$$

$$= \tan 0 = 0.$$

For $y \ne 0$, $\left|\cot \pi z\right| = \left|\cot\left(\pi N + \frac{\pi}{2} + i\pi y\right)\right|$

$$= \left|\cot\left(\tfrac{\pi}{2} + i\pi y\right)\right|$$

$$= \left|\tan\left(i\pi y\right)\right|$$

$$= \left|\frac{\sin i\pi y}{\cos i\pi y}\right|$$

$$= \left|\frac{\dfrac{e^{i\pi iy} - e^{-i\pi iy}}{2i}}{\dfrac{e^{i\pi iy} + e^{-i\pi iy}}{2}}\right|$$

$$= \left|\frac{e^{-\pi y} - e^{\pi y}}{e^{-\pi y} + e^{\pi y}}\right|$$

$$= \left|\frac{1 - e^{2\pi y}}{1 + e^{2\pi y}}\right| = \left|\frac{e^{2\pi y} - 1}{e^{2\pi y} + 1}\right|$$

For $y > 0$, $\dfrac{e^{2\pi y} - 1}{e^{2\pi y} + 1} < 1.\square$

For $y < 0$, $\dfrac{1 - e^{2\pi y}}{1 + e^{2\pi y}} = \dfrac{1 - \dfrac{1}{e^{2\pi(-y)}}}{1 + \dfrac{1}{e^{2\pi(-y)}}} < 1.\square$

$\underline{x = -N - \frac{1}{2},\ \text{and}\ -N - \frac{1}{2} \le y \le N + \frac{1}{2}} \Rightarrow$

For $y = 0$, $\left|\cot \pi z\right| = \left|\cot \pi(-N - \tfrac{1}{2})\right|$

$$= \left|\cot(-\pi N - \tfrac{\pi}{2})\right|$$

$$= \left|\cot(-\tfrac{\pi}{2})\right|$$

$$= \tan 0 = 0.$$

For $y \neq 0$, $\left|\cot \pi z\right| = \left|\cot\left(-\pi N - \tfrac{\pi}{2} + i\pi y\right)\right|$

$$= \left|\cot\left(-\tfrac{\pi}{2} + i\pi y\right)\right|$$

$$= \left|\tan\left(i\pi y\right)\right|$$

$$= \left|\frac{\sin i\pi y}{\cos i\pi y}\right|$$

$$= \left|\frac{\dfrac{e^{i\pi i y} - e^{-i\pi i y}}{2i}}{\dfrac{e^{i\pi i y} + e^{-i\pi i y}}{2}}\right|$$

$$= \left|\frac{e^{-\pi y} - e^{\pi y}}{e^{-\pi y} + e^{\pi y}}\right|$$

$$= \left|\frac{1 - e^{2\pi y}}{1 + e^{2\pi y}}\right| = \left|\frac{e^{2\pi y} - 1}{e^{2\pi y} + 1}\right|$$

For $y > 0$, $\quad \dfrac{e^{2\pi y} - 1}{e^{2\pi y} + 1} < 1.\square$

For $y < 0$, $\quad \dfrac{1 - e^{2\pi y}}{1 + e^{2\pi y}} = \dfrac{1 - \dfrac{1}{e^{2\pi(-y)}}}{1 + \dfrac{1}{e^{2\pi(-y)}}} < 1.\square$

6_{cot}.2 $\cot \pi z = \dfrac{\cos \pi z}{\sin \pi z}$ **has poles at** $z = n = ... - 2, -1, 0, 1, 2, ...$

6_{cot}.3

$$\boxed{\operatorname{Res}\left\{\pi \cot(\pi z)\right\}_{z=n} = 1}$$

Proof: $\operatorname{Res}\left\{\pi \cot(\pi z)\right\}_{z=n} = \left[(z - n)\pi \dfrac{\cos \pi z}{\sin \pi z}\right]_{z=n}$

$$= \left[\frac{D_z(\pi z - \pi n)}{D_z \sin(\pi z)} \cos(\pi z) \right]_{z=n}$$

$$= \left[\frac{\pi}{\pi \cos(\pi z)} \cos(\pi z) \right]_{z=n} = 1. \square$$

$6_{\cot}$.4 $\boxed{\text{Res}\left\{ \pi \cot(\pi z) f(z) \right\}_{z=n} = f(n)}$

$6_{\cot}$.5 $\left| f(z) \right|_{\square_{N+\frac{1}{2}}} \leq \dfrac{M}{z^k} \Rightarrow$

$$\boxed{\begin{aligned} &...f(-3) + f(-2) + f(-1) + f(0) + f(1) + f(2) + f(3) + ... \\ &\qquad + \sum \text{Res}_{-1}\left\{ \pi \cot(\pi \sigma) f(\sigma) \right\}_{\sigma = \text{pole of } f(\sigma)} = 0 \end{aligned}}$$

Proof:

$$\oint_{\square_{N+\frac{1}{2}}} \pi \cot(\pi \zeta) f(\zeta) d\zeta = \sum \text{Res}_{-1}\left\{ \pi \cot(\pi z) f(z) \right\}_{z = \text{pole of } \cot \pi z \text{ in } \square_{N+\frac{1}{2}}}$$

$$+ \sum \text{Res}_{-1}\left\{ \pi \cot(\pi \sigma) f(\sigma) \right\}_{\sigma = \text{pole of } f(\sigma) \text{ in } \square_{N+\frac{1}{2}}}$$

$$\left| \oint_{\square_{N+\frac{1}{2}}} \pi \cot(\pi \zeta) f(\zeta) d\zeta \right| \leq \pi \underbrace{\left| \cot \pi \zeta \right|}_{\leq A} \frac{M}{N^k} \underbrace{(\text{length} \square_{N+\frac{1}{2}})}_{8\left(N + \frac{1}{2}\right)} \xrightarrow[N \to \infty]{} 0$$

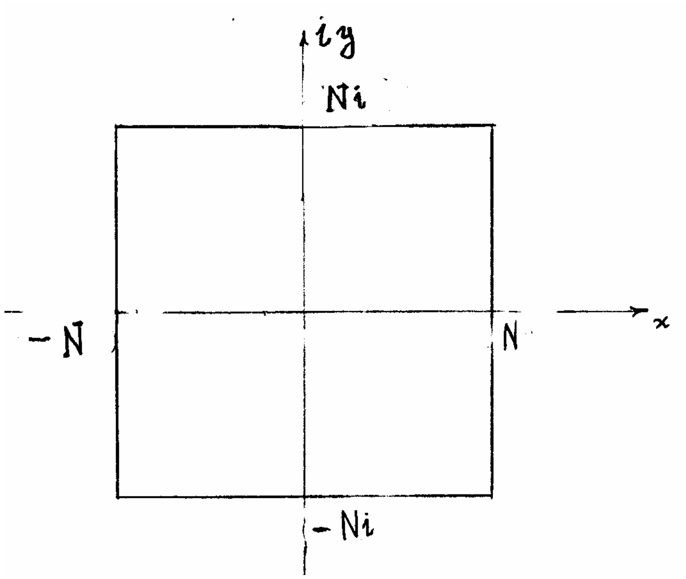
$$\text{Res}\left\{ \pi \cot(\pi z) f(z) \right\}_{z=n} = f(n). \square$$

$\mathbf{6_{tan}}$.

$$\boxed{\pi\tan(\pi z)f(z)}$$

$$\boxed{\begin{aligned}&..+f(-\tfrac{5}{2})+f(-\tfrac{3}{2})+f(-\tfrac{1}{2})\ +f(\tfrac{1}{2})+f(\tfrac{3}{2})+f(\tfrac{5}{2})+..=\\ &=\sum \text{Res}_{-1}\left\{\pi\tan(\pi\sigma)f(\sigma)\right\}_{\sigma=pole\ of\ f(\sigma)}\end{aligned}}$$

6_{tan}.1 $\left|\tan\pi z\right|\le A$ **on** $\square_N$ **for any** N



$\underline{y=-N,\text{ and } -N\le x\le N}\Big| \Rightarrow$

$$\left|\tan\pi z\right|=\left|\frac{e^{i\pi z}-e^{-i\pi z}}{e^{i\pi z}+e^{-i\pi z}}\right|$$

$$=\left|\frac{1-e^{-2i\pi z}}{1+e^{-2i\pi z}}\right|$$

$$=\left|\frac{1-e^{-2i\pi x-2\pi N}}{1+e^{-2i\pi x-2\pi N}}\right|$$

$$< \frac{1+e^{-2\pi N}}{1-e^{-2\pi N}}$$

$$< \frac{1+e^{-\pi}}{1-e^{-\pi}}.\square$$

$\underline{y = N, \text{ and } -N \le x \le N}\Big| \Rightarrow$

$$\left|\tan \pi z\right| = \left|\frac{e^{i\pi z} - e^{-i\pi z}}{e^{i\pi z} + e^{-i\pi z}}\right|$$

$$= \left|\frac{e^{2i\pi z} - 1}{e^{2i\pi z} + 1}\right|$$

$$= \left|\frac{e^{-2i\pi x+2\pi N} - 1}{e^{-2i\pi x+2\pi N} + 1}\right|$$

$$< \frac{e^{2\pi N} + 1}{e^{2\pi N} - 1}\frac{e^{-2\pi N}}{e^{-2\pi N}}$$

$$< \frac{1+\frac{1}{e^{2\pi N}}}{1-\frac{1}{e^{2\pi N}}}$$

$$< \frac{1+\frac{1}{e^{\pi}}}{1-\frac{1}{e^{\pi}}}$$

$$< \frac{e^{\pi}+1}{e^{\pi}-1}$$

$$< 1 + 2\frac{1}{e^{\pi}-1}.\square$$

$\underline{x = N, \text{ and } -N \le y \le N}\Big| \Rightarrow$

For $y = 0$, $\left|\tan \pi z\right| = \left|\tan \pi N\right|$

$$= \tan 0 = 0.$$

For $y \neq 0$, $\left|\tan \pi z\right| = \left|\tan\left(\pi N + i\pi y\right)\right|$

$$= \left|\tan\left(i\pi y\right)\right|$$

$$= \left|\frac{\sin i\pi y}{\cos i\pi y}\right|$$

$$= \left|\frac{\dfrac{e^{i\pi i y} - e^{-i\pi i y}}{2i}}{\dfrac{e^{i\pi i y} + e^{-i\pi i y}}{2}}\right|$$

$$= \left|\frac{e^{-\pi y} - e^{\pi y}}{e^{-\pi y} + e^{\pi y}}\right|$$

$$= \left|\frac{1 - e^{2\pi y}}{1 + e^{2\pi y}}\right| = \left|\frac{e^{2\pi y} - 1}{e^{2\pi y} + 1}\right|$$

For $y > 0$, $\dfrac{e^{2\pi y} - 1}{e^{2\pi y} + 1} < 1.\square$

For $y < 0$, $\dfrac{1 - e^{2\pi y}}{1 + e^{2\pi y}} = \dfrac{1 - \dfrac{1}{e^{2\pi(-y)}}}{1 + \dfrac{1}{e^{2\pi(-y)}}} < 1.\square$

$\underline{x = -N, \text{ and } -N \leq y \leq N}\Big| \Rightarrow$

For $y = 0$, $\left|\tan \pi z\right| = \left|\tan \pi(-N)\right| = \tan 0 = 0$

For $y \neq 0$, $\left|\tan \pi z\right| = \left|\tan\left(-\pi N + i\pi y\right)\right|$

$$= \left|\tan\left(i\pi y\right)\right|$$

$$= \left| \frac{\sin i\pi y}{\cos i\pi y} \right|$$

$$= \left| \frac{\dfrac{e^{i\pi i y} - e^{-i\pi i y}}{2i}}{\dfrac{e^{i\pi i y} + e^{-i\pi i y}}{2}} \right|$$

$$= \left| \frac{e^{-\pi y} - e^{\pi y}}{e^{-\pi y} + e^{\pi y}} \right|$$

$$= \left| \frac{1 - e^{2\pi y}}{1 + e^{2\pi y}} \right| = \left| \frac{e^{2\pi y} - 1}{e^{2\pi y} + 1} \right|$$

For $y > 0$, $\dfrac{e^{2\pi y} - 1}{e^{2\pi y} + 1} < 1.\square$

For $y < 0$, $\dfrac{1 - e^{2\pi y}}{1 + e^{2\pi y}} = \dfrac{1 - \dfrac{1}{e^{2\pi(-y)}}}{1 + \dfrac{1}{e^{2\pi(-y)}}} < 1.\square$

6_{tan}.2 $\pi \tan(\pi\zeta)$ **has poles at** $z = n + \frac{1}{2}$

6_{tan}.3 $\boxed{\mathrm{Res}\left\{\pi \tan(\pi z)\right\}_{z=n+\frac{1}{2}} = -1}$

Proof: $\mathrm{Res}\left\{\pi \tan(\pi z)\right\}_{z=n+\frac{1}{2}} = \left[(z - [n + \tfrac{1}{2}]) \dfrac{\pi \sin(\pi z)}{\cos(\pi z)} \right]_{z=n+\frac{1}{2}}$

$$= \pi \left[\frac{D_z(z-n)}{D_z \cos(\pi z)} \sin(\pi z) \right]_{z=n+\frac{1}{2}}$$

$$= \left[\frac{1}{-\sin(\pi z)} \sin(\pi z) \right]_{z=n+\frac{1}{2}} = -1$$

$\mathbf{6_{tan}.4}$

$$\boxed{\text{Res}\left\{\pi\tan(\pi z)f(z)\right\}_{z=n+\frac{1}{2}} = -f(n+\tfrac{1}{2})}$$

$\mathbf{6_{tan}.5}$ $\left|f(z)\right|_{\square_N} \le \dfrac{M}{z^k} \Rightarrow$

$$\boxed{\begin{aligned}&..+f(-\tfrac{5}{2})+f(-\tfrac{3}{2})+f(-\tfrac{1}{2})\ +f(\tfrac{1}{2})+f(\tfrac{3}{2})+f(\tfrac{5}{2})+..=\\ &=\sum \text{Res}_{-1}\left\{\pi\tan(\pi\sigma)f(\sigma)\right\}_{\sigma=\text{pole of } f(\sigma)}\end{aligned}}$$

Proof:

$$\oint_{\square_N} \pi\tan(\pi\zeta)f(\zeta)d\zeta = \sum \text{Res}_{-1}\left\{\pi\tan(\pi\zeta)f(z)\right\}_{z=\text{pole of } \pi\tan(\pi\zeta) \text{ in } \square_N}$$

$$+\sum \text{Res}_{-1}\left\{\pi\tan(\pi\zeta)f(\sigma)\right\}_{\sigma=\text{pole of } f(\sigma) \text{ in } \square_N}$$

$$\left|\oint_{\square_N} \pi\tan(\pi\zeta)f(\zeta)d\zeta\right| \le \underbrace{\left|\pi\tan(\pi\zeta)\right|}_{\le A}\frac{M}{N^k}\underbrace{(\text{length}\square_N)}_{8N} \xrightarrow[N\to\infty]{} 0$$

$\pi\tan(\pi\zeta)$ **has poles at** $z = n+\frac{1}{2}$

$$\text{Res}_{-1}\left\{\pi\tan(\pi z)f(z)\right\}_{z=n+\frac{1}{2}} = -f(n+\tfrac{1}{2})$$

$$\Rightarrow\ ..+f(-\tfrac{5}{2})+f(-\tfrac{3}{2})+f(-\tfrac{1}{2})\ +f(\tfrac{1}{2})+f(\tfrac{3}{2})+f(\tfrac{5}{2})+...=$$

$$=\sum \text{Res}_{-1}\left\{\pi\tan(\pi\sigma)f(\sigma)\right\}_{\sigma=pole\ \ of\ \ f(\sigma)}$$

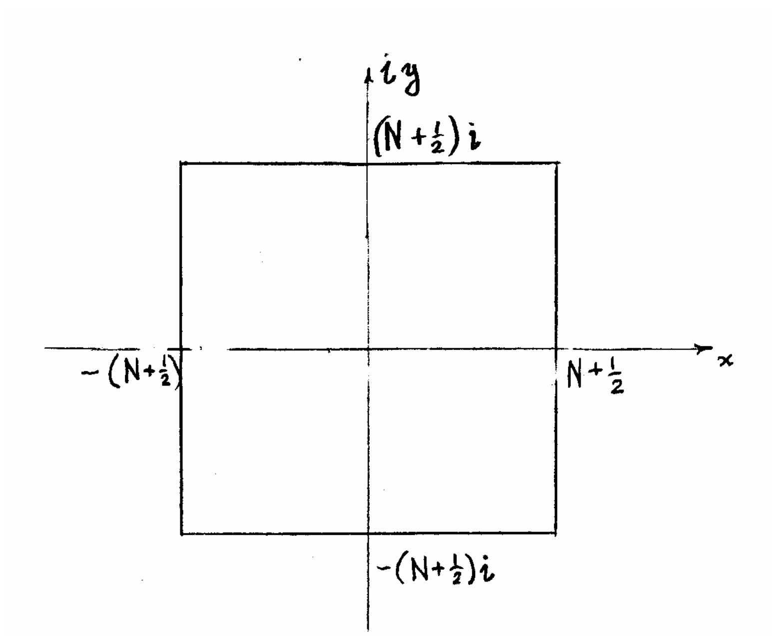
$\mathbf{6_{sin}}$.

$$\boxed{\frac{\pi}{\sin \pi z} f(z)}$$

$$\boxed{\operatorname{Res}\left\{\pi \frac{1}{\sin \pi z} f(z)\right\}_{z=n} = (-1)^n f(n)}$$

$$\boxed{\begin{array}{l} ... - f(-3) + f(-2) - f(-1) + f(0) - f(1) + f(2) - f(3) + ... \\ \qquad + \sum \operatorname{Res}_{-1}\left\{\frac{\pi}{\sin(\pi\sigma)} f(\sigma)\right\}_{\sigma=\text{pole of } f(\sigma)} = 0 \end{array}}$$

6_{sin}.1 $\left|\frac{1}{\sin \pi z}\right| \leq A$ **on** $\square_{N+\frac{1}{2}}$ **for any** N



$$\underline{y = -N - \frac{1}{2}, \text{ and } -N - \frac{1}{2} \leq x \leq N + \frac{1}{2}} \Rightarrow$$

$$\left|\frac{1}{\sin \pi z}\right| = \frac{2}{\left|e^{i\pi z} - e^{-i\pi z}\right|}$$

$$= \frac{2}{\left|e^{i\pi(x+iy)} - e^{-i\pi(x+iy)}\right|}$$

$$\leq \frac{2}{\left|\left|e^{i\pi(x+iy)}\right| - \left|e^{-i\pi(x+iy)}\right|\right|}$$

$$= \frac{2}{\left|e^{-\pi y} - e^{\pi y}\right|}$$

$$= \frac{2}{e^{\pi N + \frac{\pi}{2}} - e^{-\pi N - \frac{\pi}{2}}} \frac{e^{-\pi N + \frac{\pi}{2}}}{e^{-\pi N + \frac{\pi}{2}}}$$

$$= \frac{2e^{-\pi N + \frac{\pi}{2}}}{e^{\pi} - e^{-2\pi N}}$$

$$= \frac{\dfrac{2e^{\frac{\pi}{2}}}{e^{\pi N}}}{e^{\pi} - \dfrac{1}{e^{2\pi N}}}$$

$$< \frac{2e^{\frac{\pi}{2}}}{e^{\pi} - 1} \equiv A. \square$$

$$\underline{\left. y = N + \frac{1}{2}, \text{ and } -N - \frac{1}{2} \leq x \leq N + \frac{1}{2} \right|} \Rightarrow$$

$$\left|\frac{1}{\sin \pi z}\right| = \frac{2}{\left|e^{i\pi z} - e^{-i\pi z}\right|}$$

$$= \frac{2}{\left|e^{i\pi(x+iy)} - e^{-i\pi(x+iy)}\right|}$$

$$\leq \frac{2}{\left|\left|e^{i\pi(x+iy)}\right| - \left|e^{-i\pi(x+iy)}\right|\right|}$$

$$= \frac{2}{\left|e^{-\pi y} - e^{\pi y}\right|}$$

$$= \frac{2}{e^{\pi N + \frac{\pi}{2}} - e^{-\pi N - \frac{\pi}{2}}} \frac{e^{-\pi N + \frac{\pi}{2}}}{e^{-\pi N + \frac{\pi}{2}}}$$

$$= \frac{2e^{-\pi N + \frac{\pi}{2}}}{e^{\pi} - e^{-2\pi N}}$$

$$= \frac{\frac{2e^{\frac{\pi}{2}}}{e^{\pi N}}}{e^{\pi} - \frac{1}{e^{2\pi N}}}$$

$$< \frac{2e^{\frac{\pi}{2}}}{e^{\pi} - 1} \equiv A. \square$$

$$\underline{x = N + \frac{1}{2}, \text{ and } -N - \frac{1}{2} \leq y \leq N + \frac{1}{2}} \Big| \Rightarrow$$

$$\left|\frac{1}{\sin \pi z}\right| = \frac{1}{\left|\sin(\pi N + \frac{\pi}{2} + i\pi y)\right|}$$

$$= \left|\csc(\pi N + \tfrac{\pi}{2} + i\pi y)\right|$$

$$= \left|\csc(\tfrac{\pi}{2} + i\pi y)\right|$$

$$= \frac{1}{\left|\sin(\frac{\pi}{2} + i\pi y)\right|}$$

$$= \frac{1}{\left|\cos(i\pi y)\right|}$$

$$= \frac{1}{\left|\frac{e^{i\pi(iy)} + e^{-i\pi(iy)}}{2}\right|}$$

$$= \frac{2}{e^{-\pi y} + e^{\pi y}}$$

For $y = 0$**,** $\frac{2}{e^{-\pi y} + e^{\pi y}} = 1$

For $y > 0$**,** $\frac{2}{e^{-\pi y} + e^{\pi y}} \frac{e^{\pi y}}{e^{\pi y}} = 2\frac{e^{\pi y}}{e^{2\pi y} + 1} < 2$

For $y < 0$**,** $\frac{2}{e^{-\pi y} + e^{\pi y}} \frac{e^{-\pi y}}{e^{-\pi y}} = 2\frac{e^{\pi(-y)}}{e^{2\pi(-y)} + 1} < 2\,.\square$

$\underline{x = -N - \frac{1}{2}, \text{ and } -N - \frac{1}{2} \le y \le N + \frac{1}{2}}\Big| \Rightarrow$

$$\left|\frac{1}{\sin \pi z}\right| = \frac{1}{\left|\sin(-\pi N - \frac{\pi}{2} + i\pi y)\right|}$$

$$= \left|\csc(-\pi N - \tfrac{\pi}{2} + i\pi y)\right|$$

$$= \left|\csc(-\tfrac{\pi}{2} + i\pi y)\right|$$

$$= \frac{1}{\left|\sin(\frac{\pi}{2} + i\pi y)\right|}$$

$$= \frac{1}{\left|\cos(i\pi y)\right|}$$

$$= \frac{1}{\left|\frac{e^{i\pi(iy)} + e^{-i\pi(iy)}}{2}\right|}$$

$$= \frac{2}{e^{-\pi y} + e^{\pi y}}$$

For $y = 0$**,** $\quad \frac{2}{e^{-\pi y} + e^{\pi y}} = 1$

For $y > 0$**,** $\quad \frac{2}{e^{-\pi y} + e^{\pi y}} \frac{e^{\pi y}}{e^{\pi y}} = 2 \frac{e^{\pi y}}{e^{2\pi y} + 1} < 2$

For $y < 0$**,** $\quad \frac{2}{e^{-\pi y} + e^{\pi y}} \frac{e^{-\pi y}}{e^{-\pi y}} = 2 \frac{e^{\pi(-y)}}{e^{2\pi(-y)} + 1} < 2\,.\square$

$6_{\sin}$.2 $\quad \frac{1}{\sin \pi z}$ **has poles of order 1 at** $z = n = ... -2, -1, 0, 1, 2, ...$

$6_{\sin}$.3

$$\boxed{\text{Res}\left\{\pi \frac{1}{\sin \pi z}\right\}_{z=n} = (-1)^n}$$

Proof: $\text{Res}\left\{\pi \frac{1}{\sin \pi z}\right\}_{z=n} = \left[(z-n)\pi \frac{1}{\sin \pi z}\right]_{z=n}$

$$= \left[\frac{D_z(\pi z - \pi n)}{D_z \sin(\pi z)}\right]_{z=n}$$

$$= \left[\frac{\pi}{\pi \cos(\pi z)}\right]_{z=n} = (-1)^n.\square$$

$6_{\sin}$.4

$$\boxed{\text{Res}\left\{\pi \frac{1}{\sin \pi z} f(z)\right\}_{z=n} = (-1)^n f(n)}$$

$6_{\sin}$.5 $\left| f(z) \right|_{\square_{N+\frac{1}{2}}} \leq \frac{M}{z^k} \Rightarrow$

$$\boxed{\begin{aligned} &...-f(-3)+f(-2)-f(-1)+f(0)-f(1)+f(2)-f(3)+... \\ &\qquad + \sum \operatorname{Res}_{-1}\left\{\frac{\pi}{\sin(\pi\sigma)} f(\sigma)\right\}_{\sigma=\text{pole of } f(\sigma)} = 0 \end{aligned}}$$

Proof:

$$\oint_{\square_{N+\frac{1}{2}}} \frac{\pi}{\sin(\pi\zeta)} f(\zeta) d\zeta = \sum \operatorname{Res}_{-1}\left\{\frac{\pi}{\sin(\pi z)} f(z)\right\}_{z=\text{pole of } \frac{\pi}{\sin(\pi\zeta)} \text{ in } \square_{N+\frac{1}{2}}}$$

$$+\sum \operatorname{Res}_{-1}\left\{\frac{\pi}{\sin(\pi\sigma)} f(\sigma)\right\}_{\sigma=\text{pole of } f(\sigma) \text{ in } \square_{N+\frac{1}{2}}}$$

$$\left| \oint_{\square_{N+\frac{1}{2}}} \frac{\pi}{\sin(\pi\zeta)} f(\zeta) d\zeta \right| \leq \pi \underbrace{\left| \frac{1}{\sin(\pi\zeta)} \right|}_{\leq A} \frac{M}{N^k} \underbrace{(\text{length}\square_{N+\frac{1}{2}})}_{8\left(N+\frac{1}{2}\right)} \xrightarrow[N\to\infty]{} 0$$

$\frac{1}{\sin \pi z}$ **has poles at** $z = n$

$$\operatorname{Res}_{-1}\left\{\frac{\pi}{\sin(\pi z)} f(z)\right\}_{z=\text{pole of } \frac{\pi}{\sin(\pi z)} \text{ in } \square_{N+\frac{1}{2}}} = (-1)^n f(n).\square$$

$\Rightarrow$

$$...-f(-3)+f(-2)-f(-1)+f(0)-f(1)+f(2)-f(3)+...$$

$$+\sum \operatorname{Res}_{-1}\left\{\frac{\pi}{\sin(\pi\sigma)} f(\sigma)\right\}_{\sigma=pole\ \ of\ \ f(\sigma)} = 0$$

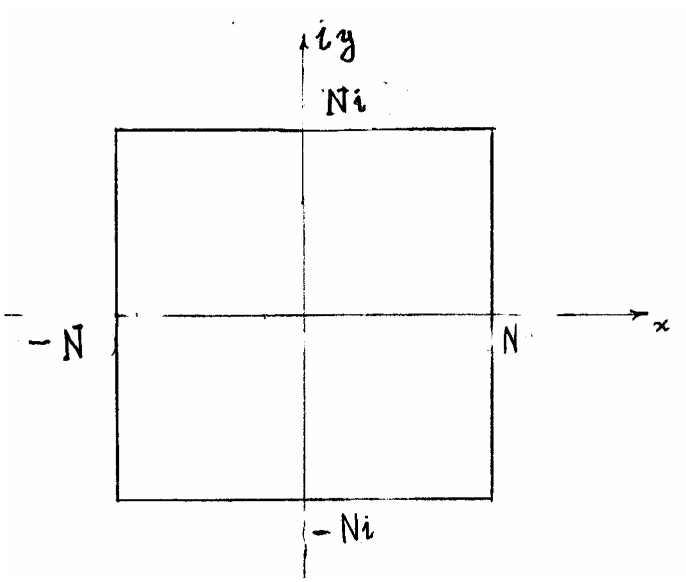
$6_{\cos}$.

$$\boxed{\frac{\pi}{\cos \pi z} f(z)}$$

$$\boxed{\text{Res}\left\{\frac{\pi}{\cos(\pi z)} f(z)\right\}_{z=n+\frac{1}{2}} = -(-1)^n f(n+\tfrac{1}{2})}$$

$$\boxed{\begin{aligned}&... - f(-\tfrac{5}{2}) + f(-\tfrac{3}{2}) - f(-\tfrac{1}{2}) + f(\tfrac{1}{2}) - f(\tfrac{3}{2}) + f(\tfrac{5}{2}) - ... = \\ &= \sum \text{Res}_{-1}\left\{\frac{\pi}{\cos(\pi\sigma)} f(\sigma)\right\}_{\sigma=\text{pole of } f(\sigma)}\end{aligned}}$$

$6_{\cos}$.1 $\left|\frac{1}{\cos \pi z}\right| \le A$ **on** $\square_N$ **for any** N



$\underline{y = -N, \text{ and } -N \le x \le N} \Rightarrow$

$$\frac{1}{|\cos \pi z|} = \frac{2}{\left|e^{i\pi z} + e^{-i\pi z}\right|}$$

$$= \frac{2}{\left|e^{i\pi(x+iy)} + e^{-i\pi(x+iy)}\right|}$$

$$\leq \frac{2}{\left|\left|e^{i\pi(x+iy)}\right| - \left|e^{-i\pi(x+iy)}\right|\right|}$$

$$= \frac{2}{\left|e^{-\pi y} - e^{\pi y}\right|}$$

$$= \frac{2}{\left|e^{-\pi(-N)} - e^{\pi(-N)}\right|}$$

$$= \frac{2}{e^{\pi N} - e^{-\pi N}}$$

$$\leq 2\frac{1}{e^{\pi} - 1}. \square$$

$\underline{y = N, \text{ and } -N \leq x \leq N}| \Rightarrow$

$$\left|\frac{1}{\cos \pi z}\right| = \frac{2}{\left|e^{-i\pi z} + e^{i\pi z}\right|}$$

$$= \frac{2}{\left|e^{i\pi(x+iy)} + e^{-i\pi(x+iy)}\right|}$$

$$\leq \frac{2}{\left|\left|e^{i\pi(x+iy)}\right| - \left|e^{-i\pi(x+iy)}\right|\right|}$$

$$= \frac{2}{\left|e^{-\pi y} - e^{\pi y}\right|}$$

$$= \frac{2}{\left|e^{-\pi N} - e^{\pi N}\right|}$$

$$\leq 2\frac{1}{e^{\pi}-1}.\square$$

$\underline{x = N, \text{ and } -N \leq y \leq N}| \Rightarrow$

$$\left|\frac{1}{\cos \pi z}\right| = \frac{1}{\left|\cos(\pi N + i\pi y)\right|}$$

$$= \left|\sec(\pi N + i\pi y)\right|$$

$$= \left|\sec(i\pi y)\right|$$

$$= \frac{1}{\left|\cos(i\pi y)\right|}$$

$$= \frac{2}{\left|e^{i(i\pi y)} + e^{-i(i\pi y)}\right|}$$

$$= \frac{2}{\left|e^{-\pi y} + e^{\pi y}\right|}$$

$$= \frac{2}{e^{-\pi y} + e^{\pi y}}$$

For $y = 0$, $\quad \frac{2}{e^{-\pi y} + e^{\pi y}} = 1$

For $y > 0$, $\quad \frac{2}{e^{-\pi y} + e^{\pi y}}\frac{e^{\pi y}}{e^{\pi y}} = 2\frac{e^{\pi y}}{e^{2\pi y}+1} < 2$

For $y < 0$, $\quad \frac{2}{e^{-\pi y} + e^{\pi y}}\frac{e^{-\pi y}}{e^{-\pi y}} = 2\frac{e^{\pi(-y)}}{e^{2\pi(-y)}+1} < 2.\square$

$\underline{x = -N, \text{ and } -N \leq y \leq N}| \Rightarrow$

$$\left|\frac{1}{\cos \pi z}\right| = \frac{1}{\left|\cos(-\pi N + i\pi y)\right|}$$

$$= \left|\sec(-\pi N + i\pi y)\right|$$

$$= \left|\sec(i\pi y)\right|$$

$$= \frac{1}{\left|\cos(i\pi y)\right|}$$

$$= \frac{2}{\left|e^{i(i\pi y)} + e^{-i(i\pi y)}\right|}$$

$$= \frac{2}{\left|e^{-\pi y} + e^{\pi y}\right|}$$

$$= \frac{2}{e^{-\pi y} + e^{\pi y}}$$

For $y = 0$, $\quad \dfrac{2}{e^{-\pi y} + e^{\pi y}} = 1$

For $y > 0$, $\quad \dfrac{2}{e^{-\pi y} + e^{\pi y}} \dfrac{e^{\pi y}}{e^{\pi y}} = 2\dfrac{e^{\pi y}}{e^{2\pi y} + 1} < 2$

For $y < 0$, $\quad \dfrac{2}{e^{-\pi y} + e^{\pi y}} \dfrac{e^{-\pi y}}{e^{-\pi y}} = 2\dfrac{e^{\pi(-y)}}{e^{2\pi(-y)} + 1} < 2 . \square$

$6_{\cos}$.2 $\quad \dfrac{1}{\cos \pi z}$ **has poles at** $z = n + \frac{1}{2} = ..., -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ...$

$6_{\cos}$.3

$$\boxed{\operatorname{Res}\left\{\frac{\pi}{\cos(\pi z)}\right\}_{z=n+\frac{1}{2}} = -(-1)^n}$$

Proof: $\operatorname{Res}\left\{\dfrac{\pi}{\cos(\pi z)}\right\}_{z=n+\frac{1}{2}} = \left[(z - [n + \tfrac{1}{2}])\dfrac{\pi}{\cos(\pi z)}\right]_{z=n+\frac{1}{2}}$

$$= \pi \left[\frac{D_z(z - n - \frac{1}{2})}{D_z \cos(\pi z)} \right]_{z=n+\frac{1}{2}}$$

$$= \left[\frac{1}{-\sin(\pi z)} \right]_{z=n+\frac{1}{2}}$$

$$= \frac{1}{-\sin(\pi n + \frac{\pi}{2})}$$

$$= -(-1)^n . \square$$

$6_{\cos}$.4

$$\boxed{\text{Res}\left\{ \frac{\pi}{\cos(\pi z)} f(z) \right\}_{z=n+\frac{1}{2}} = -(-1)^n f(n + \tfrac{1}{2})}$$

$6_{\cos}$.5 $\left| f(z) \right|_{\square_N} \leq \frac{M}{z^k} \Rightarrow$

$$\boxed{\begin{aligned} &... - f(-\tfrac{5}{2}) + f(-\tfrac{3}{2}) - f(-\tfrac{1}{2}) + f(\tfrac{1}{2}) - f(\tfrac{3}{2}) + f(\tfrac{5}{2}) - ... = \\ &= \sum \text{Res}_{-1} \left\{ \frac{\pi}{\cos(\pi\sigma)} f(\sigma) \right\}_{\sigma=\text{pole of } f(\sigma)} \end{aligned}}$$

Proof:

$$\oint_{\square_N} \frac{\pi}{\cos(\pi\zeta)} f(\zeta) d\zeta = \sum \text{Res}_{-1} \left\{ \frac{\pi}{\cos(\pi z)} f(z) \right\}_{z=\text{pole of } \frac{\pi}{\cos(\pi z)} \text{ in } \square_N}$$

$$+ \sum \text{Res}_{-1} \left\{ \frac{\pi}{\cos(\pi\sigma)} f(\sigma) \right\}_{\sigma=\text{pole of } f(\sigma) \text{ in } \square_N}$$

$$\left|\oint_{\square_N} \frac{\pi}{\cos(\pi\zeta)} f(\zeta) d\zeta\right| \le \underbrace{\left|\frac{\pi}{\cos(\pi\zeta)}\right|}_{\le A} \frac{M}{N^k} \underbrace{(\text{length}\square_N)}_{8(N)} \xrightarrow[N\to\infty]{} 0$$

$\dfrac{1}{\cos \pi z}$ **has poles at** $z = n + \frac{1}{2}$

$$\text{Res}_{-1}\left\{\frac{\pi}{\cos(\pi z)} f(z)\right\}_{z=\text{pole of } \frac{\pi}{\cos(\pi z)} \text{ in } \square_N} = \text{Res}_{-1}\left\{\frac{\pi}{\cos(\pi z)} f(z)\right\}_{z=n+\frac{1}{2}}$$

$$= -(-1)^n f(n + \tfrac{1}{2})$$

$$\Rightarrow \quad \sum (-1)^n f(n + \tfrac{1}{2}) = \sum \text{Res}_{-1}\left\{\frac{\pi}{\cos(\pi\sigma)} f(\sigma)\right\}_{\sigma=\text{pole of } f(z)}$$

$$\Rightarrow \quad ... + f(-\tfrac{7}{2}) - f(-\tfrac{5}{2}) + f(-\tfrac{3}{2}) - f(-\tfrac{1}{2}) + f(\tfrac{1}{2}) - f(\tfrac{3}{2}) + f(\tfrac{5}{2}) + ... =$$

$$= \sum \text{Res}_{-1}\left\{\frac{\pi}{\cos(\pi\sigma)} f(\sigma)\right\}_{\sigma=pole\ \ of\ \ f(z)} . \square$$

7_{cot}.

7_{cot}.1

$$\pi^3 = m^3 \frac{\sin^2(\frac{l}{m}\pi)}{\cot(\frac{l}{m}\pi)} \left\{ \frac{1}{l^3} - \frac{1}{(m-l)^3} + \frac{1}{(m+l)^3} - \frac{1}{(2m-l)^3} + \frac{1}{(2m+l)^3} - \frac{1}{(3m-l)^3} + \dots \right\}$$

$l < \frac{1}{2}m$, l and m have no common factor

$$R_{l/m} = \frac{1}{l^3} - \frac{1}{(m-l)^3} + \frac{1}{(m+l)^3} - \frac{1}{(2m-l)^3} + \frac{1}{(2m+l)^3} - \frac{1}{(3m-l)^3} + \dots = \frac{\pi^3}{m^3} \frac{\cot(\frac{l}{m}\pi)}{\sin^2(\frac{l}{m}\pi)}$$

$$= \frac{\pi^3}{m^3} \frac{\cos(\frac{l}{m}\pi)}{\sin^3(\frac{l}{m}\pi)}$$

Proof: $\pi\cot(\pi z)\frac{1}{(z-\frac{l}{m})^3}$ has poles of order 1 at $z = n$,

and a pole of order 3 at $z = \frac{l}{m}$

$$\oint_{\square_{N+\frac{1}{2}}} \pi\cot(\pi\zeta)\frac{1}{(\zeta-\frac{l}{m})^3}d\zeta = \sum \text{Res}_{-1}\left\{\pi\cot(\pi z)\frac{1}{(z-\frac{l}{m})^3}\right\}_{z=n}$$

$$+ \text{Res}_{-1}\left\{\pi\cot(\pi z)\frac{1}{(z-\frac{l}{m})^3}\right\}_{z=\frac{l}{m}}$$

$$\left| \oint_{\square_{N+\frac{1}{2}}} \pi\cot(\pi\zeta)\frac{1}{(\zeta-\frac{l}{m})^3}d\zeta \right| \le \underbrace{\lfloor \pi\cot\pi\zeta \rfloor}_{\le A} \underbrace{\oint_{\square_{N+\frac{1}{2}}} \frac{1}{(\zeta-\frac{l}{m})^3}d\zeta}_{\le\left[\frac{1}{\zeta^2}\right]_{\square_{N+\frac{1}{2}}}=0} = 0.\square$$

$$\operatorname{Res}_{-1}\left\{\pi\cot(\pi z)\frac{1}{(z-\frac{l}{m})^3}\right\}_{z=n} = \left[(z-n)\pi\frac{\cos\pi z}{\sin\pi z}\frac{1}{(z-\frac{l}{m})^3}\right]_{z=n}$$

$$= \left[\frac{\pi D_z(z-n)}{D_z\sin(\pi z)}\right]_{z=n}\left(\cos(\pi n)\frac{1}{(n-\frac{l}{m})^3}\right)$$

$$= \left[\frac{\pi}{\pi\cos(\pi z)}\right]_{z=n}\left(\cos(\pi n)\frac{1}{(n-\frac{l}{m})^3}\right)$$

$$= \frac{1}{(n-\frac{l}{m})^3}.\square$$

To find $\operatorname{Res}_{-1}\left\{\pi\cot(\pi z)\frac{1}{(z-\frac{l}{m})^3}\right\}_{z=\frac{l}{m}}$ **, divide the series**

$$\pi\frac{\cos(\pi z)}{\sin(\pi z)}\frac{1}{(z-\frac{l}{m})^3} = \pi\frac{\cos(\pi[z-\frac{l}{m}]+\frac{l}{m}\pi)}{\sin(\pi[z-\frac{l}{m}]+\frac{l}{m}\pi)}\frac{1}{(z-\frac{l}{m})^3}$$

$$= \pi\frac{\cos(\pi u+\frac{l}{m}\pi)}{\sin(\pi u+\frac{l}{m}\pi)}\frac{1}{u^3}$$

$$= \pi\frac{\cos(\pi u)\cos(\frac{l}{m}\pi)-\sin(\pi u)\sin(\frac{l}{m}\pi)}{\sin(\pi u)\cos(\frac{l}{m}\pi)+\cos(\pi u)\sin(\frac{l}{m}\pi)}\frac{1}{u^3}$$

$$\approx \pi \frac{(1-\frac{1}{2}\pi^2u^2)\cos(\frac{l}{m}\pi) - (\pi u)\sin(\frac{l}{m}\pi)}{(\pi u)\cos(\frac{l}{m}\pi) + (1-\frac{1}{2}\pi^2u^2)\sin(\frac{l}{m}\pi)} \frac{1}{u^3}$$

$$= \pi \frac{\cos(\frac{l}{m}\pi)}{\sin(\frac{l}{m}\pi)} \left(\frac{(1-\frac{1}{2}\pi^2u^2) - \pi u \tan(\frac{l}{m}\pi)}{\pi u \cot(\frac{l}{m}\pi) + (1-\frac{1}{2}\pi^2u^2)} \right) \frac{1}{u^3}$$

$$= \pi \cot(\tfrac{l}{m}\pi) \left(\frac{1 - \pi u \tan(\frac{l}{m}\pi) - \frac{1}{2}\pi^2u^2}{1 + \pi u \cot(\frac{l}{m}\pi) - \frac{1}{2}\pi^2u^2} \right) \frac{1}{u^3}$$

$$\begin{array}{l}
1 - \pi u\{\tan(\frac{l}{m}\pi) + \cot(\frac{l}{m}\pi)\} + \boxed{\pi^2u^2\{\tan(\frac{l}{m}\pi) + \cot(\frac{l}{m}\pi)\}\cot(\frac{l}{m}\pi)} - \pi^3u^3\{\tan(\frac{l}{m}\pi) + \cot(\frac{l}{m}\pi)\}\left\{\cot^2(\frac{l}{m}\pi) + \frac{1}{2}\right\} \\
\hline
1 - \pi u \tan(\frac{l}{m}\pi) - \frac{1}{2}\pi^2u^2 \qquad \Big| \; 1 + \pi u \cot(\frac{l}{m}\pi) - \frac{1}{2}\pi^2u^2 \\
1 + \pi u \cot(\frac{l}{m}\pi) - \frac{1}{2}\pi^2u^2 \\
\hline
\quad - \pi u\{\tan(\frac{l}{m}\pi) + \cot(\frac{l}{m}\pi)\} \\
\quad - \pi u\{\tan(\frac{l}{m}\pi) + \cot(\frac{l}{m}\pi)\} - \pi^2u^2\{\tan(\frac{l}{m}\pi) + \cot(\frac{l}{m}\pi)\}\cot(\frac{l}{m}\pi) + \frac{1}{2}\pi^3u^3\{\tan(\frac{l}{m}\pi) + \cot(\frac{l}{m}\pi)\} \\
\hline
\qquad \pi^2u^2\{\tan(\frac{l}{m}\pi) + \cot(\frac{l}{m}\pi)\}\cot(\frac{l}{m}\pi) - \frac{1}{2}\pi^3u^3\{\tan(\frac{l}{m}\pi) + \cot(\frac{l}{m}\pi)\} \\
\qquad \pi^2u^2\{\tan(\frac{l}{m}\pi) + \cot(\frac{l}{m}\pi)\}\cot(\frac{l}{m}\pi) + \pi^3u^3\{\tan(\frac{l}{m}\pi) + \cot(\frac{l}{m}\pi)\}\cot^2(\frac{l}{m}\pi) - \frac{1}{2}\pi^4u^4\{\tan(\frac{l}{m}\pi) + \cot(\frac{l}{m}\pi)\}\cot(\frac{l}{m}\pi) \\
\hline
\qquad\qquad - \pi^3u^3\{\tan(\frac{l}{m}\pi) + \cot(\frac{l}{m}\pi)\}\left\{\cot^2(\frac{l}{m}\pi) + \frac{1}{2}\right\} + \frac{1}{2}\pi^4u^4\{\tan(\frac{l}{m}\pi) + \cot(\frac{l}{m}\pi)\}\cot(\frac{l}{m}\pi)
\end{array}$$

$$\approx \pi \cot(\tfrac{l}{m}\pi)\left[1 - \pi u\{\tan(\tfrac{l}{m}\pi) + \cot(\tfrac{l}{m}\pi)\} + \boxed{\pi^2u^2\{\tan(\tfrac{l}{m}\pi) + \cot(\tfrac{l}{m}\pi)\}\cot(\tfrac{l}{m}\pi)}\right]\frac{1}{u^3}$$

$$= \pi \cot(\tfrac{l}{m}\pi)\frac{1}{u^3}$$

$$-\pi^2 \cot(\tfrac{l}{m}\pi)\{\tan(\tfrac{l}{m}\pi) + \cot(\tfrac{l}{m}\pi)\}\frac{1}{u^2}$$

$$+\pi^3 \cot(\tfrac{l}{m}\pi)\Big(\{\tan(\tfrac{l}{m}\pi) + \cot(\tfrac{l}{m}\pi)\}\cot(\tfrac{l}{m}\pi)\Big)\frac{1}{u}$$

$$\text{Res}_{-1}\left\{\pi\cot(\pi z)\frac{1}{(z-\frac{l}{m})^3}\right\}_{z=\frac{l}{m}} = \pi^3\cot^2(\tfrac{l}{m}\pi)\{\tan(\tfrac{l}{m}\pi) + \cot(\tfrac{l}{m}\pi)\}$$

$$= \pi^3\cot(\tfrac{l}{m}\pi)\{1 + \cot^2(\tfrac{l}{m}\pi)\}$$

$$= \pi^3\frac{\cot(\frac{l}{m}\pi)}{\sin^2(\frac{l}{m}\pi)}$$

$$= \pi^3 \frac{\cos(\frac{l}{m}\pi)}{\sin^3(\frac{l}{m}\pi)}$$

Therefore,

$$... + \frac{1}{(-3-\frac{l}{m})^3} + \frac{1}{(-2-\frac{l}{m})^3} + \frac{1}{(-1-\frac{l}{m})^3} + \frac{1}{(0-\frac{l}{m})^3} +$$

$$+\frac{1}{(1-\frac{l}{m})^3} + \frac{1}{(2-\frac{l}{m})^3} + \frac{1}{(3-\frac{l}{m})^3} + ... + \pi^3 \frac{\cot(\frac{l}{m}\pi)}{\sin^2(\frac{l}{m}\pi)} = 0$$

$$\pi^3 = m^3 \frac{\sin^2(\frac{l}{m}\pi)}{\cot(\frac{l}{m}\pi)} \left\{ \frac{1}{l^3} - \frac{1}{(m-l)^3} + \frac{1}{(m+l)^3} - \frac{1}{(2m-l)^3} + \frac{1}{(2m+l)^3} - \frac{1}{(3m-l)^3} + ... \right\}$$

$$R_{l/m} = \frac{1}{l^3} - \frac{1}{(m-l)^3} + \frac{1}{(m+l)^3} - \frac{1}{(2m-l)^3} + \frac{1}{(2m+l)^3} - \frac{1}{(3m-l)^3} + ... = \frac{\pi^3}{m^3}\frac{\cot(\frac{l}{m}\pi)}{\sin^2(\frac{l}{m}\pi)}$$

$7_{cot}.2$ Euler Almost Obtained $7._{cot}$

Euler had in his section #174[5]

[5] Leonardi Euleri, "Introductio in Analysin Infinitorum", Section #174

$$\frac{\pi^3}{8m_1^3}\frac{k^2+1}{k^3} = \frac{1}{l^3} - \frac{1}{(2m_1 - l_1)^3} + \frac{1}{(2m_1 + l_1)^3} - \frac{1}{(4m_1 - l_1)^3} + \frac{1}{(4m_1 + l_1)^3} - \frac{1}{(6m_1 - l_1)^3} + \dots$$

Euler's Notation	Our Notation
$2m_1$	m
l_1	l
k	$\tan(\frac{l_1}{2m_1}\pi) = \tan(\frac{l}{m}\pi)$

Therefore,

$$\frac{\pi^3}{8m_1^3}\frac{k^2+1}{k^3} = \frac{\pi^3}{m^3}\frac{\tan^2\left(\frac{l}{m}\pi\right)+1}{\tan^3\left(\frac{l}{m}\pi\right)}$$

$$= \frac{\pi^3}{m^3}\frac{1}{\tan^3\left(\frac{l}{m}\pi\right)}\frac{1}{\cos^2\left(\frac{l}{m}\pi\right)}$$

$$= \frac{\pi^3}{m^3}\frac{\cos\left(\frac{l}{m}\pi\right)}{\sin^3\left(\frac{l}{m}\pi\right)}$$

Replacing Euler's Notation with ours we obtain $\mathbf{7_{cot}}$.

But the notation

$$\tan(\tfrac{l}{m}\pi) = k$$

insinuates that $\tan(\frac{l}{m}\pi)$ is always an integer.

This amounts to Euler's hypothesizing that π^3 series are parameterized by three integers

$$l_1, \quad m_1, \text{ and } k,$$

In fact, π^3 series are parameterized by the first two, and the crucial role of the tangent function is hidden by a notation that is reserved to integers.

$7_{\sin}$.

$$\boxed{\pi^3 = m^3 \frac{2\sin^3(\frac{l}{m}\pi)}{\cos^2(\frac{l}{m}\pi)+1}\left\{\frac{1}{l^3}+\frac{1}{(m-l)^3} - \frac{1}{(m+l)^3}-\frac{1}{(2m-l)^3} + \frac{1}{(2m+l)^3}+\frac{1}{(3m-l)^3}-...\right\}}$$

$l < \frac{1}{2}m$, l and m have no common factor.

$$S_{l/m} = \frac{1}{l^3}+\frac{1}{(m-l)^3} - \frac{1}{(m+l)^3}-\frac{1}{(2m-l)^3} + \frac{1}{(2m+l)^3}+\frac{1}{(3m-l)^3}-... = \frac{\pi^3}{m^3}\frac{\cos^2(\frac{l}{m}\pi)+1}{2\sin^3(\frac{l}{m}\pi)}$$

$$= \frac{\pi^3}{m^3}\frac{2\cot^2(\frac{l}{m}\pi)+1}{2\sin(\frac{l}{m}\pi)}$$

$$= \frac{\pi^3}{m^3}\frac{1}{2\sin(\frac{l}{m}\pi)}\frac{3+\cos(2\frac{l}{m}\pi)}{1-\cos(2\frac{l}{m}\pi)}$$

Proof: $\pi\frac{1}{\sin(\pi z)}\frac{1}{(z-\frac{l}{m})^3}$ has poles of order 1 at $z = n$,

and a pole of order 3 at $z = \frac{l}{m}$

$$\oint_{\square_{N+\frac{1}{2}}} \pi\frac{1}{\sin(\pi\zeta)}\frac{1}{(\zeta-\frac{l}{m})^3}d\zeta = \sum \text{Res}_{-1}\left\{\pi\frac{1}{\sin(\pi z)}\frac{1}{(z-\frac{l}{m})^3}\right\}_{z=n}$$

$$+\text{Res}_{-1}\left\{\pi\frac{1}{\sin(\pi z)}\frac{1}{(z-\frac{l}{m})^3}\right\}_{z=-\frac{l}{m}}$$

$$\left| \oint_{\square_{N+\frac{1}{2}}} \frac{\pi}{\sin(\pi\zeta)} \frac{1}{(\zeta - \frac{l}{m})^3} d\zeta \right| \leq \underbrace{\left| \frac{\pi}{\sin(\pi\zeta)} \right|}_{\leq A} \underbrace{\oint_{\square_{N+\frac{1}{2}}} \frac{1}{(\zeta - \frac{l}{m})^3} d\zeta}_{\left[\frac{1}{\zeta^2}\right]_{\square_{N+\frac{1}{2}}} = 0} = 0. \square$$

$$\mathrm{Res}_{-1}\left\{\pi \frac{1}{\sin(\pi z)} \frac{1}{(z - \frac{l}{m})^3}\right\}_{z=n} = \left[(z-n)\pi \frac{1}{\sin \pi z} \frac{1}{(z - \frac{l}{m})^3}\right]_{z=n}$$

$$= \left[\frac{\pi D_z(z-n)}{D_z \sin(\pi z)}\right]_{z=n} \frac{1}{(n - \frac{l}{m})^3}$$

$$= \left[\frac{\pi}{\pi \cos(\pi z)}\right]_{z=n} \frac{1}{(n - \frac{l}{m})^3}$$

$$= \frac{(-1)^n}{(n - \frac{l}{m})^3}. \square$$

To find $\mathrm{Res}_{-1}\left\{\pi \frac{1}{\sin(\pi z)} \frac{1}{(z - \frac{l}{m})^3}\right\}_{z=\frac{l}{m}}$ **, divide the series**

$$\pi \frac{1}{\sin(\pi z)} \frac{1}{(z - \frac{l}{m})^3} = \pi \frac{1}{\sin[\pi(z - \frac{l}{m}) + \frac{l}{m}\pi)]} \frac{1}{(z - \frac{l}{m})^3}$$

$$= \pi \frac{1}{\sin(\pi u + \frac{l}{m}\pi)} \frac{1}{u^3}$$

$$= \pi \frac{1}{\sin(\pi u)\cos(\frac{l}{m}\pi) + \cos(\pi u)\sin(\frac{l}{m}\pi)} \frac{1}{u^3}$$

$$\approx \pi \frac{1}{(\pi u)\cos(\frac{l}{m}\pi) + (1 - \frac{1}{2}u^2)\sin(\frac{l}{m}\pi)} \frac{1}{u^3}$$

$$= \pi \frac{1}{\sin(\frac{l}{m}\pi)} \left(\frac{1}{1 + \pi u \cot(\frac{l}{m}\pi) - \frac{1}{2}\pi^2 u^2} \right) \frac{1}{u^3}$$

$$\begin{array}{l}
1 - \pi u \cot(\frac{l}{m}\pi) + \pi^2 u^2 \left\{ \cot^2(\frac{l}{m}\pi) + \frac{1}{2} \right\} - \pi^3 u^3 \left\{ \cot^2(\frac{l}{m}\pi) + 1 \right\} \cot(\frac{l}{m}\pi) \\
\hline
1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Big| \; 1 + \pi u \cot(\frac{l}{m}\pi) - \frac{1}{2}\pi^2 u^2 \\
\underline{1 + \pi u \cot(\frac{l}{m}\pi) - \frac{1}{2}\pi^2 u^2} \\
\quad - \pi u \cot(\frac{l}{m}\pi) + \frac{1}{2}\pi^2 u^2 \\
\quad \underline{- \pi u \cot(\frac{l}{m}\pi) - \pi^2 u^2 \cot^2(\frac{l}{m}\pi) + \frac{1}{2}\pi^3 u^3 \cot(\frac{l}{m}\pi)} \\
\qquad\qquad \pi^2 u^2 \left\{ \cot^2(\frac{l}{m}\pi) + \frac{1}{2} \right\} - \frac{1}{2}\pi^3 u^3 \cot(\frac{l}{m}\pi) \\
\qquad\qquad \underline{\pi^2 u^2 \left\{ \cot^2(\frac{l}{m}\pi) + \frac{1}{2} \right\} + \pi^3 u^3 \left\{ \cot^2(\frac{l}{m}\pi) + \frac{1}{2} \right\} \cot(\frac{l}{m}\pi) - \pi^4 u^4 \left\{ \cot^2(\frac{l}{m}\pi) + \frac{1}{2} \right\}} \\
\qquad\qquad\qquad - \pi^3 u^3 \left\{ \cot^2(\frac{l}{m}\pi) + 1 \right\} \cot(\frac{l}{m}\pi) + \pi^4 u^4 \left\{ \cot^2(\frac{l}{m}\pi) + \frac{1}{2} \right\}
\end{array}$$

$$\approx \pi \frac{1}{\sin(\frac{l}{m}\pi)} \left\{ 1 - \pi u \cot(\frac{l}{m}\pi) + \pi^2 u^2 \left\{ \cot^2(\frac{l}{m}\pi) + \frac{1}{2} \right\} \right\} \frac{1}{u^3}$$

$$= \pi \frac{1}{\sin(\frac{l}{m}\pi)} \frac{1}{u^3} - \pi^2 \frac{\cot(\frac{l}{m}\pi)}{\sin(\frac{l}{m}\pi)} \frac{1}{u^2} + \pi^3 \frac{\cot^2(\frac{l}{m}\pi) + \frac{1}{2}}{\sin(\frac{l}{m}\pi)} \frac{1}{u}$$

$$\text{Res}_{-1} \left\{ \pi \frac{1}{\sin(\pi z)} \frac{1}{(z - \frac{l}{m})^3} \right\}_{z = \frac{l}{m}} = \pi^3 \frac{2\cot^2(\frac{l}{m}\pi) + 1}{2\sin(\frac{l}{m}\pi)}$$

$$= \pi^3 \frac{\cos^2(\frac{l}{m}\pi) + 1}{2\sin^3(\frac{l}{m}\pi)}$$

$$= \pi^3 2 \frac{1}{\sin(\frac{l}{m}\pi)} \frac{3 + \cos(2\frac{l}{m}\pi)}{1 - \cos(2\frac{l}{m}\pi)}$$

Therefore,

$$\ldots + \frac{(-1)^{-3}}{(-3 - \frac{l}{m})^3} + \frac{(-1)^{-2}}{(-2 - \frac{l}{m})^3} + \frac{(-1)^{-1}}{(-1 - \frac{l}{m})^3} + \frac{(-1)^0}{(0 - \frac{l}{m})^3} + \frac{(-1)^1}{(1 - \frac{l}{m})^3} +$$

$$+\frac{(-1)^2}{(2-\frac{l}{m})^3}+\frac{(-1)^3}{(3-\frac{l}{m})^3}+...+\pi^3\frac{\cos(2\frac{l}{m}\pi)+3}{2\sin(\frac{l}{m}\pi)[1-\cos(2\frac{l}{m}\pi)]}=0$$

$$\pi^3 = m^3\frac{2\sin(\frac{l}{m}\pi)[1-\cos(2\frac{l}{m}\pi)]}{\cos(2\frac{l}{m}\pi)+3}\left\{\frac{1}{l^3}+\frac{1}{(m-l)^3}-\frac{1}{(m+l)^3}-\frac{1}{(2m-l)^3}+\frac{1}{(2m+l)^3}+\frac{1}{(3m-l)^3}-...\right\}$$

And the Associated Series is

$$S_{l/m}=\frac{1}{l^3}+\frac{1}{(m-l)^3}-\frac{1}{(m+l)^3}-\frac{1}{(2m-l)^3}+\frac{1}{(2m+l)^3}+\frac{1}{(3m-l)^3}-...=\frac{\pi^3}{m^3}\frac{\cos(2\frac{l}{m}\pi)+3}{2\sin(\frac{l}{m}\pi)[1-\cos(2\frac{l}{m}\pi)]}$$

$\mathbf{7_{cos}}$.

$$\pi^3 = 8m^3 \frac{2\cos^3(\frac{l}{m}\pi)}{\sin^2(\frac{l}{m}\pi)+1}\left\{\frac{1}{(m-2l)^3}+\frac{1}{(m+2l)^3} - \frac{1}{(3m-2l)^3}-\frac{1}{(3m+2l)^2} + \frac{1}{(5m-2l)^2}+\frac{1}{(5m+2l)^2}-...\right\}$$

$l < \frac{1}{2}m$, l and m have no common factor.

$$C_{l/m} = \frac{1}{(m-2l)^2}+\frac{1}{(m+2l)^2} - \frac{1}{(3m-2l)^2}-\frac{1}{(3m+2l)^2} + \frac{1}{(5m-2l)^2}+\frac{1}{(5m+2l)^2}-... = \frac{\pi^3}{8m^3}\frac{\sin^2(\frac{l}{m}\pi)+1}{2\cos^3(\frac{l}{m}\pi)}$$

$$= \frac{\pi^3}{8m^3}\frac{2\tan^2(\frac{l}{m}\pi)+1}{2\cos(\frac{l}{m}\pi)}$$

$$= \frac{\pi^3}{8m^3}\frac{1}{2\cos(\frac{l}{m}\pi)}\frac{3-\cos(2\frac{l}{m}\pi)}{1+\cos(2\frac{l}{m}\pi)}$$

Proof: $\pi\frac{1}{\cos(\pi z)}\frac{1}{(z-\frac{l}{m})^3}$ has poles of order 1 at $z = n+\frac{1}{2}$,

and a pole of order 3 at $z = \frac{l}{m}$

$$\oint_{\square_N}\pi\frac{1}{\cos(\pi\zeta)}\frac{1}{(\zeta-\frac{l}{m})^3}d\zeta = \sum \text{Res}_{-1}\left\{\pi\frac{1}{\cos(\pi z)}\frac{1}{(z-\frac{l}{m})^3}\right\}_{z=n+\frac{1}{2}}$$

$$+\text{Res}_{-1}\left\{\pi\frac{1}{\cos(\pi z)}\frac{1}{(z-\frac{l}{m})^3}\right\}_{z=\frac{l}{m}}$$

$$\left|\oint_{\square_N} \frac{\pi}{\cos(\pi\zeta)} \frac{1}{(\zeta - \frac{l}{m})^3} d\zeta\right| \leq \underbrace{\left|\frac{\pi}{\cos(\pi\zeta)}\right|}_{\leq A} \underbrace{\oint_{\square_N} \frac{1}{(\zeta - \frac{l}{m})^3} d\zeta}_{\leq \left[\frac{1}{|\zeta|^2}\right]_{\square_N} = 0} = 0. \square$$

$$\text{Res}_{-1}\left\{\pi \frac{1}{\cos(\pi z)} \frac{1}{(z - \frac{l}{m})^3}\right\}_{z=n+\frac{1}{2}} = \left[(z - n - \tfrac{1}{2})\pi \frac{1}{\cos(\pi z)} \frac{1}{(z - \frac{l}{m})^3}\right]_{z=n+\frac{1}{2}}$$

$$= \left[\frac{\pi D_z(z - n - \frac{1}{2})}{D_z \cos(\pi z)}\right]_{z=n+\frac{1}{2}} \frac{1}{(n + \frac{1}{2} - \frac{l}{m})^3}$$

$$= \left[\frac{\pi}{-\pi \sin(\pi z)}\right]_{z=n+\frac{1}{2}} \frac{1}{(n + \frac{1}{2} - \frac{l}{m})^3}$$

$$= \frac{1}{-\sin \pi(n + \frac{1}{2})} \frac{1}{(n + \frac{1}{2} - \frac{l}{m})^3}$$

$$= -(-1)^n \frac{1}{(n + \frac{1}{2} - \frac{l}{m})^3}$$

$$= (-1)^{n+1} \frac{1}{(n + \frac{1}{2} - \frac{l}{m})^3}. \square$$

To find $\text{Res}_{-1}\left\{\pi \frac{1}{\cos(\pi z)} \frac{1}{(z - \frac{l}{m})^3}\right\}_{z=\frac{l}{m}}$**, divide the series**

$$\pi \frac{1}{\cos(\pi z)} \frac{1}{(z - \frac{l}{m})^3} = \pi \frac{1}{\cos(\pi[z - \frac{l}{m}] + \frac{l}{m}\pi)} \frac{1}{(z - \frac{l}{m})^3}$$

$$= \pi \frac{1}{\cos(\pi u + \frac{l}{m}\pi)} \frac{1}{u^3}$$

$$= \pi \frac{1}{\cos(\pi u)\cos(\frac{l}{m}\pi) - \sin(\pi u)\sin(\frac{l}{m}\pi)} \frac{1}{u^3}$$

$$\approx \pi \frac{1}{(1 - \frac{1}{2}\pi^2 u^2)\cos(\frac{l}{m}\pi) - (\pi u)\sin(\frac{l}{m}\pi)} \frac{1}{u^3}$$

$$= \pi \frac{1}{\cos(\frac{l}{m}\pi)} \left(\frac{1}{1 - \pi u \tan(\frac{l}{m}\pi) - \frac{1}{2}\pi^2 u^2} \right) \frac{1}{u^3}$$

$$1 + \pi u \tan(\tfrac{l}{m}\pi) + \pi^2 u^2 \left\{ \tan^2(\tfrac{l}{m}\pi) + \tfrac{1}{2} \right\} + \pi^3 u^3 \tan(\tfrac{l}{m}\pi) \left\{ \tan^2(\tfrac{l}{m}\pi) + 1 \right\}$$

$$1 \;\Big|\; 1 - \pi u \tan(\tfrac{l}{m}\pi) - \tfrac{1}{2}\pi^2 u^2$$

$$\underline{1 - \pi u \tan(\tfrac{l}{m}\pi) - \tfrac{1}{2}\pi^2 u^2}$$

$$\pi u \tan(\tfrac{l}{m}\pi) + \tfrac{1}{2}\pi^2 u^2$$

$$\underline{\pi u \tan(\tfrac{l}{m}\pi) - \pi^2 u^2 \tan^2(\tfrac{l}{m}\pi) - \tfrac{1}{2}\pi^3 u^3 \tan(\tfrac{l}{m}\pi)}$$

$$\pi^2 u^2 \left\{ \tan^2(\tfrac{l}{m}\pi) + \tfrac{1}{2} \right\} + \tfrac{1}{2}\pi^3 u^3 \tan(\tfrac{l}{m}\pi)$$

$$\underline{\pi^2 u^2 \left\{ \tan^2(\tfrac{l}{m}\pi) + \tfrac{1}{2} \right\} - \pi^3 u^3 \tan(\tfrac{l}{m}\pi) \left\{ \tan^2(\tfrac{l}{m}\pi) + \tfrac{1}{2} \right\} - \tfrac{1}{2}\pi^4 u^4 \left\{ \tan^2(\tfrac{l}{m}\pi) + \tfrac{1}{2} \right\}}$$

$$\pi^3 u^3 \tan(\tfrac{l}{m}\pi) \left\{ \tan^2(\tfrac{l}{m}\pi) + 1 \right\} + \tfrac{1}{2}\pi^4 u^4 \left\{ \tan^2(\tfrac{l}{m}\pi) + \tfrac{1}{2} \right\}$$

$$= \pi \frac{1}{\cos(\frac{l}{m}\pi)} \left(1 + \pi u \tan(\tfrac{l}{m}\pi) + \pi^2 u^2 \left\{ \tan^2(\tfrac{l}{m}\pi) + \tfrac{1}{2} \right\} \right) \frac{1}{u^3}$$

$$= \pi \frac{1}{\cos(\frac{l}{m}\pi)} \frac{1}{u^3} + \pi^2 \frac{\tan(\frac{l}{m}\pi)}{\cos(\frac{l}{m}\pi)} \frac{1}{u^2} + \pi^3 \frac{2\tan^2(\frac{l}{m}\pi) + 1}{2\cos(\frac{l}{m}\pi)} \frac{1}{u}$$

$$\operatorname{Res}_{-1} \left\{ \pi \frac{1}{\cos(\pi z)} \frac{1}{(z - \frac{l}{m})^3} \right\}_{z = \frac{l}{m}} = \pi^3 \frac{2\tan^2(\frac{l}{m}\pi) + 1}{2\cos(\frac{l}{m}\pi)}$$

$$= \pi^3 \frac{\sin^2(\frac{l}{m}\pi) + 1}{2\cos^3(\frac{l}{m}\pi)}$$

$$= \pi^3 \frac{3 - \cos(2\frac{l}{m}\pi)}{2\cos(\frac{l}{m}\pi)[1 + \cos(2\frac{l}{m}\pi)]}$$

Therefore,

$$... + \frac{-(-1)^{-3}}{(-3+\frac{1}{2}-\frac{l}{m})^3} + \frac{-(-1)^{-2}}{(-2+\frac{1}{2}-\frac{l}{m})^3} + \frac{-(-1)^{-1}}{(-1+\frac{1}{2}-\frac{l}{m})^3} + \frac{-(-1)^{0}}{(0+\frac{1}{2}-\frac{l}{m})^3} +$$

$$+ \frac{-(-1)^{3}}{(1+\frac{1}{2}-\frac{l}{m})^3} + \frac{-(-1)^{2}}{(2+\frac{1}{2}-\frac{l}{m})^3} + \frac{-(-1)^{3}}{(3+\frac{1}{2}-\frac{l}{m})^3} + ... + \pi^3 \frac{2\tan^2(\frac{l}{m}\pi)+1}{2\cos(\frac{l}{m}\pi)} = 0$$

$$\pi^3 = 8m^3 \underbrace{\frac{2\cos(\frac{l}{m}\pi)}{2\tan^2(\frac{l}{m}\pi)+1}}_{\frac{2\cos(\frac{l}{m}\pi)[1+\cos(2\frac{l}{m}\pi)]}{3-\cos(2\frac{l}{m}\pi)}} \left\{ \frac{1}{(m-2l)^3} + \frac{1}{(m+2l)^3} - \frac{1}{(3m-2l)^3} - \frac{1}{(3m+2l)^2} + \frac{1}{(5m-2l)^2} + \frac{1}{(5m+2l)^2} - ... \right\}$$

The Associated Series is

$$C_{l/m} = \frac{1}{(m-2l)^2} + \frac{1}{(m+2l)^2} - \frac{1}{(3m-2l)^2} - \frac{1}{(3m+2l)^2} + \frac{1}{(5m-2l)^2} + \frac{1}{(5m+2l)^2} - ... = \frac{\pi^3}{8m^3} \frac{2\tan^2(\frac{l}{m}\pi)+1}{2\cos(\frac{l}{m}\pi)} . \square$$

$7_{\tan}$.

$7_{\tan}.1$

$$\boxed{\pi^3 = 8m^3 \frac{\cos^2(\frac{l}{m}\pi)}{\tan(\frac{l}{m}\pi)}\left\{\frac{1}{(m-2l)^3} - \frac{1}{(m+2l)^3} + \frac{1}{(3m-2l)^3} - \frac{1}{(3m+2l)^3} + \frac{1}{(5m-2l)^2} - \frac{1}{(5m+2l)^2} + ...\right\}}$$

$l < \frac{1}{2}m$, l and m have no common factor.

$$T_{l/m} = \frac{1}{(m-2l)^3} - \frac{1}{(m+2l)^3} + \frac{1}{(3m-2l)^3} - \frac{1}{(3m+2l)^3} + \frac{1}{(5m-2l)^3} - \frac{1}{(5m+2l)^3} + ... = \frac{\pi^3}{8m^3}\frac{\tan(\frac{l}{m}\pi)}{\cos^2(\frac{l}{m}\pi)}$$

$$= \frac{\pi^3}{8m^3}\frac{\sin(\frac{l}{m}\pi)}{\cos^3(\frac{l}{m}\pi)}$$

Proof: $\pi\tan(\pi z)\frac{1}{(z-\frac{l}{m})^3}$ has poles of order 1 at $z = n + \frac{1}{2}$,

and a pole of order 3 at $z = \frac{l}{m}$

$$\oint_{\square_N} \pi\tan(\pi\zeta)\frac{1}{(\zeta-\frac{l}{m})^3}d\zeta = \sum \text{Res}_{-1}\left\{\pi\tan(\pi z)\frac{1}{(z-\frac{l}{m})^3}\right\}_{z=n+\frac{1}{2}}$$

$$+\text{Res}_{-1}\left\{\pi\tan(\pi z)\frac{1}{(z-\frac{l}{m})^3}\right\}_{z=\frac{l}{m}}$$

$$\left|\oint_{\square_N} \pi\tan(\pi\zeta)\frac{1}{(\zeta-\frac{l}{m})^3}d\zeta\right| \le \underbrace{\lfloor \pi\tan(\pi\zeta)\rfloor}_{\le A} \underbrace{\oint_{\square_N}\frac{1}{(\zeta-\frac{l}{m})^3}d\zeta}_{\le\left[\frac{1}{|\zeta|^2}\right]_{\square_N}=0} = 0\,.\square$$

$$\mathrm{Res}_{-1}\left\{\pi\tan(\pi z)\frac{1}{(z-\frac{l}{m})^3}\right\}_{z=n+\frac{1}{2}} = \left[(z-n-\tfrac{1}{2})\pi\frac{\sin(\pi z)}{\cos(\pi z)}\frac{1}{(z-\frac{l}{m})^3}\right]_{z=n+\frac{1}{2}}$$

$$= \left[\frac{\pi D_z(z-n-\frac{1}{2})}{D_z\cos(\pi z)}\right]_{z=n+\frac{1}{2}} \sin\pi(n+\tfrac{1}{2})\frac{1}{(n+\frac{1}{2}-\frac{l}{m})^3}$$

$$= \left[\frac{\pi}{-\pi\sin(\pi z)}\right]_{z=n+\frac{1}{2}} \sin\pi(n+\tfrac{1}{2})\frac{1}{(n+\frac{1}{2}-\frac{l}{m})^3}$$

$$= \frac{-1}{(n+\frac{1}{2}-\frac{l}{m})^3}\,.\square$$

To find $\mathrm{Res}_{-1}\left\{\pi\tan(\pi z)\frac{1}{(z-\frac{l}{m})^3}\right\}_{z=\frac{l}{m}}$**, divide the series**

$$\pi\tan(\pi z)\frac{1}{[z-\frac{l}{m}]^3} = \pi\frac{\sin(\pi[z-\frac{l}{m}]+\pi\frac{l}{m})}{\cos(\pi[z-\frac{l}{m}]+\pi\frac{l}{m})}\frac{1}{[z-\frac{l}{m}]^3}$$

$$= \pi\frac{\sin(\pi u+\pi\frac{l}{m})}{\cos(\pi u+\pi\frac{l}{m})}\frac{1}{u^3}$$

$$= \pi\frac{\sin(\pi u)\cos(\pi\frac{l}{m})+\cos(\pi u)\sin(\pi\frac{l}{m})}{\cos(\pi u)\cos(\pi\frac{l}{m})-\sin(\pi u)\sin(\pi\frac{l}{m})}\frac{1}{u^3}$$

$$\approx \pi\frac{(\pi u)\cos(\pi\frac{l}{m})+(1-\frac{1}{2}\pi^2u^2)\sin(\pi\frac{l}{m})}{(1-\frac{1}{2}\pi^2u^2)\cos(\pi\frac{l}{m})-(\pi u)\sin(\pi\frac{l}{m})}\frac{1}{u^3}$$

$$= \pi \frac{\sin(\pi \frac{l}{m})}{\cos(\pi \frac{l}{m})} \left(\frac{(\pi u)\cot(\pi \frac{l}{m}) + (1 - \frac{1}{2}\pi^2 u^2)}{(1 - \frac{1}{2}\pi^2 u^2) - (\pi u)\tan(\pi \frac{l}{m})} \right) \frac{1}{u^3}$$

$$= \pi \tan(\tfrac{l}{m}\pi) \left(\frac{1 + \pi u \cot(\frac{l}{m}\pi) - \frac{1}{2}\pi^2 u^2}{1 - \pi u \tan(\frac{l}{m}\pi) - \frac{1}{2}\pi^2 u^2} \right) \frac{1}{u^3}$$

$$1 + \pi u\{\cot(\tfrac{l}{m}\pi) + \tan(\tfrac{l}{m}\pi)\} + \boxed{\pi^2 u^2\{\cot(\tfrac{l}{m}\pi) + \tan(\tfrac{l}{m}\pi)\}\tan(\tfrac{l}{m}\pi)} + \pi^3 u^3\{\cot(\tfrac{l}{m}\pi) + \tan(\tfrac{l}{m}\pi)\}\left\{\tfrac{1}{2} + \tan^2(\tfrac{l}{m}\pi)\right\}$$

$$1 + \pi u \cot(\tfrac{l}{m}\pi) - \tfrac{1}{2}\pi^2 u^2 \quad \Big| \quad 1 - \pi u \tan(\tfrac{l}{m}\pi) - \tfrac{1}{2}\pi^2 u^2$$

$$\underline{1 - \pi u \tan(\tfrac{l}{m}\pi) - \tfrac{1}{2}\pi^2 u^2}$$

$$\pi u\{\cot(\tfrac{l}{m}\pi) + \tan(\tfrac{l}{m}\pi)\}$$

$$\underline{\pi u\{\cot(\tfrac{l}{m}\pi) + \tan(\tfrac{l}{m}\pi)\} - \pi^2 u^2\{\cot(\tfrac{l}{m}\pi) + \tan(\tfrac{l}{m}\pi)\}\tan(\tfrac{l}{m}\pi) - \tfrac{1}{2}\pi^3 u^3\{\cot(\tfrac{l}{m}\pi) + \tan(\tfrac{l}{m}\pi)\}}$$

$$\pi^2 u^2\{\cot(\tfrac{l}{m}\pi) + \tan(\tfrac{l}{m}\pi)\}\tan(\tfrac{l}{m}\pi) + \tfrac{1}{2}\pi^3 u^3\{\cot(\tfrac{l}{m}\pi) + \tan(\tfrac{l}{m}\pi)\}$$

$$\underline{\pi^2 u^2\{\cot(\tfrac{l}{m}\pi) + \tan(\tfrac{l}{m}\pi)\}\tan(\tfrac{l}{m}\pi) - \pi^3 u^3\{\cot(\tfrac{l}{m}\pi) + \tan(\tfrac{l}{m}\pi)\}\tan^2(\tfrac{l}{m}\pi) - \tfrac{1}{2}\pi^4 u^4\{\cot(\tfrac{l}{m}\pi) + \tan(\tfrac{l}{m}\pi)\}\tan(\tfrac{l}{m}\pi)}$$

$$\pi^3 u^3\{\cot(\tfrac{l}{m}\pi) + \tan(\tfrac{l}{m}\pi)\}\left\{\tfrac{1}{2} + \tan^2(\tfrac{l}{m}\pi)\right\} + \tfrac{1}{2}\pi^4 u^4\{\cot(\tfrac{l}{m}\pi) + \tan(\tfrac{l}{m}\pi)\}\tan(\tfrac{l}{m}\pi)$$

$$= \pi \tan(\tfrac{l}{m}\pi)\frac{1}{u^3}$$

$$+\pi^2 \tan(\tfrac{l}{m}\pi)\{\cot(\tfrac{l}{m}\pi) + \tan(\tfrac{l}{m}\pi)\}\frac{1}{u^2}$$

$$+\pi^3 \tan(\tfrac{l}{m}\pi)\Big[\pi^2 u^2\{\cot(\tfrac{l}{m}\pi) + \tan(\tfrac{l}{m}\pi)\}\tan(\tfrac{l}{m}\pi)\Big]\frac{1}{u}$$

$$\text{Res}_{-1}\left\{\pi \tan(\tfrac{l}{m}\pi)\frac{1}{(z - \frac{l}{m})^2}\right\}_{z=\frac{l}{m}} = \pi^3 \tan^2(\tfrac{l}{m}\pi)\{\cot(\tfrac{l}{m}\pi) + \tan(\tfrac{l}{m}\pi)\}$$

$$= \pi^3 \tan(\tfrac{l}{m}\pi)\{1 + \tan^2(\tfrac{l}{m}\pi)\}$$

$$= \pi^3 \frac{\tan(\frac{l}{m}\pi)}{\cos^2(\frac{l}{m}\pi)}$$

$$= \pi^3 \frac{\sin(\frac{l}{m}\pi)}{\cos^3(\frac{l}{m}\pi)}$$

Then,

$$...-\frac{1}{(-3+\frac{1}{2}-\frac{l}{m})^3}-\frac{1}{(-2+\frac{1}{2}-\frac{l}{m})^3}-\frac{1}{(-1+\frac{1}{2}-\frac{l}{m})^3}-\frac{1}{(0+\frac{1}{2}-\frac{l}{m})^3}-$$

$$-\frac{1}{(1+\frac{1}{2}-\frac{l}{m})^3}-\frac{1}{(2+\frac{1}{2}-\frac{l}{m})^3}-\frac{1}{(3+\frac{1}{2}-\frac{l}{m})^3}-...+\pi^3\frac{\tan(\frac{l}{m}\pi)}{\cos^2(\frac{l}{m}\pi)}=0$$

$$\pi^3 = 8m^3\frac{\cos^2(\frac{l}{m}\pi)}{\tan(\frac{l}{m}\pi)}\Big\{\frac{1}{(m-2l)^3}-\frac{1}{(m+2l)^3}+\frac{1}{(3m-2l)^3}-\frac{1}{(3m+2l)^3}+\frac{1}{(5m-2l)^2}-\frac{1}{(5m+2l)^2}+...\Big\}$$

The Associated Series is

$$T_{l/m}=\frac{1}{(m-2l)^3}-\frac{1}{(m+2l)^3}+\frac{1}{(3m-2l)^3}-\frac{1}{(3m+2l)^3}+\frac{1}{(5m-2l)^3}-\frac{1}{(5m+2l)^3}+...=\frac{\pi^3}{8m^3}\frac{\tan(\frac{l}{m}\pi)}{\cos^2(\frac{l}{m}\pi)}$$

7_{tan}.2 Euler Almost Obtained $7._{\text{tan}}$

Euler had in his section #175[6]

$$\frac{\pi^3}{m_1^3}\frac{k^3+k}{8}=\frac{1}{(m_1-l_1)^3}-\frac{1}{(m_1+l_1)^3}+\frac{1}{(3m_1-l_1)^3}-\frac{1}{(3m_1+l_1)^3}+\frac{1}{(5m_1-l_1)^3}-\frac{1}{(5m_1+l_1)^3}+...$$

[6] Leonardi Euleri, "Introductio in Analysin Infinitorum", Section #175

Euler's Notation	Our Notation
m_1	m
l_1	$2l$
k	$\tan(\frac{l_1}{2m_1}\pi) = \tan(\frac{l}{m}\pi)$

Therefore,

$$k^3 + k = \tan\left(\frac{l}{m}\pi\right)[\tan^2\left(\frac{l}{m}\pi\right) + 1]$$

$$= \tan\left(\frac{l}{m}\pi\right)\frac{1}{\cos^2\left(\frac{l}{m}\pi\right)}$$

Replacing Euler's Notation with ours we obtain $\mathbf{7_{tan}}$.

But the notation

$$\tan(\frac{l}{m}\pi) = k$$

insinuates that $\tan(\frac{l}{m}\pi)$ is always an integer.

This amounts to Euler's hypothesizing that π^3 series are parameterized by three integers

$$n_1,\quad m_1, \text{ and } k,$$

In fact, π series are parameterized by the first two, and the crucial role of the tangent function is hidden by a notation that is reserved to integers.

8.

8.1

$$\pi^3 = 38^3 \frac{\sin^3(\frac{15}{38}\pi)}{\cos(\frac{15}{38}\pi)} \left\{ \frac{1}{15^3} - \frac{1}{(38-15)^3} + \frac{1}{(38+15)^3} - \frac{1}{[2(38)-15]^3} + \frac{1}{[2(38)+15]^3} - \frac{1}{[3(38)-15]^3} + \ldots \right\}$$

$$R_{15/38} = \frac{1}{15^3} - \frac{1}{(38-15)^3} + \frac{1}{(38+15)^3} - \frac{1}{[2(38)-15]^3} + \frac{1}{[2(38)+15]^3} - \frac{1}{[3(38)-15]^3} + \ldots = \frac{\pi^3}{38^3} \frac{\cos(\frac{15}{38}\pi)}{\sin^3(\frac{15}{38}\pi)}$$

Proof: By $\mathbf{7_{cot}}$. based on $\pi \cot(\pi z) \dfrac{1}{(z - \frac{15}{38})^3}$. □

8.2

$$\pi^3 = 38^3 \frac{2\sin^3(\frac{15}{38}\pi)}{\cos^2(\frac{15}{38}\pi)+1} \left\{ \frac{1}{15^3} + \frac{1}{(38-15)^3} - \frac{1}{(38+15)^3} - \frac{1}{[2(38)-15]^3} + \frac{1}{[2(38)+15]^3} + \frac{1}{[3(38)-15]^3} - \ldots \right\}$$

$$S_{15/38} = \frac{1}{15^3} + \frac{1}{(38-15)^3} - \frac{1}{(38+15)^3} - \frac{1}{[2(38)-15]^3} + \frac{1}{[2(38)+15]^3} + \frac{1}{[3(38)-15]^3} - \ldots = \frac{\pi^3}{38^3} \frac{\cos^2(\frac{15}{38}\pi)+1}{2\sin^3(\frac{15}{38}\pi)}$$

Proof: By $\mathbf{7_{sin}}$. based on $\pi\frac{1}{\sin(\pi z)}\frac{1}{(z-\frac{15}{38})^3}$.□

8.3
$$\boxed{\begin{aligned}\pi^3 = 8(38)^3\frac{2\cos^3(\frac{15}{38}\pi)}{\sin^2(\frac{15}{38}\pi)+1}\Bigg(&\frac{1}{(38-2(15))^3}+\frac{1}{(38+2(15))^3}\\ &-\frac{1}{(3(38)-2(15))^3}-\frac{1}{(3(38)+2(15))^3}\\ &+\frac{1}{(5(38)-2(15))^3}+\frac{1}{(5(38)+2(15))^3}-...\Bigg)\end{aligned}}$$

$$\begin{aligned}C_{15/38} &= \frac{1}{(38-2(15))^3}+\frac{1}{(38+2(15))^3}\\ &-\frac{1}{(3(38)-2(15))^3}-\frac{1}{(3(38)+2(15))^3}\\ &+\frac{1}{(5(38)-2(15))^3}+\frac{1}{(5(38)+2(15))^3}-... = \frac{\pi^3}{8(38)^3}\frac{\sin^2(\frac{15}{38}\pi)+1}{2\cos^3(\frac{15}{38}\pi)}\end{aligned}$$

Proof: By $\mathbf{7_{cos}}$. based on $\pi\frac{1}{\cos(\pi z)}\frac{1}{(z-\frac{15}{38})^3}$.□

8.4
$$\boxed{\begin{aligned}\pi^3 = 8(38^3)\frac{\cos^3(\frac{15}{38}\pi)}{\sin(\frac{15}{38}\pi)}\Bigg\{&\frac{1}{(38-2(15))^3}-\frac{1}{(38+2(15))^3}\\ &+\frac{1}{(3(38-2(15))^3}-\frac{1}{(3(38)+2(15))^3}\\ &+\frac{1}{(5(38)-2(15))^3}-\frac{1}{(5(38)+2(15))^3}+...\Bigg\}\end{aligned}}$$

$$T_{15/38} = \frac{1}{(38-2(15))^3}+\frac{1}{(38+2(15))^3}$$

$$+\frac{1}{(3(38)-2(15))^3}+\frac{1}{(3(38)+2(15))^3}$$

$$+\frac{1}{(5(38)-2(15))^3}+\frac{1}{(5(38)+2(15))^3}+...=\frac{\pi^3}{8(38)^3}\frac{\sin(\frac{15}{38}\pi)}{\cos^3(\frac{15}{38}\pi)}$$

Proof: By $\mathbf{7_{tan}}$. based on $\pi\tan(\pi z)\dfrac{1}{(z-\frac{15}{38})^3}$.□

9.

9.1

$$\pi^3 = 2^{3n}\frac{\sin^3(\frac{2k+1}{2^n}\pi)}{\cos(\frac{2k+1}{2^n}\pi)}\left\{\frac{1}{(2k+1)^3}-\frac{1}{(2^n-2k-1)^3}+\frac{1}{(2^n+2k+1)^3}-\frac{1}{(2\cdot 2^n-2k-1)^3}+\frac{1}{(2\cdot 2^n+2k+1)^3}-\frac{1}{(3\cdot 2^n-2k-1)^3}+\ldots\right\}$$

$$R_{(2k+1)/2^n} = \frac{1}{(2k+1)^3}-\frac{1}{(2^n-2k-1)^3}+\frac{1}{(2^n+2k+1)^3}-\frac{1}{(2\cdot 2^n-2k-1)^3}+\frac{1}{(2\cdot 2^n+2k+1)^3}-\frac{1}{(3\cdot 2^n-2k-1)^3}+\ldots = \frac{\pi^3}{2^{3n}}\frac{\cos(\frac{2k+1}{2^n}\pi)}{\sin^3(\frac{2k+1}{2^n}\pi)}$$

$$2k+1 < 2^{n-1}$$

$$\sin(\tfrac{2k+1}{2^n}\pi) = \sqrt{\tfrac{1}{2}[1-\cos(\tfrac{2k+1}{2^{n-1}}\pi)]}$$

$$= \sqrt{1-\tfrac{1}{\sqrt{2}}\sqrt{1+\cos(\tfrac{2k+1}{2^{n-2}}\pi)}} = ...$$

$$= \text{ a number composed of } \sqrt{2}\text{'s}$$

$$\cos(\tfrac{2k+1}{2^n}\pi) = \sqrt{\tfrac{1}{2}[1+\cos(\tfrac{2k+1}{2^{n-1}}\pi)]}$$

$$= \text{ a number composed of } \sqrt{2}\text{'s}$$

Proof: By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\dfrac{1}{(z-\frac{2k+1}{2^n})^3}$. □

9.2
$$\pi^3 = 2^{3n}\frac{2\sin^3(\frac{2k+1}{2^n}\pi)}{\cos^2(\frac{2k+1}{2^n}\pi)+1}\left\{\frac{1}{(2k+1)^2}+\frac{1}{(2^n-2k-1)^2}-\frac{1}{(2^n+2k+1)^2}-\frac{1}{(2\cdot 2^n-2k-1)^2}+\frac{1}{(2\cdot 2^n+2k+1)^2}+\frac{1}{(3\cdot 2^n-2k-1)^2}-...\right\}$$

$$S_{(2k+1)/2^n} = \frac{1}{(2k+1)^3}+\frac{1}{(2^n-2k-1)^3}-\frac{1}{(2^n+2k+1)^3}-\frac{1}{(2\cdot 2^n-2k-1)^3}+\frac{1}{(2\cdot 2^n+2k+1)^3}+\frac{1}{(3\cdot 2^n-2k-1)^3}-... = \frac{\pi^3}{2^{3n}}\frac{\cos^2(\frac{2k+1}{2^n}\pi)+1}{2\sin^3(\frac{2k+1}{2^n}\pi)}$$

Proof: By $\mathbf{7_{sin}}$. based on $\pi\frac{1}{\sin(\pi z)}\frac{1}{(z-\frac{2k+1}{2^n})^3}$.□

9.3
$$\pi^3 = 8(2^{3n})\frac{2\cos^3(\frac{2k+1}{2^n}\pi)}{\sin^2(\frac{2k+1}{2^n}\pi)+1}\left(\frac{1}{(2^n-2(2k+1))^3}+\frac{1}{(2^n+2(2k+1))^3}-\frac{1}{(3(2^n)-2(2k+1))^3}-\frac{1}{(3(2^n)+2(2k+1))^3}+\frac{1}{(5(2^n)-2(2k+1))^3}+\frac{1}{(5(2^n)+2(2k+1))^3}-..\right)$$

$$C_{l/m} = \frac{1}{(2^n-2(2k+1))^3}+\frac{1}{(2^n+2(2k+1))^3}-\frac{1}{(3(2^n)-2(sk+1))^3}-\frac{1}{(3(2^n)+2(2k+1))^3}+\frac{1}{(5(2^n)-2(2k+1))^3}+\frac{1}{(5(2^n)+2(2k+1))^3}-... = \frac{\pi^3}{8m^3}\frac{\sin^2(\frac{2k+1}{2^n}\pi)+1}{2\cos^3(\frac{2k+1}{2^n}\pi)}$$

Proof: By $\mathbf{7_{cos}}$. based on $\pi \frac{1}{\cos(\pi z)} \frac{1}{(z - \frac{2k+1}{2^n})^3}$.□

9.4
$$\boxed{\begin{aligned}\pi^2 = 8(2^{3n})\frac{\cos^3(\frac{2k+1}{2^n}\pi)}{\sin(\frac{2k+1}{2^n}\pi)}\Bigg(&\frac{1}{(2^n - 2(2k+1))^3} - \frac{1}{(2^n + 2(2k+1))^3}\\ &+ \frac{1}{(3(2^n) - 2(2k+1))^2} - \frac{1}{(3(2^n) + 2(2k+1))^3}\\ &+ \frac{1}{(5(2^n) - 2(2k+1))^3} - \frac{1}{(5(2^n) + 2(2k+1))^3} - ...\Bigg)\end{aligned}}$$

$$\begin{aligned}T_{l/m} &= \frac{1}{(2^n - 2(2k+1))^3} - \frac{1}{(2^n + 2(2k+1))^3}\\ &+ \frac{1}{(3(2^n) - 2(2k+1))^3} - \frac{1}{(3(2^n) + 2(2k+1))^3}\\ &+ \frac{1}{(5(2^n) - 2(2k+1))^3} - \frac{1}{(5(2^n) + 2(2k+1))^3} - ... =\\ &= \frac{\pi^3}{8(2^{3n})}\frac{\sin(\frac{2k+1}{2^n}\pi)}{\cos^3(\frac{2k+1}{2^n}\pi)}\end{aligned}$$

Proof: By $\mathbf{7_{tan}}$. based on $\pi \tan(\pi z) \frac{1}{(z - \frac{2k+1}{2^n})^3}$.□

10.

10.1
$$\boxed{\pi^3 = 32\left(\frac{1}{1}-\frac{1}{3^3} + \frac{1}{5^3}-\frac{1}{7^3} + \frac{1}{9^3}-\frac{1}{11^3}+...\right)}\textbf{(Euler)}$$

$$R_{1/4} = \frac{1}{1}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{11^3}+... = \frac{\pi^3}{32}\textbf{(Euler)}$$

Proof: By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\dfrac{1}{(z-\frac{1}{4})^3}$.

$$\pi^3 = 4^3\underbrace{\frac{\sin^2(\frac{1}{4}\pi)}{\cot(\frac{1}{4}\pi)}}_{1/2}\left(\frac{1}{1^3}-\frac{1}{(4-1)^3} + \frac{1}{(4+1)^3}-\frac{1}{(2(4)-1)^3} + \frac{1}{(2(4)+1)^3}-\frac{1}{(3(4)-1)^3}+...\right)$$

Or, by $\mathbf{7_{tan}}$. based on $\pi\tan(\pi z)\dfrac{1}{(z-\frac{1}{4})^3}$,

$$\pi^3 = 8(4)^3\underbrace{\frac{\cos^2(\frac{1}{4}\pi)}{\tan(\frac{1}{4}\pi)}}_{1/2} = \left(\frac{1}{(4-2)^3}-\frac{1}{(4+2)^3} + \frac{1}{(3(4)-2)^3}-\frac{1}{(3(4)+2)^3} + \frac{1}{(5(4)-2)^3}-\frac{1}{(5(4)+2)^3}-...\right).\square$$

10.2

$$\boxed{\pi^3 = \frac{64\sqrt{2}}{3}\left(\frac{1}{1}+\frac{1}{3^3} - \frac{1}{5^3}-\frac{1}{7^3} + \frac{1}{9^3}+\frac{1}{11^3}-...\right)}$$

$$S_{1/4} = \frac{1}{1^3}+\frac{1}{3^3}-\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}+\frac{1}{11^3}-... = 3\frac{\pi^3}{64\sqrt{2}}$$

Proof: By $\mathbf{7_{sin}}$. based on $\pi\frac{1}{\sin(\pi z)}\frac{1}{(z-\frac{1}{4})^3}$,

$$\pi^3 = 4^3\frac{2\sin(\frac{1}{4}\pi)}{2\cot^2(\frac{1}{4}\pi)+1}\left\{\frac{1}{1^3}+\frac{1}{(4-1)^3} - \frac{1}{(4+1)^3}-\frac{1}{(2(4)-1)^3} + \frac{1}{(2(4)+1)^3}+\frac{1}{(3(4)-1)^3}-...\right\}. \square$$

Or, by $\mathbf{7_{cos}}$. based on $\pi\frac{1}{\cos(\pi z)}\frac{1}{(z-\frac{1}{4})^3}. \square$

$$\pi^3 = 8(4^3)\frac{2\cos(\frac{1}{4}\pi)}{2\tan^2(\frac{1}{4}\pi)+1}\left\{\frac{1}{(4-2)^3}+\frac{1}{(4+2)^3} - \frac{1}{(3(4)-2)^3}-\frac{1}{(3(4)+2)^2} + \frac{1}{(5(4)-2)^2}+\frac{1}{(5(4)+2)^2}-...\right\}. \square$$

10.3

$$\boxed{\sqrt{2} = \frac{3}{2}\frac{1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + ...}{1 + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} + \frac{1}{11^3} - ...}}$$

11.

11.1

$$\pi^3 = 8^3 \frac{\sin^2(\frac{1}{8}\pi)}{\cot(\frac{1}{8}\pi)}\left(\frac{1}{1} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{15^3} + \frac{1}{17^3} - \frac{1}{23^3} + ...\right)$$

$$R_{1/8} = \frac{1}{1} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{15^3} + \frac{1}{17^3} - \frac{1}{23^3} + ... = \frac{\pi^3}{8^3}\frac{\cot(\frac{1}{8}\pi)}{\sin^3(\frac{1}{8}\pi)}$$

Proof: By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\frac{1}{(z-\frac{1}{8})^3}$,

$$\pi^3 = 8^3 \frac{\sin^3(\frac{1}{8}\pi)}{\cot(\frac{1}{8}\pi)}\left(\frac{1}{1^3} - \frac{1}{(8-1)^3} + \frac{1}{(8+1)^3} - \frac{1}{(2(8)-1)^3} + \frac{1}{(2(8)+1)^3} - \frac{1}{(3(8)-1)^3} + ...\right). \square$$

11.2

$$\pi^3 = 8^3 \frac{2\sin^3(\frac{1}{8}\pi)}{\cos^2(\frac{1}{8}\pi)+1}\left(\frac{1}{1} + \frac{1}{7^3} - \frac{1}{9^3} - \frac{1}{15^3} + \frac{1}{17^3} + \frac{1}{23^3} - ...\right)$$

$$S_{1/8} = \frac{1}{1} + \frac{1}{7^3} - \frac{1}{9^3} - \frac{1}{15^3} + \frac{1}{17^3} + \frac{1}{23^3} - ... = \frac{\pi^3}{8^3}\frac{\cos^2(\frac{1}{8}\pi)+1}{2\sin^3(\frac{1}{8}\pi)}$$

$$\sin^2(\tfrac{1}{8}\pi) = \frac{2-\sqrt{2}}{4}$$

$$\cos(\tfrac{1}{8}\pi) = \frac{\sqrt{2+\sqrt{2}}}{2}$$

__Proof__: By $\mathbf{7_{sin}}$. based on $\pi\frac{1}{\sin(\pi z)}\frac{1}{(z-\frac{1}{8})^3}$,

$$\pi^3 = 8^3\frac{1}{2\sin(\frac{1}{8}\pi)}\underbrace{\frac{3+\cos(2\frac{1}{8}\pi)}{1-\cos(2\frac{1}{8}\pi)}}_{\frac{6+\sqrt{2}}{2-\sqrt{2}}}\left(\frac{1}{1^3}+\frac{1}{(8-1)^3} - \frac{1}{(8+1)^3} - \frac{1}{(2(8)-1)^3} + \frac{1}{(2(8)+1)^3} + \frac{1}{(3(8)-1)^3} - ...\right).\square$$

$$\sin^2(\tfrac{1}{8}\pi) = \tfrac{1}{2}\left[1-\cos(\tfrac{1}{4}\pi)\right] = \frac{2-\sqrt{2}}{4}.\square$$

$$\cos(\tfrac{1}{8}\pi) = \sqrt{\tfrac{1}{2}\left[1+\cos(\tfrac{1}{4}\pi)\right]}$$

$$= \sqrt{\tfrac{1}{2}\left[1+\tfrac{\sqrt{2}}{2}\right]}$$

$$= \frac{\sqrt{2+\sqrt{2}}}{2}.\square$$

11.3

$$\boxed{\pi^3 = 8^3\frac{2\cos^3(\frac{1}{8}\pi)}{\sin^2(\frac{1}{8}\pi)+1}\left(\frac{1}{3^3}+\frac{1}{5^3} - \frac{1}{11^2} - \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{21^2} - ...\right)}$$

$$C_{1/8} = \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{11^2} - \frac{1}{13^2} + \frac{1}{19^2} + \frac{1}{21^2} - ... = \frac{\pi^3}{8^3}\frac{\sin^2(\frac{1}{8}\pi) + 1}{2\cos^3(\frac{1}{8}\pi)}$$

$$\cos^2(\tfrac{1}{8}\pi) = \frac{2+\sqrt{2}}{4}$$

$$\sin(\tfrac{1}{8}\pi) = \frac{\sqrt{2-\sqrt{2}}}{2}$$

<u>Proof</u>: By **$7_{\cos}$.** based on $\pi\frac{1}{\cos(\pi z)}\frac{1}{(z-\frac{1}{8})^3}$.□

$$\pi^3 = 8(8^3)\frac{2\cos^3(\frac{1}{8}\pi)}{\sin^2(\frac{1}{8}\pi)+1}\Bigg(\frac{1}{(8-2)^3} + \frac{1}{(8+2)^3} - \frac{1}{(3(8)-2)^3} - \frac{1}{(3(8)+2)^3} + \frac{1}{(5(8)-2)^3} + \frac{1}{(5(8)+2)^3} - ...\Bigg).\square$$

$$\sin(\tfrac{1}{8}\pi) = \sqrt{\tfrac{1}{2}\left[1-\cos(\tfrac{1}{4}\pi)\right]} = \sqrt{\tfrac{1}{2}\left[1-\tfrac{\sqrt{2}}{2}\right]} = \frac{\sqrt{2-\sqrt{2}}}{2}.\square$$

11.4

$$\boxed{\pi^3 = 8^3\frac{\cos^3(\frac{1}{8}\pi)}{\sin(\frac{1}{8}\pi)}\Bigg(\frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{11^2} - \frac{1}{13^2} + \frac{1}{19^2} - \frac{1}{21^2} - ...\Bigg)}$$

$$T_{1/8} = \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{11^2} - \frac{1}{13^2} + \frac{1}{19^2} - \frac{1}{21^2} - ... = \frac{\pi^3 \sin(\frac{1}{8}\pi)}{8^3 \cos^3(\frac{1}{8}\pi)}$$

$$\tan(\tfrac{1}{8}\pi) = \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}} = \textbf{depends on } \sqrt{2}\textbf{'s}$$

Proof: By $\mathbf{7_{tan}}$. based on $\pi \tan(\pi z)\dfrac{1}{(z-\frac{1}{8})^3}$,

$$\pi^3 = 8(8^3)\frac{\cos^3(\frac{1}{8}\pi)}{\sin(\frac{1}{8}\pi)}\left(\frac{1}{(8-2)^3} - \frac{1}{(8+2)^3} + \frac{1}{(3(8)-2)^3} - \frac{1}{(3(8)+2)^3} + \frac{1}{(5(8)-2)^3} - \frac{1}{(5(8)+2)^3} - ...\right). \square$$

$$\cot(\tfrac{1}{8}\pi) = \frac{\cos(\frac{1}{8}\pi)}{\sin(\frac{1}{8}\pi)}$$

$$= \frac{\sqrt{\frac{1}{2}\left[1+\cos(\frac{1}{4}\pi)\right]}}{\sqrt{\frac{1}{2}\left[1-\cos(\frac{1}{4}\pi)\right]}}$$

$$= \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}}. \square$$

12.

12.1

$$\pi^3 = 8^3 \frac{\sin^3(\frac{3}{8}\pi)}{\cos(\frac{3}{8}\pi)}\left(\frac{1}{3^3} - \frac{1}{5^3} + \frac{1}{11^3} - \frac{1}{13^3} + \frac{1}{19^3} - \frac{1}{21^3} + ...\right)$$

$$R_{3/8} = \frac{1}{3^3} - \frac{1}{5^3} + \frac{1}{11^3} - \frac{1}{13^3} + \frac{1}{19^3} - \frac{1}{21^3} + ... = \frac{\pi^3}{8^3}\frac{\cos(\frac{3}{8}\pi)}{\sin^3(\frac{3}{8}\pi)}$$

$$\sin^2(\tfrac{3}{8}\pi) = \frac{2+\sqrt{2}}{4} = \textbf{depends on } \sqrt{2}\text{'s}$$

Proof: By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\dfrac{1}{(z-\frac{3}{8})^3}$,

$$\pi^3 = 8^3 \frac{\sin^3(\frac{1}{8}\pi)}{\cos(\frac{1}{8}\pi)}\left(\frac{1}{3^3} - \frac{1}{(8-3)^3} + \frac{1}{(8+3)^3} - \frac{1}{(2(8)-3)^3} + \frac{1}{(2(8)+3)^3} - \frac{1}{(3(8)-3)^3} + ...\right). \square$$

12.2

$$\pi^3 = 8^3 2\sin(\tfrac{3}{8}\pi)\frac{2+\sqrt{2}}{6-\sqrt{2}}\left(\frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{11^3} - \frac{1}{13^3} + \frac{1}{19^3} + \frac{1}{21^3} + ...\right)$$

$$S_{3/8} = \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{11^3} - \frac{1}{13^3} + \frac{1}{19^3} + \frac{1}{21^3} - ... = \frac{\pi^3}{8^3}\frac{1}{2\sin(\frac{3}{8}\pi)}\frac{6-\sqrt{2}}{2+\sqrt{2}}$$

$$\sin^2(\tfrac{3}{8}\pi) = \frac{2+\sqrt{2}}{4} = \text{depends on } \sqrt{2}\text{'s}$$

$$\cos(\tfrac{3}{8}\pi) = \frac{\sqrt{2-\sqrt{2}}}{2} = \text{ depends on } \sqrt{2}\text{'s}$$

Proof By $\mathbf{7_{sin}}$. based on $\pi\frac{1}{\sin(\pi z)}\frac{1}{(z-\frac{3}{8})^3}$,

$$\pi^3 = 8^3 2\sin(\tfrac{3}{8}\pi)\underbrace{\frac{1-\cos(2\frac{3}{8}\pi)}{3+\cos(2\frac{3}{8}\pi)}}_{\frac{2+\sqrt{2}}{6-\sqrt{2}}}\left(\frac{1}{3^3} + \frac{1}{(8-3)^3} - \frac{1}{(8+3)^3} - \frac{1}{(2(8)-3)^3} + \frac{1}{(2(8)+3)^3} + \frac{1}{(3(8)-3)^3} - ...\right).\square$$

$$\sin^2(\tfrac{3}{8}\pi) = \tfrac{1}{2}\left[1-\cos(\tfrac{3}{4}\pi)\right] = \frac{2+\sqrt{2}}{4}.\square$$

$$\cos(\tfrac{3}{8}\pi) = \sqrt{\tfrac{1}{2}\left[1+\cos(\tfrac{3}{4}\pi)\right]} = \frac{\sqrt{2-\sqrt{2}}}{2}.\square$$

12.3

$$\boxed{\pi^3 = 8^3\frac{2\cos^3(\frac{3}{8}\pi)}{\sin^2(\frac{3}{8}\pi)+1}\left(\frac{1}{1^3} + \frac{1}{7^3} - \frac{1}{9^3} - \frac{1}{15^3} + \frac{1}{17^3} + \frac{1}{23^3} - ...\right)}$$

$$C_{3/8} = \frac{1}{1^3} + \frac{1}{7^3} - \frac{1}{9^3} - \frac{1}{15^3} + \frac{1}{17^3} + \frac{1}{23^3} - ... = \frac{\pi^3}{8^3}\frac{\sin^2(\frac{1}{8}\pi) + 1}{2\cos^3(\frac{1}{8}\pi)}$$

$$\cos^2(\tfrac{3}{8}\pi) = \frac{2-\sqrt{2}}{4} = \text{depends on } \sqrt{2}\text{'s}$$

$$\sin(\tfrac{3}{8}\pi) = \frac{\sqrt{2+\sqrt{2}}}{2} = \text{ depends on } \sqrt{2}\text{'s}$$

Proof By $\mathbf{7_{cos}}$. based on $\pi\frac{1}{\cos(\pi z)}\frac{1}{(z-\frac{3}{8})^3}$,

$$\pi^3 = 8(8^3)\frac{2\cos^3(\frac{3}{8}\pi)}{\sin^2(\frac{3}{8}\pi)+1}\left(\frac{1}{(8-2(3))^3} + \frac{1}{(8+2(3))^3} - \frac{1}{(3(8)-2(3))^3} - \frac{1}{(3(8)+2(3))^3} + \frac{1}{(5(8)-2(3))^3} + \frac{1}{(5(8)+2(3))^3} - ...\right).\square$$

$$\cos^2(\tfrac{3}{8}\pi) = \tfrac{1}{2}\left[1+\cos(\tfrac{3}{4}\pi)\right] = \frac{2-\sqrt{2}}{4}.\square$$

$$\sin(\tfrac{3}{8}\pi) = \sqrt{\tfrac{1}{2}\left[1+\cos(\tfrac{3}{4}\pi)\right]} = \frac{\sqrt{2+\sqrt{2}}}{2}.\square$$

12.4

$$\boxed{\pi^3 = (8^3)\frac{\cos^3(\frac{3}{8}\pi)}{\sin(\frac{3}{8}\pi)}\left(\frac{1}{1^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{15^3} + \frac{1}{17^3} - \frac{1}{23^3} - ...\right)}$$

$$T_{3/8} = \frac{1}{1^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{15^3} + \frac{1}{17^3} - \frac{1}{23^3} + ... = \frac{\pi^3}{8^3}\frac{\sin(\frac{3}{8}\pi)}{\cos^3(\frac{3}{8}\pi)}$$

$$\cos^2(\tfrac{3}{8}\pi) = \frac{2-\sqrt{2}}{4} = \text{depends on } \sqrt{2}\text{'s}$$

$$\sin(\tfrac{3}{8}\pi) = \frac{\sqrt{2+\sqrt{2}}}{2} = \text{ depends on } \sqrt{2}\text{'s}$$

Proof By $\mathbf{7_{tan}}$. based on $\pi\tan(\pi z)\dfrac{1}{(z-\frac{3}{8})^3}$,

$$\pi^3 = 8(8^3)\frac{\cos^3(\frac{3}{8}\pi)}{\sin(\frac{3}{8}\pi)}\left(\frac{1}{(8-2(3))^3} - \frac{1}{(8+2(3))^3} + \frac{1}{(3(8)-2(3))^3} - \frac{1}{(3(8)+2(3))^3} + \frac{1}{(5(8)-2(3))^3} - \frac{1}{(5(8)+2(3))^3} + ...\right).\square$$

$$\cos^2(\tfrac{3}{8}\pi) = \tfrac{1}{2}\left[1+\cos(\tfrac{3}{4}\pi)\right] = \frac{2-\sqrt{2}}{4}.\square$$

$$\sin(\tfrac{3}{8}\pi) = \sqrt{\tfrac{1}{2}\left[1+\cos(\tfrac{3}{4}\pi)\right]} = \frac{\sqrt{2+\sqrt{2}}}{2}.\square$$

13.

$R_{1/8} + R_{3/8} = S_{1/4}$

$$\boxed{R_{1/8} + R_{3/8} = \frac{1}{1} + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} + \frac{1}{11^3} ... = S_{1/4}}$$

Proof: By $\mathbf{7_{cot}}$.

$$R_{1/8} = \frac{1}{1} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{15^3} + \frac{1}{17^3} - \frac{1}{23^3} + ... = \frac{\pi^3}{8^3} \frac{\cot(\frac{1}{8}\pi)}{\sin^2(\frac{1}{8}\pi)}$$

$$R_{3/8} = \frac{1}{3^3} - \frac{1}{5^3} + \frac{1}{11^3} - \frac{1}{13^3} + \frac{1}{19^3} - \frac{1}{21^3} + ... = \frac{\pi^3}{8^3} \frac{\cot(\frac{3}{8}\pi)}{\sin^2(\frac{3}{8}\pi)}$$

$$\Rightarrow \quad R_{1/8} + R_{3/8} = \frac{1}{1} + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} + \frac{1}{11^3} - ... = S_{1/4}$$

Or,

$$R_{1/8} + R_{3/8} = \frac{\pi^3}{8^3} \left\{ \frac{\cot(\frac{1}{8}\pi)}{\sin^2(\frac{1}{8}\pi)} + \frac{\cot(\frac{3}{8}\pi)}{\sin^2(\frac{3}{8}\pi)} \right\}$$

$$= \frac{\pi^3}{8^3} \left\{ \frac{\cos(\frac{1}{8}\pi)}{\sin^3(\frac{1}{8}\pi)} + \frac{\cos(\frac{3}{8}\pi)}{\sin^3(\frac{3}{8}\pi)} \right\}$$

$$= \frac{\pi^3}{8^3} \left\{ \frac{\sin(\frac{3}{8}\pi)}{\sin^3(\frac{1}{8}\pi)} + \frac{\sin(\frac{1}{8}\pi)}{\sin^3(\frac{3}{8}\pi)} \right\}$$

$$= \frac{\pi^3}{8^3} \frac{\sin^4(\frac{3}{8}\pi) + \sin^4(\frac{1}{8}\pi)}{[\sin(\frac{1}{8}\pi)\sin(\frac{3}{8}\pi)]^3}$$

$$= \frac{\pi^3}{8^3} \frac{\left[\frac{1}{2}\{1 - \cos(\frac{3}{4}\pi)\}\right]^2 + \left[\frac{1}{2}\{1 - \cos(\frac{1}{4}\pi)\}\right]^2}{\left[\sin(\frac{1}{8}\pi)\cos(\frac{1}{8}\pi)\right]^3}$$

$$= \frac{\pi^3}{8^3} \frac{\left[\frac{1}{2}\{1+\frac{\sqrt{2}}{2}\}\right]^2 + \left[\frac{1}{2}\{1-\frac{\sqrt{2}}{2}\right]^2}{\left[\frac{1}{2}\sin(\frac{1}{4}\pi)\right]^3}$$

$$= \frac{\pi^3}{8^3} \frac{\frac{3}{4}}{\left[\frac{1}{2}\frac{\sqrt{2}}{2}\right]^3}$$

$$= \frac{\pi^3}{8^2}\frac{3}{\sqrt{2}} = \frac{\pi^3}{4^3}\frac{3}{\sqrt{2}} = S_{1/4}$$

14.

$S_{1/8} - S_{3/8}$

$$1 - \frac{1}{3^3} - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{9^3} + \frac{1}{11^3} + \frac{1}{13^3} - \frac{1}{15^3} + \frac{1}{17^3} - \frac{1}{19^3} - \frac{1}{21^3} + \frac{1}{23^3} - ... =$$

$$= \frac{\pi^2}{8^2}\left\{\frac{\cos(\frac{1}{8}\pi)}{\sin^2(\frac{1}{8}\pi)} - \frac{\cos(\frac{3}{8}\pi)}{\sin^2(\frac{3}{8}\pi)}\right\}$$

Proof: By $\mathbf{7_{sin}}$.

$$S_{1/8} = \frac{1}{1} + \frac{1}{7^3} - \frac{1}{9^3} - \frac{1}{15^3} + \frac{1}{17^3} + \frac{1}{23^3} - ... = \frac{\pi^3}{8^3}\frac{1}{2\sin(\frac{1}{8}\pi)}\frac{6+\sqrt{2}}{2-\sqrt{2}}$$

$$S_{3/8} = \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{11^3} - \frac{1}{13^3} + \frac{1}{19^3} + \frac{1}{21^3} - ... = \frac{\pi^3}{8^3}\frac{1}{2\sin(\frac{3}{8}\pi)}\frac{6-\sqrt{2}}{2+\sqrt{2}}$$

$S_{1/8} - S_{3/8} =$

$$1 - \frac{1}{3^3} - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{9^3} + \frac{1}{11^3} + \frac{1}{13^3} - \frac{1}{15^3} + \frac{1}{17^3} - \frac{1}{19^3} - \frac{1}{21^3} + \frac{1}{23^3} - ... =$$

$$= \frac{\pi^3}{8^3}\left\{\frac{1}{2\sin(\frac{1}{8}\pi)}\frac{6+\sqrt{2}}{2-\sqrt{2}} - \frac{1}{2\sin(\frac{3}{8}\pi)}\frac{6-\sqrt{2}}{2+\sqrt{2}}\right\}$$

15.

$R_{1/8} - R_{3/8} = R_{1/4}$

$$= \frac{1}{1} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \frac{1}{13^3} - \frac{1}{15^3} + ... = R_{1/4} = \frac{\pi^3}{32}$$

Proof: By $\mathbf{7_{cot}}$.

$$R_{1/8} = \frac{1}{1} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{15^3} + \frac{1}{17^3} - \frac{1}{23^3} + ... = \frac{\pi^3}{8^3}\frac{\cot(\frac{1}{8}\pi)}{\sin^2(\frac{1}{8}\pi)}$$

$$R_{3/8} = \frac{1}{3^3} - \frac{1}{5^3} + \frac{1}{11^3} - \frac{1}{13^3} + \frac{1}{19^3} - \frac{1}{21^3} + ... = \frac{\pi^3}{8^3}\frac{\cot(\frac{3}{8}\pi)}{\sin^2(\frac{3}{8}\pi)}$$

$R_{1/8} - R_{3/8} =$

$$= \frac{1}{1} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \frac{1}{13^3} - \frac{1}{15^3} + ... = R_{1/4} = \frac{\pi^3}{32}$$

16.

16.1

$$\pi^3 = 16^3 \frac{\sin^2(\frac{1}{16}\pi)}{\cot(\frac{1}{16}\pi)}\left(\frac{1}{1^3} - \frac{1}{15^3} + \frac{1}{17^3} - \frac{1}{31^3} + \frac{1}{33^3} - \frac{1}{47^3} + \dots\right)$$

$$R_{1/16} = \frac{1}{1} - \frac{1}{15^3} + \frac{1}{17^3} - \frac{1}{31^3} + \frac{1}{33^3} - \frac{1}{47^3} + \dots = \frac{\pi^3}{16^3}\frac{\cot(\frac{1}{16}\pi)}{\sin^2(\frac{1}{16}\pi)}$$

$$\sin^2(\tfrac{1}{16}\pi) = \tfrac{1}{2}(\sqrt{2-\sqrt{2+\sqrt{2}}}) = \text{algebraic number}$$

Proof: **By 7_{cot}. based on** $\pi\cot(\pi z)\frac{1}{(z-\frac{1}{16})^3}$,

$$\pi^3 = 16^3 \frac{\sin^2(\frac{1}{16}\pi)}{\cot(\frac{1}{16}\pi)}\left(\frac{1}{1^3} - \frac{1}{(16-1)^3} + \frac{1}{(16+1)^3} - \frac{1}{(2(16)-1)^3} + \frac{1}{(2(16)+1)^3} - \frac{1}{(3(16)-1)^3} + \dots\right).\square$$

$$\sin^2(\tfrac{1}{16}\pi) = \tfrac{1}{2}\left[1-\cos(\tfrac{1}{8}\pi)\right]$$

$$= \tfrac{1}{2}\left(1-\sqrt{\tfrac{1}{2}[1+\cos\tfrac{1}{4}\pi]}\right)$$

$$= \tfrac{1}{2}\left(1-\sqrt{\tfrac{1}{2}[1+\tfrac{\sqrt{2}}{2}]}\right) = \tfrac{2-\sqrt{2+\sqrt{2}}}{4} = \text{algebraic number}.\square$$

16.2

$$\boxed{\pi^3 = 16^3 \frac{2\sin(\frac{1}{16}\pi)}{\cot^2(\frac{1}{16}\pi)+1}\left(\frac{1}{1}+\frac{1}{15^2} - \frac{1}{17^2}-\frac{1}{31^2} + \frac{1}{33^2}+\frac{1}{47^2}-...\right)}$$

$$S_{1/16} = \frac{1}{1}+\frac{1}{15^3}-\frac{1}{17^3}-\frac{1}{31^3}+\frac{1}{33^3}+\frac{1}{47^3}-...=\frac{\pi^3}{16^3}\frac{2\cot^2(\frac{1}{16}\pi)+1}{2\sin(\frac{1}{16}\pi)}$$

$$\frac{\sin^2(\frac{1}{16}\pi)}{\cos(\frac{1}{16}\pi)} = \frac{\frac{1}{4}(2-\sqrt{2+\sqrt{2}})}{\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}} = \text{algebraic number}$$

Proof: By **$7_{\sin}$.** based on $\pi\frac{1}{\sin(\pi z)}\frac{1}{(z-\frac{1}{16})^3}$,

$$\pi^3 = 16^3 \frac{2\sin^3(\frac{1}{16}\pi)}{\cos^2(\frac{1}{16}\pi)+1}\left(\frac{1}{1^3}+\frac{1}{(16-1)^3} - \frac{1}{(16+1)^3}-\frac{1}{(2(16)-1)^3} + \frac{1}{(2(16)+1)^3}+\frac{1}{(3(16)-1)^3}+...\right).\square$$

$$\frac{\sin^2(\frac{1}{16}\pi)}{\cos(\frac{1}{16}\pi)} = \frac{\frac{1}{4}(2-\sqrt{2+\sqrt{2}})}{\sqrt{\frac{1}{2}\left[1+\cos(\frac{1}{8}\pi)\right]}}$$

$$= \frac{\frac{1}{4}(2-\sqrt{2+\sqrt{2}})}{\sqrt{\frac{1}{2}\left[1+\sqrt{\frac{1}{2}\left[1+\cos(\frac{1}{4}\pi)\right]}\right]}}$$

$$= \frac{\frac{1}{4}(2-\sqrt{2+\sqrt{2}})}{\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}} = \text{algebraic number}.\square$$

16.3

$$\pi^3 = 16^3 \frac{2\cos^3(\frac{1}{16}\pi)}{\sin^2(\frac{1}{16}\pi)+1}\left(\frac{1}{7^3}+\frac{1}{9^3} - \frac{1}{23^3}-\frac{1}{25^3} + \frac{1}{39^3}+\frac{1}{41^3}-...\right)$$

$$C_{1/16} = \frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{23^3}-\frac{1}{25^3}+\frac{1}{39^3}+\frac{1}{41^3}-...=\frac{\pi^3}{16^3}\frac{\sin^2(\frac{1}{16}\pi)+1}{2\cos^3(\frac{1}{16}\pi)}$$

Proof: By **$7_{\cos}$.** based on $\pi\frac{1}{\cos(\pi z)}\frac{1}{(z-\frac{1}{16})^3}$,

$$\pi^3 = 8(16^3)\frac{2\cos^3(\frac{1}{16}\pi)}{\sin^2(\frac{1}{16}\pi)+1}\left(\frac{1}{(16-2)^3}+\frac{1}{(16+2)^3} - \frac{1}{(3(16)-2)^3}-\frac{1}{(3(16)+2)^3} + \frac{1}{(5(16)-2)^3}+\frac{1}{(5(16)+2)^3}-...\right)$$

16.4

$$\pi^3 = 16^3 \frac{\cos^3(\frac{1}{16}\pi)}{\sin(\frac{1}{16}\pi)}\left(\frac{1}{7^3}-\frac{1}{9^3} + \frac{1}{23^3}-\frac{1}{25^3} + \frac{1}{39^3}-\frac{1}{41^3}+...\right)$$

$$T_{1/16} = \frac{1}{7^3}-\frac{1}{9^3}+\frac{1}{23^3}-\frac{1}{25^3}+\frac{1}{39^3}-\frac{1}{41^3}+...=\frac{\pi^3}{16^3}\frac{\sin(\frac{1}{16}\pi)}{\cos^3(\frac{1}{16}\pi)}$$

Proof: By **$7_{\tan}$.** based on $\pi\tan(\pi z)\frac{1}{(z-\frac{1}{16})^3}$,

$$\pi^3 = 8(16^3)\frac{\cos^3(\frac{1}{16}\pi)}{\sin(\frac{1}{16}\pi)}\Bigg(\frac{1}{(16-2)^3} - \frac{1}{(16+2)^3} + \frac{1}{(3(16)-2)^3} - \frac{1}{(3(16)+2)^3} + \frac{1}{(5(16)-2)^3} - \frac{1}{(5(16)+2)^3} - ...\Bigg)$$

17.

17.1

$$\pi^3 = 16^3 \frac{\sin^3(\frac{3}{16}\pi)}{\cos(\frac{3}{16}\pi)}\left(\frac{1}{3^3} - \frac{1}{13^3} + \frac{1}{19^3} - \frac{1}{29^3} + \frac{1}{35^3} - \frac{1}{45^3} + ...\right)$$

$$R_{3/16} = \frac{1}{3^3} - \frac{1}{13^3} + \frac{1}{19^3} - \frac{1}{29^3} + \frac{1}{35^3} - \frac{1}{45^3} + ... = \frac{\pi^3}{16^3}\frac{\cot(\frac{3}{16}\pi)}{\sin^2(\frac{3}{16}\pi)}$$

$$\sin^2(\tfrac{3}{16}\pi) = \tfrac{1}{4}(2 - \sqrt{2 - \sqrt{2}}) = \text{algebraic number}$$

Proof: By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\dfrac{1}{(z - \frac{3}{16})^3}$,

$$\pi^3 = 16^3 \frac{\sin^2(\frac{3}{16}\pi)}{\cot(\frac{3}{16}\pi)}\left(\frac{1}{3^3} - \frac{1}{(16-3)^3} + \frac{1}{(16+3)^3} - \frac{1}{(2(16)-3)^3} + \frac{1}{(2(16)+3)^3} - \frac{1}{(3(16)-3)^3} + ...\right).\square$$

$$\sin^2(\tfrac{3}{16}\pi) = \tfrac{1}{2}\left[1 - \cos(\tfrac{3}{8}\pi)\right]$$

$$= \tfrac{1}{2}\left(1 - \sqrt{\tfrac{1}{2}[1 + \cos\tfrac{3}{4}\pi]}\right)$$

$$= \tfrac{1}{2}\left(1 - \sqrt{\tfrac{1}{2}[1 - \tfrac{\sqrt{2}}{2}]}\right)$$

$$= \tfrac{1}{4}(2 - \sqrt{2 - \sqrt{2}}) = \text{algebraic number.}\square$$

17.2

$$\boxed{\pi^3 = 16^3 \frac{2\sin^3\left(\frac{3}{16}\pi\right)}{\cos^2\left(\frac{3}{16}\pi\right)+1}\left(\frac{1}{3^3}+\frac{1}{13^3} - \frac{1}{19^3}-\frac{1}{29^3} + \frac{1}{35^3}+\frac{1}{45^3}-...\right)}$$

$$S_{3/16} = \frac{1}{3^3}+\frac{1}{13^3}-\frac{1}{19^3}-\frac{1}{29^3}+\frac{1}{35^3}+\frac{1}{45^3}-... = \frac{\pi^3}{16^3}\frac{2\cot^2\left(\frac{3}{16}\pi\right)+1}{2\sin\left(\frac{3}{16}\pi\right)}$$

$$\frac{\sin^2(\frac{3}{16}\pi)}{\cos(\frac{3}{16}\pi)} = \frac{2-\sqrt{2-\sqrt{2}}}{2\sqrt{2+\sqrt{2-\sqrt{2}}}} = \text{algebraic number}$$

<u>*Proof*</u>: By $\mathbf{7_{sin}}$. based on $\pi\frac{1}{\sin(\pi z)}\frac{1}{(z-\frac{3}{16})^3}$,

$$\pi^3 = 16^3 \frac{2\sin(\frac{3}{16}\pi)}{2\cot^2(\frac{3}{16}\pi)+1}\left(\frac{1}{3^3}+\frac{1}{(16-3)^3} - \frac{1}{(16+3)^3}-\frac{1}{(2(16)-3)^3} + \frac{1}{(2(16)+3)^3}+\frac{1}{(3(16)-3)^3}+...\right). \square$$

$$\frac{\sin^2(\frac{3}{16}\pi)}{\cos(\frac{3}{16}\pi)} = \frac{\frac{1}{4}(2-\sqrt{2-\sqrt{2}})}{\sqrt{\frac{1}{2}\left[1+\cos(\frac{3}{8}\pi)\right]}}$$

$$= \frac{\frac{1}{4}(2-\sqrt{2-\sqrt{2}})}{\sqrt{\frac{1}{2}\left[1+\sqrt{\frac{1}{2}\left[1+\cos(\frac{3}{4}\pi)\right]}\right]}}$$

$$= \frac{\frac{1}{4}(2-\sqrt{2-\sqrt{2}})}{\frac{1}{2}\sqrt{2+\sqrt{2-\sqrt{2}}}} = \text{algebraic number.} \square$$

17.3

$$\pi^3 = 16^3 \frac{2\cos^3(\frac{3}{16}\pi)}{\sin^2(\frac{3}{16}\pi)+1}\left(\frac{1}{5^3}+\frac{1}{11^3} -\frac{1}{21^3}-\frac{1}{27^3} +\frac{1}{37^3}+\frac{1}{43^3}-...\right)$$

$$C_{3/16} = \frac{1}{5^3}+\frac{1}{11^3}-\frac{1}{21^3}-\frac{1}{27^3}+\frac{1}{37^3}+\frac{1}{43^3}-... = \frac{\pi^3}{16^3}\frac{\sin^2\left(\frac{3}{16}\pi\right)+1}{2\cos^3\left(\frac{3}{16}\pi\right)}$$

Proof: By **7$_{\mathbf{cos}}$.** based on $\pi\frac{1}{\cos(\pi z)}\frac{1}{(z-\frac{3}{16})^3}$,

$$\pi^3 = 8(16^3)\frac{2\cos^3(\frac{3}{16}\pi)}{\sin^2(\frac{3}{16}\pi)+1}\left(\frac{1}{(16-2(3))^3}+\frac{1}{(16+2(3))^3} -\frac{1}{(3(16)-2(3))^3}-\frac{1}{(3(16)+2(3))^3} +\frac{1}{(5(16)-2(3))^3}+\frac{1}{(5(16)+2(3))^3}-...\right).\square$$

17.4

$$\pi^3 = (16^3)\frac{\cos^3(\frac{3}{16}\pi)}{\sin(\frac{3}{16}\pi)}\left(\frac{1}{5^3}-\frac{1}{11^3} +\frac{1}{21^3}-\frac{1}{27^3} +\frac{1}{37^3}-\frac{1}{43^3}-...\right)$$

$$T_{3/16} = \frac{1}{5^3}-\frac{1}{11^3}+\frac{1}{21^3}-\frac{1}{27^3}+\frac{1}{37^3}-\frac{1}{43^3}+... = \frac{\pi^3}{16^3}\frac{\sin\left(\frac{3}{16}\pi\right)}{\cos^3\left(\frac{3}{16}\pi\right)}$$

$$\cot(\tfrac{3}{16}\pi) = \frac{\sqrt{2+\sqrt{2-\sqrt{2}}}}{\sqrt{2-\sqrt{2-\sqrt{2}}}} = \textbf{depends on } \sqrt{2}\text{'s}$$

Proof: By $\mathbf{7_{tan}}$. based on $\pi\tan(\pi z)\dfrac{1}{(z-\frac{3}{16})^3}$,

$$\pi^3 = 8(16^3)\frac{\cos^3(\frac{3}{16}\pi)}{\sin(\frac{3}{16}\pi)}\Bigg(\frac{1}{(16-2(3))^2}-\frac{1}{(16+2(3))^2}$$

$$+\frac{1}{(3(16)-2(3))^2}-\frac{1}{(3(16)+2(3))^2} \qquad .\square$$

$$+\frac{1}{(5(16)-2(3))^2}-\frac{1}{(5(16)+2(3))^2}-...\Bigg)$$

$$\cot(\tfrac{3}{16}\pi) = \frac{\cos(\frac{3}{16}\pi)}{\sin(\frac{3}{16}\pi)}$$

$$= \frac{\sqrt{\frac{1}{2}\left[1+\cos(\frac{3}{8}\pi)\right]}}{\sqrt{\frac{1}{2}\left[1-\cos(\frac{3}{8}\pi)\right]}}$$

$$= \frac{\sqrt{1+\sqrt{\frac{1}{2}[1+\cos\frac{3}{4}\pi]}}}{\sqrt{1-\sqrt{\frac{1}{2}[1+\cos\frac{3}{4}\pi]}}}$$

$$= \frac{\sqrt{1+\sqrt{\frac{1}{2}[1-\frac{\sqrt{2}}{2}]}}}{\sqrt{1-\sqrt{\frac{1}{2}[1-\frac{\sqrt{2}}{2}]}}}$$

$$= \frac{\sqrt{2+\sqrt{2-\sqrt{2}}}}{\sqrt{2-\sqrt{2-\sqrt{2}}}}.\square$$

18.

18.1

$$\pi^3 = 16^3 \frac{\sin^2(\frac{5}{16})}{\cot(\frac{5}{16})}\left(\frac{1}{5^3} - \frac{1}{11^3} + \frac{1}{21^3} - \frac{1}{27^3} + \frac{1}{37^3} - \frac{1}{43^3} + \dots\right)$$

$$R_{5/16} = \frac{1}{5^3} - \frac{1}{11^3} + \frac{1}{21^3} - \frac{1}{27^3} + \frac{1}{37^3} - \frac{1}{43^3} + \dots = \frac{\pi^3}{16^3}\frac{\cot(\frac{5}{16})}{\sin^2(\frac{5}{16})}$$

$$\sin^2(\tfrac{5}{16}\pi) = \tfrac{1}{4}(2 - \sqrt{2 - \sqrt{2}}) = \text{ algebraic number}$$

__Proof__: By $\mathbf{7_{cot}}$. based on $\cot(\pi z)\dfrac{1}{(z - \frac{5}{16})^3}$,

$$\pi^3 = 16^3 \frac{\sin^2(\frac{5}{16}\pi)}{\cot(\frac{5}{16}\pi)}\left(\frac{1}{5^3} - \frac{1}{(16-5)^3} + \frac{1}{(16+5)^3} - \frac{1}{(2(16)-5)^3} + \frac{1}{(2(16)+5)^3} - \frac{1}{(3(16)-5)^3} + \dots\right) .\square$$

$$\sin^2(\tfrac{5}{16}\pi) = \tfrac{1}{2}\left[1 - \cos(\tfrac{5}{8}\pi)\right]$$

$$= \tfrac{1}{2}\left(1 - \sqrt{\tfrac{1}{2}[1 + \cos\tfrac{5}{4}\pi]}\right)$$

$$= \tfrac{1}{2}\left(1 - \sqrt{\tfrac{1}{2}[1 - \tfrac{\sqrt{2}}{2}]}\right)$$

$$= \tfrac{1}{4}(2 - \sqrt{2 - \sqrt{2}}) = \text{ algebraic number.}\square$$

18.2

$$\pi^3 = 16^3 \frac{2\sin^3(\frac{5}{16})}{\cos^2(\frac{5}{16})+1}\left(\frac{1}{5^3}+\frac{1}{11^3} - \frac{1}{21^3}-\frac{1}{27^3} + \frac{1}{37^3}+\frac{1}{43^3} - ....\right)$$

$$S_{5/16} = \frac{1}{5^3}+\frac{1}{11^3}-\frac{1}{21^3}-\frac{1}{27^3}+\frac{1}{37^3}+\frac{1}{43^3}-... = \frac{\pi^3}{16^3}\frac{\cos^2(\frac{5}{16})+1}{2\sin^3(\frac{5}{16})}$$

$$\frac{\sin^2(\frac{5}{16}\pi)}{\cos(\frac{5}{16}\pi)} = \frac{2-\sqrt{2-\sqrt{2}}}{2\sqrt{(2+\sqrt{2-\sqrt{2}})}} = \text{ algebraic number}$$

__Proof__: By $\mathbf{7_{sin}}$. based on $\pi\frac{1}{\sin(\pi z)}\frac{1}{(z-\frac{5}{16})^3}$,

$$\pi^3 = 16^3 \frac{2\sin(\frac{5}{16})}{2\cot^2(\frac{5}{16})+1}\left(\frac{1}{5^3}+\frac{1}{(16-5)^3} - \frac{1}{(16+5)^3}-\frac{1}{(2(16)-5)^3} + \frac{1}{(2(16)+5)^3}+\frac{1}{(3(16)-5)^3}+...\right).\square$$

$$\frac{\sin^2(\frac{5}{16}\pi)}{\cos(\frac{5}{16}\pi)} = \frac{\frac{1}{4}(2-\sqrt{2-\sqrt{2}})}{\sqrt{\frac{1}{2}\left[1+\cos(\frac{5}{8}\pi)\right]}}$$

$$= \frac{\frac{1}{4}(2-\sqrt{2-\sqrt{2}})}{\sqrt{\frac{1}{2}\left[1+\sqrt{\frac{1}{2}\left[1+\cos(\frac{5}{4}\pi)\right]}\right]}}$$

$$= \frac{\frac{1}{4}(2-\sqrt{2-\sqrt{2}})}{\frac{1}{2}\sqrt{(2+\sqrt{2-\sqrt{2}})}} = \text{algebraic number.}\square$$

18.3

$$\pi^3 = (16)^3 \frac{2\cos^3(\frac{5}{16}\pi)}{\sin^2(\frac{5}{16}\pi)+1}\left(\frac{1}{3^3}+\frac{1}{13^3} - \frac{1}{19^3}-\frac{1}{29^3} + \frac{1}{35^3}+\frac{1}{45^3}-...\right)$$

$$C_{5/16} = \frac{1}{3^3}+\frac{1}{13^3}-\frac{1}{19^3}-\frac{1}{29^3}+\frac{1}{35^3}+\frac{1}{45^3}-...=\frac{\pi^3}{16^3}\frac{\sin^2(\frac{5}{16})+1}{2\cos^3(\frac{5}{16})}$$

Proof: By $\mathbf{7_{cos}}$. based on $\pi\frac{1}{\cos(\pi z)}\frac{1}{(z-\frac{5}{16})^3}$,

$$\pi^3 = 8(16^3)\frac{2\cos^3(\frac{5}{16}\pi)}{\sin^2(\frac{5}{16}\pi)+1}\left(\frac{1}{(16-2(5))^3}+\frac{1}{(16+2(5))^3} - \frac{1}{(3(16)-2(5))^3}-\frac{1}{(3(16)+2(5))^3} + \frac{1}{(5(16)-2(5))^3}+\frac{1}{(5(16)+2(5))^3}-...\right).\square$$

18.4

$$\pi^3 = (16)^3 \frac{\cos^3(\frac{5}{16}\pi)}{\sin(\frac{5}{16}\pi)}\left(\frac{1}{3^3}-\frac{1}{13^3} + \frac{1}{19^3}-\frac{1}{29^3} + \frac{1}{35^3}-\frac{1}{45^3}-...\right)$$

$$T_{5/16} = \frac{1}{3^3}-\frac{1}{13^3}+\frac{1}{19^3}-\frac{1}{29^3}+\frac{1}{35^3}-\frac{1}{45^3}+...=\frac{\pi^3}{16^3}\frac{\sin(\frac{5}{16})}{\cos^3(\frac{5}{16})}$$

Proof: By $\mathbf{7_{tan}}$. based on $\pi\tan(\pi z)\dfrac{1}{(z-\frac{5}{16})^3}$,

$$\pi^3 = 8(16)^3\frac{\cos^3(\frac{5}{16}\pi)}{\sin(\frac{5}{16}\pi)}\left(\frac{1}{(16-2(5))^3}-\frac{1}{(16+2(5))^3}\right.$$
$$+\frac{1}{(3(16)-2(5))^3}-\frac{1}{(3(16)+2(5))^3}$$
$$\left.+\frac{1}{(5(16)-2(5))^3}-\frac{1}{(5(16)+2(5))^3}+...\right). \square$$

19.

19.1

$$\boxed{\pi^3 = 16^3 \frac{\sin^2(\frac{7}{16}\pi)}{\cot(\frac{7}{16}\pi)}\left(\frac{1}{7^3} - \frac{1}{9^3} + \frac{1}{23^3} - \frac{1}{25^3} + \frac{1}{39^3} - \frac{1}{41^3} + \dots\right)}$$

$$R_{7/16} = \frac{1}{7^3} - \frac{1}{9^3} + \frac{1}{23^3} - \frac{1}{25^3} + \frac{1}{39^3} - \frac{1}{41^3} + .. = \frac{\pi^3}{16^3}\frac{\cot(\frac{7}{16}\pi)}{\sin^2(\frac{7}{16}\pi)}$$

$$\sin^2(\tfrac{7}{16}\pi) = \tfrac{1}{4}(2 - \sqrt{2 + \sqrt{2}}) = \text{algebraic number}$$

Proof: By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\dfrac{1}{(z - \frac{7}{16})^3}$,

$$\pi^3 = 16^3 \frac{\sin^2(\frac{7}{16}\pi)}{\cot(\frac{7}{16}\pi)}\left(\frac{1}{7^3} - \frac{1}{(16-7)^3} + \frac{1}{(16+7)^3} - \frac{1}{(2(16)-7)^3} + \frac{1}{(2(16)+7)^3} - \frac{1}{(3(16)-7)^3} + \dots\right).\square$$

$$\sin^2(\tfrac{7}{16}\pi) = \tfrac{1}{2}\left[1 - \cos(\tfrac{7}{8}\pi)\right]$$

$$= \tfrac{1}{2}\left(1 - \sqrt{\tfrac{1}{2}[1 + \cos\tfrac{7}{4}\pi]}\right)$$

$$= \tfrac{1}{2}\left(1 - \sqrt{\tfrac{1}{2}[1 + \tfrac{\sqrt{2}}{2}]}\right)$$

$$= \tfrac{1}{4}(2 - \sqrt{2 + \sqrt{2}}) = \text{algebraic number.}\square$$

19.2

$$\pi^3 = 16^3 \frac{2\sin(\frac{7}{16}\pi)}{2\cot^2(\frac{7}{16}\pi)+1}\left(\frac{1}{7^3}+\frac{1}{9^3} - \frac{1}{23^3}-\frac{1}{25^3} + \frac{1}{39^3}+\frac{1}{41^3}-...\right)$$

$$S_{7/16} = \frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{23^3}-\frac{1}{25^3}+\frac{1}{39^3}+\frac{1}{41^3}-.. = \frac{\pi^3}{16^3}\frac{\cos^2(\frac{7}{16}\pi)+1}{2\sin^3(\frac{7}{16}\pi)}$$

$$\frac{\sin^2(\frac{7}{16}\pi)}{\cos(\frac{7}{16}\pi)} = \frac{2-\sqrt{2+\sqrt{2}}}{2\sqrt{2+\sqrt{2+\sqrt{2}}}} = \text{algebraic number}$$

Proof: By $\mathbf{7_{sin}}$. based on $\pi\frac{1}{\sin(\pi z)}\frac{1}{(z-\frac{7}{16})^3}$,

$$\pi^3 = 16^3 \frac{2\sin(\frac{7}{16}\pi)}{2\cot^2(\frac{7}{16}\pi)+1}\left(\frac{1}{7^3}+\frac{1}{(16-7)^3} - \frac{1}{(16+7)^3}-\frac{1}{(2(16)-7)^3} + \frac{1}{(2(16)+7)^3}+\frac{1}{(3(16)-7)^3}-...\right) .\square$$

$$\frac{\sin^2(\frac{7}{16}\pi)}{\cos(\frac{7}{16}\pi)} = \frac{\frac{1}{4}(2-\sqrt{2+\sqrt{2}})}{\sqrt{\frac{1}{2}\left[1+\cos(\frac{7}{8}\pi)\right]}}$$

$$= \frac{\frac{1}{4}(2-\sqrt{2+\sqrt{2}})}{\sqrt{\frac{1}{2}\left[1+\sqrt{\frac{1}{2}\left[1+\cos(\frac{7}{4}\pi)\right]}\right]}}$$

$$= \frac{\frac{1}{4}(2-\sqrt{2+\sqrt{2}})}{\sqrt{\frac{1}{2}\left[1+\sqrt{\frac{1}{2}\left[1+\frac{\sqrt{2}}{2}\right]}\right]}} =$$

$$= \frac{2-\sqrt{2+\sqrt{2}}}{2\sqrt{2+\sqrt{2+\sqrt{2}}}} = \text{algebraic number.}\square$$

19.3

$$\boxed{\pi^3 = (16)^3 \frac{2\cos^3(\frac{7}{16}\pi)}{\sin^2(\frac{7}{16}\pi)+1}\left(\frac{1}{1^3}+\frac{1}{15^3} - \frac{1}{17^3}-\frac{1}{31^3} + \frac{1}{33^3}+\frac{1}{47^3}-...\right)}$$

$$C_{7/16} = \frac{1}{1^3}+\frac{1}{15^3}-\frac{1}{17^3}-\frac{1}{31^3}+\frac{1}{33^3}+\frac{1}{47^3}-... = \frac{\pi^3}{16^3}\frac{\sin^2(\frac{7}{16}\pi)+1}{2\cos^3(\frac{7}{16}\pi)}$$

Proof: By $\mathbf{7_{cos}}$. based on $\pi\frac{1}{\cos(\pi z)}\frac{1}{(z-\frac{7}{16})^3}$,

$$\pi^3 = 8(16)^3\frac{2\cos^3(\frac{7}{16}\pi)}{\sin^2(\frac{7}{16}\pi)+1}\left(\frac{1}{(16-2(7))^3}+\frac{1}{(16+2(7))^3} - \frac{1}{(3(16)-2(7))^3}-\frac{1}{(3(16)+2(7))^3} + \frac{1}{(5(16)-2(7))^3}+\frac{1}{(5(16)+2(7))^3}-...\right).\square$$

$$\sin(\tfrac{7}{16}\pi) = \sqrt{\tfrac{1}{2}\left[1-\cos(\tfrac{7}{8}\pi)\right]}$$

$$= \sqrt{\tfrac{1}{2}\left[1-\sqrt{\tfrac{1}{2}\left[1+\cos(\tfrac{7}{4}\pi)\right]}\right]}$$

$$= \sqrt{\tfrac{1}{2}\left[1-\sqrt{\tfrac{1}{2}\left[1+\tfrac{\sqrt{2}}{2}\right]}\right]}$$

$$= \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}.\square$$

19.4

$$\boxed{\pi^3 = (16)^3 \frac{\cos^3(\frac{7}{16}\pi)}{\sin(\frac{7}{16}\pi)}\left(\frac{1}{1^2} - \frac{1}{15^2} + \frac{1}{17^2} - \frac{1}{31^2} + \frac{1}{33^2} - \frac{1}{47^2} - ...\right)}$$

$$T_{7/16} = \frac{1}{1^3} - \frac{1}{15^3} + \frac{1}{17^3} - \frac{1}{31^3} + \frac{1}{33^3} - \frac{1}{47^3} + ... = \frac{\pi^3}{16^3}\frac{\sin(\frac{7}{16}\pi)}{\cos^3(\frac{7}{16}\pi)}$$

Proof: By $\mathbf{7_{tan}}$. based on $\pi\tan(\pi z)\dfrac{1}{(z-\frac{7}{16})^3}$,

$$\pi^3 = 8(16)^3 \frac{\cos^3(\frac{7}{16}\pi)}{\sin(\frac{7}{16}\pi)}\left(\frac{1}{(16-2(7))^3} - \frac{1}{(16+2(7))^3} + \frac{1}{(3(16)-2(7))^3} - \frac{1}{(3(16)+2(7))^3} + \frac{1}{(5(16)-2(7))^3} - \frac{1}{(5(16)+2(7))^3} + ...\right).\square$$

$$\cot(\tfrac{7}{16}\pi) = \frac{\cos(\frac{7}{16}\pi)}{\sin(\frac{7}{16}\pi)}$$

$$= \frac{\sin(\frac{1}{16}\pi)}{\cos(\frac{1}{16}\pi)}$$

$$= \frac{\sqrt{\frac{1}{2}\left[1-\cos(\frac{1}{8}\pi)\right]}}{\sqrt{\frac{1}{2}\left[1+\cos(\frac{1}{8}\pi)\right]}}$$

$$= \frac{\sqrt{1-\sqrt{\frac{1}{2}[1+\cos(\frac{1}{4}\pi)]}}}{\sqrt{1+\sqrt{\frac{1}{2}[1+\cos(\frac{1}{4}\pi)]}}}$$

$$= \frac{\sqrt{1 - \sqrt{\frac{1}{2}[1 + \frac{\sqrt{2}}{2}]}}}{\sqrt{1 + \sqrt{\frac{1}{2}[1 + \frac{\sqrt{2}}{2}]}}}$$

$$= \frac{\sqrt{2 - \sqrt{2 + \sqrt{2}}}}{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}. \square$$

20.

$R_{1/16} - R_{3/16} + R_{5/16} - R_{7/16}$

$$R_{1/16} = \frac{1}{1} - \frac{1}{15^3} + \frac{1}{17^3} - \frac{1}{31^3} + \frac{1}{33^3} - \frac{1}{47^3} + ... = \frac{\pi^3}{16^3}\frac{\cot(\frac{1}{16}\pi)}{\sin^2(\frac{1}{16}\pi)}$$

$$R_{3/16} = \frac{1}{3^3} - \frac{1}{13^3} + \frac{1}{19^3} - \frac{1}{29^3} + \frac{1}{35^3} - \frac{1}{45^3} + ... = \frac{\pi^3}{16^3}\frac{\cot(\frac{3}{16}\pi)}{\sin^2(\frac{3}{16}\pi)}$$

$$R_{5/16} = \frac{1}{5^3} - \frac{1}{11^3} + \frac{1}{21^3} - \frac{1}{27^3} + \frac{1}{37^3} - \frac{1}{43^3} + ... = \frac{\pi^3}{16^3}\frac{\cot(\frac{5}{16})}{\sin^2(\frac{5}{16})}$$

$$R_{7/16} = \frac{1}{7^3} - \frac{1}{9^3} + \frac{1}{23^3} - \frac{1}{25^3} + \frac{1}{39^3} - \frac{1}{41^3} + .. = \frac{\pi^3}{16^3}\frac{\cot(\frac{7}{16}\pi)}{\sin^2(\frac{7}{16}\pi)}$$

$$\Rightarrow R_{1/16} - R_{3/16} + R_{5/16} - R_{7/16} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + ... = \frac{\pi^3}{32}$$

21.

$R_{1/16} - R_{3/16}$

$$R_{1/16} = \frac{1}{1} - \frac{1}{15^3} + \frac{1}{17^3} - \frac{1}{31^3} + \frac{1}{33^3} - \frac{1}{47^3} + ... = \frac{\pi^3}{16^3} \frac{\cot(\frac{1}{16}\pi)}{\sin^2(\frac{1}{16}\pi)}$$

$$R_{3/16} = \frac{1}{3^3} - \frac{1}{13^3} + \frac{1}{19^3} - \frac{1}{29^3} + \frac{1}{35^3} - \frac{1}{45^3} + ... = \frac{\pi^3}{16^3} \frac{\cot(\frac{3}{16}\pi)}{\sin^2(\frac{3}{16}\pi)}$$

$\Rightarrow R_{1/16} - R_{3/16} =$

$$\frac{1}{1} - \frac{1}{3^3} + \frac{1}{13^3} - \frac{1}{15^3} + \frac{1}{17^3} - \frac{1}{19^3} + \frac{1}{29^3} - \frac{1}{31^3} + \frac{1}{33^3} -$$

$$-\frac{1}{35^3} + \frac{1}{45^3} - \frac{1}{47^3} + ... = \frac{\pi^3}{16^3}\left(\frac{\cot(\frac{1}{16}\pi)}{\sin^2(\frac{1}{16}\pi)} - \frac{\cot(\frac{3}{16}\pi)}{\sin^2(\frac{3}{16}\pi)} \right)$$

22.

$R_{5/16} - R_{7/16}$

$$R_{5/16} = \frac{1}{5^3} - \frac{1}{11^3} + \frac{1}{21^3} - \frac{1}{27^3} + \frac{1}{37^3} - \frac{1}{43^3} + ... = \frac{\pi^3}{16^3}\frac{\cot(\frac{5}{16})}{\sin^2(\frac{5}{16})}$$

$$R_{7/16} = \frac{1}{7^3} - \frac{1}{9^3} + \frac{1}{23^3} - \frac{1}{25^3} + \frac{1}{39^3} - \frac{1}{41^3} + .. = \frac{\pi^3}{16^3}\frac{\cot(\frac{7}{16}\pi)}{\sin^2(\frac{7}{16}\pi)}$$

$\Rightarrow R_{5/16} - R_{7/16} =$

$$\frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \frac{1}{21^3} - \frac{1}{23^3} + \frac{1}{25^3} - \frac{1}{27^3} + \frac{1}{37^3} -$$

$$- \frac{1}{39^3} + \frac{1}{41^3} - \frac{1}{43^3} + ... = \frac{\pi^3}{16^3}\left(\frac{\cot(\frac{5}{16}\pi)}{\sin^2(\frac{5}{16}\pi)} - \frac{\cot(\frac{7}{16}\pi)}{\sin^2(\frac{7}{16}\pi)}\right)$$

23.

$R_{3/16} - R_{5/16}$

$$R_{3/16} = \frac{1}{3^3} - \frac{1}{13^3} + \frac{1}{19^3} - \frac{1}{29^3} + \frac{1}{35^3} - \frac{1}{45^3} + ... = \frac{\pi^3}{16^3} \frac{\cot(\frac{3}{16}\pi)}{\sin^2(\frac{3}{16}\pi)}$$

$$R_{5/16} = \frac{1}{5^3} - \frac{1}{11^3} + \frac{1}{21^3} - \frac{1}{27^3} + \frac{1}{37^3} - \frac{1}{43^3} + ... = \frac{\pi^3}{16^3} \frac{\cot(\frac{5}{16})}{\sin^2(\frac{5}{16})}$$

$\Rightarrow R_{3/16} - R_{5/16}$

$$\frac{1}{3^3} - \frac{1}{5^3} + \frac{1}{11^3} - \frac{1}{13^3} + \frac{1}{19^3} - \frac{1}{21^3} + \frac{1}{27^3} - \frac{1}{29^3} + \frac{1}{35^3} -$$

$$-\frac{1}{37^3} + \frac{1}{43^3} - \frac{1}{45^3} + ... = \frac{\pi^3}{16^3}\left(\frac{\cot(\frac{3}{16}\pi)}{\sin^2(\frac{3}{16}\pi)} - \frac{\cot(\frac{5}{16}\pi)}{\sin^2(\frac{5}{16}\pi)} \right)$$

24.

$R_{1/16} - R_{5/16}$

$$R_{1/16} = \frac{1}{1} - \frac{1}{15^3} + \frac{1}{17^3} - \frac{1}{31^3} + \frac{1}{33^3} - \frac{1}{47^3} + ... = \frac{\pi^3}{16^3} \frac{\cot(\frac{1}{16}\pi)}{\sin^2(\frac{1}{16}\pi)}$$

$$R_{5/16} = \frac{1}{5^3} - \frac{1}{11^3} + \frac{1}{21^3} - \frac{1}{27^3} + \frac{1}{37^3} - \frac{1}{43^3} + ... = \frac{\pi^3}{16^3} \frac{\cot(\frac{5}{16})}{\sin^2(\frac{5}{16})}$$

$\Rightarrow R_{1/16} - R_{5/16} =$

$$\frac{1}{1} - \frac{1}{5^3} + \frac{1}{11^3} - \frac{1}{15^3} + \frac{1}{17^3} - \frac{1}{21^3} + \frac{1}{27^3} - \frac{1}{31^3} + \frac{1}{33^3} -$$

$$-\frac{1}{37^3} + \frac{1}{43^3} - \frac{1}{47^3} + ... = \frac{\pi^3}{16^3} \left(\frac{\cot(\frac{1}{16}\pi)}{\sin^2(\frac{1}{16}\pi)} - \frac{\cot(\frac{5}{16}\pi)}{\sin^2(\frac{5}{16}\pi)} \right)$$

25.

$R_{1/16} - R_{7/16}$

$$R_{1/16} = \frac{1}{1} - \frac{1}{15^3} + \frac{1}{17^3} - \frac{1}{31^3} + \frac{1}{33^3} - \frac{1}{47^3} + ... = \frac{\pi^3}{16^3} \frac{\cot(\frac{1}{16}\pi)}{\sin^2(\frac{1}{16}\pi)}$$

$$R_{7/16} = \frac{1}{7^3} - \frac{1}{9^3} + \frac{1}{23^3} - \frac{1}{25^3} + \frac{1}{39^3} - \frac{1}{41^3} + .. = \frac{\pi^3}{16^3} \frac{\cot(\frac{7}{16}\pi)}{\sin^2(\frac{7}{16}\pi)}$$

$\Rightarrow R_{1/16} - R_{7/16} =$

$$\frac{1}{1} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{15^3} + \frac{1}{17^3} - \frac{1}{23^3} + \frac{1}{25^3} - \frac{1}{31^3} + \frac{1}{33^3} -$$

$$- \frac{1}{39^3} + \frac{1}{41^3} - \frac{1}{47^3} + ... = \frac{\pi^3}{16^3} \left(\frac{\cot(\frac{1}{16}\pi)}{\sin^2(\frac{1}{16}\pi)} - \frac{\cot(\frac{7}{16}\pi)}{\sin^2(\frac{7}{16}\pi)} \right)$$

26.

$R_{3/16} - R_{7/16}$

$$R_{3/16} = \frac{1}{3^3} - \frac{1}{13^3} + \frac{1}{19^3} - \frac{1}{29^3} + \frac{1}{35^3} - \frac{1}{45^3} + ... = \frac{\pi^3}{16^3}\frac{\cot(\frac{3}{16}\pi)}{\sin^2(\frac{3}{16}\pi)}$$

$$R_{7/16} = \frac{1}{7^3} - \frac{1}{9^3} + \frac{1}{23^3} - \frac{1}{25^3} + \frac{1}{39^3} - \frac{1}{41^3} + .. = \frac{\pi^3}{16^3}\frac{\cot(\frac{7}{16}\pi)}{\sin^2(\frac{7}{16}\pi)}$$

$\Rightarrow R_{3/16} - R_{7/16} =$

$$\frac{1}{3^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{13^3} + \frac{1}{19^3} - \frac{1}{23^3} + \frac{1}{25^3} - \frac{1}{29^3} + \frac{1}{35^3} -$$

$$-\frac{1}{39^3} + \frac{1}{41^3} - \frac{1}{45^3} + ... = \frac{\pi^3}{16^3}\left(\frac{\cot(\frac{3}{16}\pi)}{\sin^2(\frac{3}{16}\pi)} - \frac{\cot(\frac{7}{16}\pi)}{\sin^2(\frac{7}{16}\pi)}\right)$$

27.

27.1

$$\pi^3 = p^3 \frac{\sin^2(\frac{1}{p}\pi)}{\cot(\frac{1}{p}\pi)}\left\{1 - \frac{1}{(p-1)^2} + \frac{1}{(p+1)^2} - \frac{1}{(2p-1)^2} + \frac{1}{(2p+1)^2} - \frac{1}{(3p-1)^2} + \dots\right\}$$

$p =$ prime.

$$R_{1/p} = 1 - \frac{1}{(p-1)^3} + \frac{1}{(p+1)^3} - \frac{1}{(2p-1)^3} + \frac{1}{(2p+1)^3} - \frac{1}{(3p-1)^3} + \frac{1}{(3p+1)^3} - \dots = \frac{\pi^3}{p^3}\frac{\cot(\frac{1}{p}\pi)}{\sin^2(\frac{1}{p}\pi)}$$

Proof: By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\frac{1}{(z-\frac{1}{p})^3}$.□

27.2

$$\pi^3 = p^3 \frac{2\sin(\frac{1}{p}\pi)}{2\cot^2(\frac{1}{p}\pi)+1}\left\{1 + \frac{1}{(p-1)^3} - \frac{1}{(p+1)^3} - \frac{1}{(2p-1)^3} + \frac{1}{(2p+1)^3} + \frac{1}{(3p-1)^3} - \dots\right\}$$

$$S_{1/p} = 1 + \frac{1}{(p-1)^3} - \frac{1}{(p+1)^3} - \frac{1}{(2p-1)^3} + \frac{1}{(2p+1)^3} + \frac{1}{(3p-1)^3} + \dots = \frac{\pi^3}{p^3}\frac{2\cot^2(\frac{1}{p}\pi)+1}{2\sin(\frac{1}{p}\pi)}$$

<u>*Proof*</u>: By $\mathbf{7_{sin}}$. based on $\pi\frac{1}{\sin(\pi z)}\frac{1}{(z-\frac{1}{p})^3}$.□

27.3

$$\pi^3 = 8p^3\frac{2\cos^3(\frac{1}{p}\pi)}{\sin^2(\frac{1}{p}\pi)+1}\left(\frac{1}{(p-2)^3}+\frac{1}{(p+2)^3} - \frac{1}{(3p-2)^3}-\frac{1}{(3p+2)^3} + \frac{1}{(5p-2)^3}+\frac{1}{(5p+2)^3}-...\right)$$

$$C_{1/p} = \frac{1}{(p-2)^3}+\frac{1}{(p+2)^3} - \frac{1}{(3p-2)^3}-\frac{1}{(3p+2)^3} + \frac{1}{(5p-2)^3}+\frac{1}{(5p+2p)^3}-... = \frac{\pi^3}{8p^3}\frac{2\sin^2(\frac{1}{p}\pi)+1}{\cos^3(\frac{1}{p}\pi)}$$

<u>*Proof*</u>: By $\mathbf{7_{cos}}$. based on $\pi\frac{1}{\cos(\pi z)}\frac{1}{(z-\frac{1}{p})^3}$.□

27.4

$$\pi^3 = 8p^3\frac{\cos^3(\frac{1}{p}\pi)}{\sin(\frac{1}{p}\pi)}\left(\frac{1}{(p-2)^3}-\frac{1}{(p+2)^3} + \frac{1}{(3p-2)^3}-\frac{1}{(3p+2)^3} + \frac{1}{(5p-2)^3}-\frac{1}{(5p+2)^3}-...\right)$$

$$T_{1/p} = \frac{1}{(p-2)^3} - \frac{1}{(p+2)^3} + \frac{1}{(3p-2)^3} - \frac{1}{(3p+2)^3} + \frac{1}{(5p-2)^3} - \frac{1}{(5p+2p)^3} - ... = \frac{\pi^3}{8p^3}\frac{\sin(\frac{1}{p}\pi)}{\cos^3(\frac{1}{p}\pi)}$$

Proof: By **7_{tan}.** based on $\pi\tan(\pi z)\frac{1}{(z-\frac{1}{p})^3}$.□

28.

28.1

$$\pi^3 = 3^3 \frac{3\sqrt{3}}{4}\left(\frac{1}{1} - \frac{1}{2^3} + \frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{8^3} + \dots\right)$$

$$R_{1/3} = \frac{1}{1} - \frac{1}{2^3} + \frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{8^3} \dots = 4\frac{\pi^3}{3^4\sqrt{3}}$$

Proof: By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\frac{1}{(z-\frac{1}{3})^3}$

$$\pi^3 = 3^3 \underbrace{\frac{\sin^2(\frac{1}{3}\pi)}{\cot(\frac{1}{3}\pi)}}_{(3/4)/(1/\sqrt{3})}\left(\frac{1}{1^3} - \frac{1}{(3-1)^3} + \frac{1}{(3+1)^3} - \frac{1}{(2(3)-1)^3} + \frac{1}{(2(3)+1)^3} - \frac{1}{(3(3)-1)^3} + \dots\right)$$

28.2

$$\pi^3 = \frac{3^4\sqrt{3}}{5}\left(\frac{1}{1} + \frac{1}{2^3} - \frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{8^3} - \dots\right)$$

$$S_{1/3} = \frac{1}{1} + \frac{1}{2^3} - \frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{8^3} - \dots = 5\frac{\pi^3}{3^4\sqrt{3}}.$$

Proof: By $\mathbf{7_{sin}}$. based on $\dfrac{\pi}{\sin(\pi z)}\dfrac{1}{(z-\frac{1}{3})^3}$,

$$\pi^3 = 3^3 \underbrace{\frac{2\sin(\frac{1}{3}\pi)}{2\cot^2(\frac{1}{3}\pi)+1}}_{\sqrt{3}(3/5)} \left(\frac{1}{1^3} + \frac{1}{(3-1)^3} - \frac{1}{(3+1)^3} - \frac{1}{(2(3)-1)^3} + \frac{1}{(2(3)+1)^3} + \frac{1}{(3(3)-1)^3} - \dots \right) . \square$$

28.3

$$\boxed{\pi^3 = 8(3^3)7\left(\frac{1}{1} + \frac{1}{5^3} - \frac{1}{7^3} - \frac{1}{11^3} + \frac{1}{13^3} + \frac{1}{17^3} - \dots \right)}$$

$$C_{1/3} = \frac{1}{1} + \frac{1}{5^3} - \frac{1}{7^3} - \frac{1}{11^3} + \frac{1}{13^3} + \frac{1}{17^3} - \dots = \frac{\pi^3}{8(3^3)}\frac{1}{7}.$$

Proof: By $\mathbf{7_{cos}}$. based on $\dfrac{\pi}{\cos(\pi z)}\dfrac{1}{(z-\frac{1}{3})^3}$,

$$\pi^3 = 8(3^3) \underbrace{\frac{2\cos^3(\frac{1}{3}\pi)}{\sin^2(\frac{1}{3}\pi)+1}}_{[(3/4)+1]/(1/4)=7} \left(\frac{1}{(3-2)^3} + \frac{1}{(3+2)^3} - \frac{1}{(3(3)-2)^3} - \frac{1}{(3(3)+2)^3} + \frac{1}{(5(3)-2)^3} + \frac{1}{(5(3)+2)^3} - \dots \right) . \square$$

28.4

$$\boxed{\pi^3 = 3^3 2\frac{1}{\sqrt{3}}\left(\frac{1}{1^3}-\frac{1}{5^3}+\frac{1}{7^3}-\frac{1}{11^3}+\frac{1}{13^3}-\frac{1}{17^3}+\ldots\right)}$$

$$T_{1/3} = \frac{1}{1}-\frac{1}{5^3}+\frac{1}{7^3}-\frac{1}{11^3}+\frac{1}{13^3}-\frac{1}{17^3}+\ldots = \pi^3\frac{\sqrt{3}}{3^3 2}.$$

<u>*Proof*</u>: By $\mathbf{7_{tan}}$. based on $\pi\tan(\pi z)\dfrac{1}{(z-\frac{1}{3})^3}$,

$$\pi^3 = 8(3^3)\underbrace{\frac{\cos^3(\frac{1}{3}\pi)}{\sin(\frac{1}{3}\pi)}}_{(1/8)/(\sqrt{3}/2)=1/(4\sqrt{3})}\left(\frac{1}{(3-2)^3}-\frac{1}{(3+2)^3}+\frac{1}{(3(3)-2)^3}-\frac{1}{(3(3)+2)^3}+\frac{1}{(5(3)-2)^3}-\frac{1}{(5(3)+2)^3}-\ldots\right).\square$$

29.

29.1

$$\pi^3 = p^{3n}\frac{\sin^2(\frac{l}{p^n}\pi)}{\cot(\frac{l}{p^n}\pi)}\left\{\frac{1}{l^3} - \frac{1}{(p^n - l)^3} + \frac{1}{(p^n + l)^3} - \frac{1}{(2p^n - l)^3} + \frac{1}{(2p^n + l)^3} - \frac{1}{(3p^n - l)^3} + ...\right\}$$

$$R_{l/p^n} = \frac{1}{l^3} - \frac{1}{(p^n - l)^3} + \frac{1}{(p^n + l)^3} - \frac{1}{(2p^n - l)^3} + \frac{1}{(2p^n + l)^3} - \frac{1}{(3p^n - l)^3} + ... = \frac{\pi^3}{p^{3n}}\frac{\cot(\frac{l}{p^n}\pi)}{\sin^2(\frac{l}{p^n}\pi)}$$

Proof: By **$7_{\cot}$.** based on $\pi\cot(\pi z)\dfrac{1}{(z - \frac{l}{p^n})^3}$.□

29.2

$$\pi^3 = p^{3n}\frac{2\sin^3(\frac{l}{p^n}\pi)}{\cos^2(\frac{l}{p^n}\pi) + 1}\left\{\frac{1}{l^3} + \frac{1}{(p^n - l)^3} - \frac{1}{(p^n + l)^3} - \frac{1}{(2p^n - l)^3} + \frac{1}{(2p^n + l)^3} + \frac{1}{(3p^n - l)^3} - ....\right\}$$

$$S_{l/p^n} = \frac{1}{l^3} + \frac{1}{(p^n - l)^3}$$
$$- \frac{1}{(p^n + l)^3} - \frac{1}{(2p^n - l)^3}$$
$$+ \frac{1}{(2p^n + l)^3} + \frac{1}{(3p^n - l)^3} - ... = \frac{\pi^3}{p^{3n}} \frac{\cos^2(\frac{l}{p^n}\pi) + 1}{2\sin^3(\frac{l}{p^n}\pi)}$$

Proof: By $\mathbf{7_{sin}}$. based on $\pi \frac{1}{\sin(\pi z)} \frac{1}{(z - \frac{l}{p^n})^3}$.□

29.3
$$\boxed{\pi^3 = 8p^{3n} \frac{2\cos^3(\frac{l}{p^n}\pi)}{\sin^2(\frac{l}{p^n}\pi) + 1} \left(\frac{1}{(p^n - 2l)^2} + \frac{1}{(p^n + 2l)^2} - \frac{1}{(3p^n - 2l)^2} - \frac{1}{(3p^n + 2l)^2} + \frac{1}{(5p^n - 2l)^2} + \frac{1}{(5p^n + 2l)^2} - ... \right)}$$

$$C_{l/p^n} = \frac{1}{(p^n - 2l)^3} + \frac{1}{(p^n + 2l)^3}$$
$$- \frac{1}{(3p^n - 2l)^3} - \frac{1}{(3p^n + 2l)^3}$$
$$+ \frac{1}{(5p^n - 2l)^3} + \frac{1}{(5p^n + 2l)^3} - ... = \frac{\pi^3}{8p^{3n}} \frac{\sin^2(\frac{l}{p^n}\pi) + 1}{2\cos^3(\frac{l}{p^n}\pi)}$$

Proof: By $\mathbf{7_{cos}}$. based on $\pi \frac{1}{\cos(\pi z)} \frac{1}{(z - \frac{l}{p^n})^3}$.□

29.4

$$\pi^3 = 8p^{3n}\frac{\cos^3(\frac{l}{p^n}\pi)}{\sin(\frac{l}{p^n}\pi)}\left(\frac{1}{(p^n-2l)^3}-\frac{1}{(p^n+2l)^3} + \frac{1}{(3p^n-2l)^3}-\frac{1}{(3p^n+2l)^3} + \frac{1}{(5p^n-2l)^3}-\frac{1}{(5p^n+2l)^3}+\ldots\right)$$

$$T_{l/p^n} = \frac{1}{(p^n-2l)^3}-\frac{1}{(p^n+2l)^3} + \frac{1}{(3p^n-2l)^3}-\frac{1}{(3p^n+2l)^3} + \frac{1}{(5p^n-2l)^3}-\frac{1}{(5p^n+2l)^3}-\ldots = \frac{\pi^3}{8p^{3n}}\frac{\sin(\frac{l}{p^n}\pi)}{\cos^3(\frac{l}{p^n}\pi)}$$

Proof: By $\mathbf{7_{tan}}$. based on $\pi\tan(\pi z)\frac{1}{(z-\frac{l}{p^n})^3}$. □

30.

30.1
$$\boxed{\pi^3 = 9^3\frac{\sin^2(\frac{1}{9}\pi)}{\cot(\frac{1}{9}\pi)}\left(\frac{1}{1}-\frac{1}{8^3}+\frac{1}{10^3}-\frac{1}{17^3}+\frac{1}{19^3}-\frac{1}{26^3}+...\right)}$$

$$R_{1/9} = \frac{1}{1}-\frac{1}{8^3}+\frac{1}{10^3}-\frac{1}{17^3}+\frac{1}{19^3}-\frac{1}{26^3}+... = \frac{\pi^3}{9^3}\frac{\cot(\frac{1}{9}\pi)}{\sin^2(\frac{1}{9}\pi)}$$

$\sin(\frac{1}{9}\pi)$ **solves the cubic** $4x^3 - 3x + \frac{1}{2}\sqrt{3} = 0$

Proof: By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\dfrac{1}{(z-\frac{1}{9})^3}$,

$$\pi^3 = 9^3\frac{\sin^2(\frac{1}{9}\pi)}{\cot(\frac{1}{9}\pi)}\left(\frac{1}{1^3}-\frac{1}{(9-1)^3}+\frac{1}{(9+1)^3}-\frac{1}{(2(9)-1)^3}+\frac{1}{(2(9)+1)^3}-\frac{1}{(3(9)-1)^3}+...\right).\square$$

$$\sin 3A = -4\sin^3 A + 3\sin A \Rightarrow$$

$$\Rightarrow \underbrace{\sin(\tfrac{1}{3}\pi)}_{\frac{1}{2}\sqrt{3}} = -4\sin^3(\tfrac{1}{9}\pi) + 3\sin(\tfrac{1}{9}\pi) \Rightarrow$$

$\Rightarrow \sin(\frac{1}{9}\pi)$ **solves the cubic** $4x^3 - 3x + \frac{1}{2}\sqrt{3} = 0$.$\square$

30.2

$$\boxed{\pi^3 = 9^3 \frac{2\sin^3(\frac{1}{9}\pi)}{\cos^2(\frac{1}{9}\pi)+1}\left(\frac{1}{1}+\frac{1}{8^3} - \frac{1}{10^3}-\frac{1}{17^3} + \frac{1}{19^3}+\frac{1}{26^3}-...\right)}$$

$$S_{1/9} = \frac{1}{1}+\frac{1}{8^3}-\frac{1}{10^3}-\frac{1}{17^3}+\frac{1}{19^3}+\frac{1}{26^3}-... = \frac{\pi^3}{9^3}\frac{\cos^2(\frac{1}{9}\pi)+1}{2\sin^3(\frac{1}{9}\pi)}$$

Proof : By $\mathbf{7_{sin}}$. based on $\pi\frac{1}{\sin(\pi z)}\frac{1}{(z-\frac{1}{9})^3}$

$$\pi^3 = 9^3 \frac{2\sin^3(\frac{1}{9}\pi)}{\cos^2(\frac{1}{9}\pi)+1}\left(\frac{1}{1^3}+\frac{1}{(9-1)^3} - \frac{1}{(9+1)^3}-\frac{1}{(2(9)-1)^3} + \frac{1}{(2(9)+1)^3}+\frac{1}{(3(9)-1)^3}-...\right). \square$$

30.3

$$\boxed{\pi^3 = 8(9^3)\frac{2\cos^3(\frac{1}{9}\pi)}{\sin^2(\frac{1}{9}\pi)+1}\left(\frac{1}{7^3}+\frac{1}{11^3} - \frac{1}{25^3}-\frac{1}{29^3} + \frac{1}{43^3}+\frac{1}{47^3}-...\right)}$$

$$C_{1/9} = \frac{1}{7^3}+\frac{1}{11^3}-\frac{1}{25^3}-\frac{1}{29^3}+\frac{1}{43^3}+\frac{1}{47^3}-... = \frac{\pi^3}{8(9^3)}\frac{\sin^2(\frac{1}{9}\pi)+1}{2\cos^3(\frac{1}{9}\pi)}$$

Proof : By $\mathbf{7_{cos}}$. based on $\pi \dfrac{1}{\cos(\pi z)} \dfrac{1}{(z-\frac{1}{9})^3}$

$$\pi^3 = 8(9^3)\frac{2\cos^3(\frac{1}{9}\pi)}{\sin^2(\frac{1}{9}\pi)+1}\left(\frac{1}{(9-2)^3}+\frac{1}{(9+2)^3} - \frac{1}{(3(9)-2)^3} - \frac{1}{(3(9)+2)^3} + \frac{1}{(5(9)-2)^3} + \frac{1}{(5(9)+2)^3} - ...\right). \square$$

30.4

$$\boxed{\pi^3 = 8(9^3)\frac{\cos^3(\frac{1}{9}\pi)}{\sin(\frac{1}{9}\pi)}\left(\frac{1}{7^3}-\frac{1}{11^3} + \frac{1}{25^3}-\frac{1}{29^3} + \frac{1}{43^3}-\frac{1}{47^3}+...\right)}$$

$$T_{1/9} = \frac{1}{7^3}-\frac{1}{11^3}+\frac{1}{25^3}-\frac{1}{29^3}+\frac{1}{43^3}-\frac{1}{47^3}+... = \frac{\pi^3}{8(9^3)}\frac{\sin(\frac{1}{9}\pi)}{\cos^3(\frac{1}{9}\pi)}$$

Proof : By $\mathbf{7_{tan}}$. based on $\pi \tan(\pi z)\dfrac{1}{(z-\frac{1}{9})^3}$

$$\pi^3 = 8(9^3)\frac{\cos^3(\frac{1}{9}\pi)}{\sin(\frac{1}{9}\pi)}\left(\frac{1}{(9-2)^3}-\frac{1}{(9+2)^3} + \frac{1}{(3(9)-2)^3} - \frac{1}{(3(9)+2)^3} + \frac{1}{(5(9)-2)^3} - \frac{1}{(5(9)+2)^3} + ...\right). \square$$

31.

31.1

$$\pi^3 = 9^3 \frac{\sin^2(\frac{2}{9}\pi)}{\cot(\frac{2}{9}\pi)}\left(\frac{1}{2^3} - \frac{1}{7^3} + \frac{1}{11^3} - \frac{1}{16^3} + \frac{1}{20^3} - \frac{1}{25^3} + ...\right)$$

$$R_{2/9} = \frac{1}{2^3} - \frac{1}{7^3} + \frac{1}{11^3} - \frac{1}{16^3} + \frac{1}{20^3} - \frac{1}{25^3} + ... = \frac{\pi^3}{9^3}\frac{\cot(\frac{2}{9}\pi)}{\sin^2(\frac{2}{9}\pi)}$$

$\sin(\frac{2}{9}\pi)$ **solves the cubic** $4x^3 - 3x + \frac{1}{2} = 0$

Proof: By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\frac{1}{(z-\frac{2}{9})^3}$,

$$\pi^3 = 9^3 \frac{\sin^2(\frac{2}{9}\pi)}{\cot(\frac{2}{9}\pi)}\left(\frac{1}{1^3} - \frac{1}{(9-2)^3} + \frac{1}{(9+2)^3} - \frac{1}{(2(9)-2)^3} + \frac{1}{(2(9)+2)^3} - \frac{1}{(3(9)-2)^3} + ...\right) . \square$$

$$\sin 3A = -4\sin^3 A + 3\sin A \Rightarrow$$

$$\Rightarrow \underbrace{\sin(3\tfrac{2}{9}\pi)}_{\frac{1}{2}} = -4\sin^3(\tfrac{2}{9}\pi) + 3\sin(\tfrac{2}{9}\pi) \Rightarrow$$

$\Rightarrow \sin(\frac{2}{9}\pi)$ **solves the cubic** $4x^3 - 3x + \frac{1}{2} = 0 . \square$

31.2

$$\pi^3 = 9^3 \frac{2\sin^3(\frac{2}{9}\pi)}{\cos^2(\frac{2}{9}\pi)+1}\left(\frac{1}{2^3}+\frac{1}{7^3} - \frac{1}{11^3}-\frac{1}{16^3} + \frac{1}{20^3}+\frac{1}{25^3}-...\right)$$

$$S_{2/9} = \frac{1}{2^3}+\frac{1}{7^3}-\frac{1}{11^3}-\frac{1}{16^3}+\frac{1}{20^3}+\frac{1}{25^3}-... = \frac{\pi^3}{9^3}\frac{\cos^2(\frac{2}{9}\pi)+1}{2\sin^3(\frac{2}{9}\pi)}$$

__Proof__: By $\mathbf{7_{sin}}$. based on $\pi\frac{1}{\sin(\pi z)}\frac{1}{(z-\frac{2}{9})^3}$,

$$\pi^3 = 9^3 \frac{2\sin^3(\frac{2}{9}\pi)}{\cos^2(\frac{2}{9}\pi)+1}\left(\frac{1}{1^2}+\frac{1}{(9-2)^3} - \frac{1}{(9+2)^3}-\frac{1}{(2(9)-2)^3} + \frac{1}{(2(9)+2)^3}+\frac{1}{(3(9)-2)^3}-...\right).\square$$

31.3

$$\pi^3 = 8(9^3)\frac{2\cos^3(\frac{2}{9}\pi)}{\sin^2(\frac{2}{9}\pi)+1}\left(\frac{1}{5^3}+\frac{1}{13^3} - \frac{1}{23^3}-\frac{1}{31^3} + \frac{1}{41^3}+\frac{1}{49^3}-...\right)$$

$$C_{2/9} = \frac{1}{5^3}+\frac{1}{13^3}-\frac{1}{23^3}-\frac{1}{31^3}+\frac{1}{41^3}+\frac{1}{49^3}-... = \frac{\pi^3}{8(9^3)}\frac{\sin^2(\frac{2}{9}\pi)+1}{2\cos^3(\frac{2}{9}\pi)}$$

Proof: By $\mathbf{7_{cos}}$. based on $\pi\dfrac{1}{\cos(\pi z)}\dfrac{1}{(z-\frac{2}{9})^3}$,

$$\pi^3 = 8(9^3)\frac{2\cos^3(\frac{2}{9}\pi)}{\sin^2(\frac{2}{9}\pi)+1}\left(\frac{1}{(9-2(2))^3}+\frac{1}{(9+2(2))^3}\right.$$
$$-\frac{1}{(3(9)-2(2))^3}-\frac{1}{(3(9)+2(2))^3}$$
$$\left.+\frac{1}{(5(9)-2(2))^3}+\frac{1}{(5(9)+2(2))^3}-...\right) .\square$$

31.4

$$\boxed{\begin{aligned}\pi^3 = 8(9^3)\frac{\cos^3(\frac{2}{9}\pi)}{\sin(\frac{2}{9}\pi)}\Big(&\frac{1}{5^3}-\frac{1}{13^3}\\ &+\frac{1}{23^3}-\frac{1}{31^3}\\ &+\frac{1}{41^3}-\frac{1}{49^3}-...\Big)\end{aligned}}$$

$$T_{2/9} = \frac{1}{5^3}-\frac{1}{13^3}+\frac{1}{23^3}-\frac{1}{31^3}+\frac{1}{41^3}-\frac{1}{49^3}+... = \frac{\pi^3}{8(9^3)}\frac{\sin(\frac{2}{9}\pi)}{\cos^3(\frac{2}{9}\pi)}$$

Proof: By $\mathbf{7_{tan}}$. based on $\pi\tan(\pi z)\dfrac{1}{(z-\frac{2}{9})^3}$,

$$\pi^3 = 8(9^3)\frac{\cos^3(\frac{2}{9}\pi)}{\sin(\frac{2}{9}\pi)}\left(\frac{1}{(9-2(2))^3}-\frac{1}{(9+2(2))^3}\right.$$
$$+\frac{1}{(3(9)-2(2))^3}-\frac{1}{(3(9)+2(2))^3}$$
$$\left.+\frac{1}{(5(9)-2(2))^3}-\frac{1}{(5(9)+2(2))^3}-...\right) .\square$$

32.

32.1

$$\boxed{\pi^3 = 9^3 \frac{\sin^2(\frac{4}{9}\pi)}{\cot(\frac{4}{9}\pi)}\left(\frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{13^3} - \frac{1}{14^3} + \frac{1}{22^3} - \frac{1}{23^3} + \ldots\right)}$$

$$R_{4/9} = \frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{13^3} - \frac{1}{14^3} + \frac{1}{22^3} - \frac{1}{23^3} + \ldots = \frac{\pi^3}{9^3}\frac{\cot(\frac{4}{9}\pi)}{\sin^2(\frac{4}{9}\pi)}$$

$\sin(\frac{4}{9}\pi)$ **solves the cubic** $4x^3 - 3x - \frac{1}{2}\sqrt{3} = 0$

Proof By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\dfrac{1}{(z-\frac{4}{9})^3}$

$$\pi^3 = 9^3 \frac{\sin^2(\frac{4}{9}\pi)}{\cot(\frac{4}{9}\pi)}\left(\frac{1}{4^3} - \frac{1}{(9-4)^3} + \frac{1}{(9+4)^3} - \frac{1}{(2(9)-4)^3} + \frac{1}{(2(9)+4)^3} - \frac{1}{(3(9)-4)^3} + \ldots\right)$$

$$\sin 3A = -4\sin^3 A + 3\sin A \Rightarrow$$

$$\Rightarrow \underbrace{\sin(3\tfrac{4}{9}\pi)}_{-\frac{1}{2}\sqrt{3}} = -4\sin^3(\tfrac{4}{9}\pi) + 3\sin(\tfrac{4}{9}\pi) \Rightarrow$$

$\Rightarrow \sin(\frac{4}{9}\pi)$ **solves the cubic** $4x^3 - 3x - \frac{1}{2}\sqrt{3} = 0. \Box$

32.2

$$\pi^3 = 9^3 \frac{2\sin^3(\frac{4}{9}\pi)}{\cos^2(\frac{4}{9}\pi)+1}\left(\frac{1}{4^3}+\frac{1}{5^3} - \frac{1}{13^3}-\frac{1}{14^3} + \frac{1}{22^3}+\frac{1}{23^3}-...\right)$$

$$S_{4/9} = \frac{1}{4^3}+\frac{1}{5^3}-\frac{1}{13^3}-\frac{1}{14^3}+\frac{1}{22^3}+\frac{1}{23^3}-... = \frac{\pi^3}{9^3}\frac{\cos^2(\frac{4}{9}\pi)+1}{2\sin^3(\frac{4}{9}\pi)}$$

Proof By **7**$_{\mathbf{sin}}$**.** based on $\pi\frac{1}{\sin(\pi z)}\frac{1}{(z-\frac{4}{9})^3}$,

$$\pi^3 = 9^3 \frac{2\sin^3(\frac{4}{9}\pi)}{\cos^2(\frac{4}{9}\pi)+1}\left(\frac{1}{4^3}+\frac{1}{(9-4)^3} - \frac{1}{(9+4)^3}-\frac{1}{(2(9)-4)^3} + \frac{1}{(2(9)+4)^3}+\frac{1}{(3(9)-4)^3}-...\right).\square$$

32.3

$$\pi^3 = 8(9)^3 \frac{2\cos^3(\frac{4}{9}\pi)}{\sin^2(\frac{4}{9}\pi)+1}\left(\frac{1}{1^3}+\frac{1}{17^3} - \frac{1}{19^3}-\frac{1}{35^3} + \frac{1}{37^3}+\frac{1}{53^3}-...\right)$$

$$C_{4/9} = \frac{1}{1^3}+\frac{1}{17^3}-\frac{1}{19^3}-\frac{1}{35^3}+\frac{1}{37^3}+\frac{1}{53^3}-... = \frac{\pi^3}{8(9^3)}\frac{\sin^2(\frac{4}{9}\pi)+1}{2\cos^3(\frac{4}{9}\pi)}$$

Proof By $\mathbf{7_{cos}}$. based on $\pi \frac{1}{\cos(\pi z)} \frac{1}{(z-\frac{4}{9})^3}$,

$$\pi^3 = 8(9)^3 \frac{2\cos^3(\frac{4}{9}\pi)}{\sin^2(\frac{4}{9}\pi)+1} \Bigg(\frac{1}{(9-2(4))^3} + \frac{1}{(9+2(4))^3} - \frac{1}{(3(9)-2(4))^3} - \frac{1}{(3(9)+2(4))^3} + \frac{1}{(5(9)-2(4))^3} + \frac{1}{(5(9)+2(4))^3} - ... \Bigg) .\square$$

32.4

$$\boxed{\pi^3 = 8(9)^3 \frac{\cos^3(\frac{4}{9}\pi)}{\sin(\frac{4}{9}\pi)} \Bigg(\frac{1}{1^3} - \frac{1}{17^3} + \frac{1}{19^3} - \frac{1}{35^3} + \frac{1}{37^3} - \frac{1}{53^3} + ... \Bigg)}$$

$$T_{4/9} = \frac{1}{1^3} - \frac{1}{17^3} + \frac{1}{19^3} - \frac{1}{35^3} + \frac{1}{37^3} - \frac{1}{53^3} + ... = \frac{\pi^3}{8(9^3)} \frac{\sin(\frac{4}{9}\pi)}{\cos^3(\frac{4}{9}\pi)}$$

Proof By $\mathbf{7_{tan}}$. based on $\pi \tan(\pi z) \frac{1}{(z-\frac{4}{9})^3}$,

$$\pi^3 = 8(9)^3 \frac{\cos^3(\frac{4}{9}\pi)}{\sin(\frac{4}{9}\pi)} \Bigg(\frac{1}{(9-2(4))^3} - \frac{1}{(9+2(4))^3} + \frac{1}{(3(9)-2(4))^3} - \frac{1}{(3(9)+2(4))^3} + \frac{1}{(5(9)-2(4))^3} - \frac{1}{(5(9)+2(4))^3} + ... \Bigg) .\square$$

$$\cot(\tfrac{4}{9}\pi) = \frac{\cos(\tfrac{4}{9}\pi)}{\sin(\tfrac{4}{9}\pi)}$$

$$= \frac{\sqrt{\tfrac{1}{2}[1+\cos(\tfrac{8}{9}\pi)]}}{\sqrt{\tfrac{1}{2}[1-\cos(\tfrac{8}{9}\pi)]}}$$

$$= \frac{\sqrt{1-\cos(\tfrac{1}{9}\pi)}}{\sqrt{1+\cos(\tfrac{1}{9}\pi)}}.\square$$

33.

33.1

$$\boxed{\pi^3 = 3^3 2\sqrt{3}\left(\frac{1}{1} - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{11^3} + \frac{1}{13^3} - \frac{1}{17^3} + ...\right)}$$

$$R_{1/6} = \frac{1}{1} - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{11^3} + \frac{1}{13^3} - \frac{1}{17^3} + ... = \frac{\pi^3}{3^3}\frac{1}{2\sqrt{3}}$$

Proof: By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\dfrac{1}{(z-\frac{1}{6})^3}$

$$\pi^3 = 6^3 \underbrace{\frac{\sin^2(\frac{1}{6}\pi)}{\cot(\frac{1}{6}\pi)}}_{(1/2^2)/\sqrt{3}}\left(\frac{1}{1^3} - \frac{1}{(6-1)^3} + \frac{1}{(6+1)^3} - \frac{1}{(2(6)-1)^3} + \frac{1}{(2(6)+1)^3} - \frac{1}{(3(6)-1)^3} + ...\right)$$

33.2

$$\boxed{\pi^3 = 6^3 7\left(\frac{1}{1} + \frac{1}{5^3} - \frac{1}{7^3} - \frac{1}{11^3} + \frac{1}{13^3} + \frac{1}{17^3} - ...\right)}$$

$$S_{1/6} = \frac{1}{1} + \frac{1}{5^3} - \frac{1}{7^3} - \frac{1}{11^3} + \frac{1}{13^3} + \frac{1}{17^3} - ... = \frac{\pi^3}{6^3}\frac{1}{7}$$

Proof: By $\mathbf{7_{sin}}$. based on $\pi \frac{1}{\sin(\pi z)} \frac{1}{(z-\frac{1}{6})^3}$,

$$\pi^3 = 6^3 \underbrace{\frac{2\sin(\frac{1}{6}\pi)}{2\cot^2(\frac{1}{6}\pi)+1}}_{1/(6+1)} \left(\frac{1}{1^3} + \frac{1}{(6-1)^3} - \frac{1}{(6+1)^3} - \frac{1}{(2(6)-1)^3} + \frac{1}{(2(6)+1)^3} + \frac{1}{(3(6)-1)^3} - \ldots \right). \square$$

33.3

$$\sqrt{3} = 2^2 7 \frac{\frac{1}{1} + \frac{1}{5^3} - \frac{1}{7^3} - \frac{1}{11^3} + \frac{1}{13^3} + \frac{1}{17^3} - \ldots}{\frac{1}{1} - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{11^3} + \frac{1}{13^3} - \frac{1}{17^3} + \ldots}$$

Proof: $\frac{S_{1/6}}{R_{1/6}}$. $\square$

33.4

$$\boxed{\pi^3 = \frac{3^4\sqrt{3}}{5}\left(\frac{1}{1} + \frac{1}{2^3} - \frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{8^3} - \ldots \right)}$$

$$C_{1/6} = \frac{1}{1} + \frac{1}{2^3} - \frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{8^3} - \ldots = \frac{2}{3^3}\pi^3$$

Proof: By $\mathbf{7_{cos}}$. based on $\pi \frac{1}{\cos(\pi z)} \frac{1}{(z-\frac{1}{6})^3}$,

$$\pi^3 = 8(6^3)\underbrace{\frac{2\cos^3(\frac{1}{6}\pi)}{\sin^2(\frac{1}{6}\pi)+1}}_{(3\sqrt{3}/4)/(5/4)}\left(\frac{1}{(6-2)^3}+\frac{1}{(6+2)^3}\right.$$
$$-\frac{1}{(3(6)-2)^3}-\frac{1}{(3(6)+2)^3}$$
$$\left.+\frac{1}{(5(6)-2)^3}+\frac{1}{(5(6)+2)^3}-...\right)$$

33.5

$$\boxed{\pi^3 = 3^4\sqrt{3}\left(\frac{1}{1}-\frac{1}{2^3}+\frac{1}{4^3}-\frac{1}{5^3}+\frac{1}{7^3}-\frac{1}{8^3}-...\right)}$$

$$T_{1/6} = \frac{1}{1}-\frac{1}{2^3}+\frac{1}{4^3}-\frac{1}{5^3}+\frac{1}{7^3}-\frac{1}{8^3}+... = \frac{1}{3^4\sqrt{3}}\pi^3$$

Proof: By **$7_{\tan}$.** based on $\pi\tan(\pi z)\dfrac{1}{(z-\frac{1}{6})^3}$,

$$\pi^3 = \underbrace{8(6^3)}_{2^6 3^3}\underbrace{\frac{\cos^3(\frac{1}{6}\pi)}{\sin(\frac{1}{6}\pi)}}_{[(3\sqrt{3})/8]/(1/2)=3\sqrt{3}/4}\left(\frac{1}{(6-2)^3}-\frac{1}{(6+2)^3}\right.$$
$$+\frac{1}{(3(6)-2)^3}-\frac{1}{(3(6)+2)^3}$$
$$\left.+\frac{1}{(5(6)-2)^3}-\frac{1}{(5(6)+2)^3}-...\right)$$

34.

34.1

$$\pi^3 = 18^3 \frac{\sin^2(\frac{1}{18}\pi)}{\cot(\frac{1}{18}\pi)}\Big(\frac{1}{1} - \frac{1}{17^3} + \frac{1}{19^3} - \frac{1}{35^3} + \frac{1}{37^3} - \frac{1}{53^3} + \dots\Big)$$

$$R_{1/18} = \frac{1}{1} - \frac{1}{17^3} + \frac{1}{19^3} - \frac{1}{35^3} + \frac{1}{37^3} - \frac{1}{53^3} + \dots = \frac{\pi^3}{18^3}\frac{\cot(\frac{1}{18}\pi)}{\sin^2(\frac{1}{18}\pi)}$$

Proof: By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\frac{1}{(z-\frac{1}{18})^3}$

$$\pi^3 = 18^3 \frac{\sin^2(\frac{1}{18}\pi)}{\cot(\frac{1}{18}\pi)}\Big(\frac{1}{1^3} - \frac{1}{(18-1)^3} + \frac{1}{(18+1)^3} - \frac{1}{(2(18)-1)^3} + \frac{1}{(2(18)+1)^3} - \frac{1}{(3(18)-1)^3} + \dots\Big)$$

34.2

$$\pi^3 = 18^3 \frac{2\sin(\frac{1}{18}\pi)}{2\cot^2(\frac{1}{18}\pi)+1}\Big(\frac{1}{1} + \frac{1}{17^3} - \frac{1}{19^3} - \frac{1}{35^3} + \frac{1}{37^3} + \frac{1}{53^3} - \dots\Big)$$

$$S_{1/18} = \frac{1}{1} + \frac{1}{17^3} - \frac{1}{19^3} - \frac{1}{35^3} + \frac{1}{37^3} + \frac{1}{53^3} - \dots = \frac{\pi^3}{18^3}\frac{2\cot^2(\frac{1}{18}\pi)+1}{2\sin(\frac{1}{18}\pi)}$$

$$\sin(\tfrac{1}{18}\pi) = \sqrt{\tfrac{1}{2}[1-\cos(\tfrac{1}{9}\pi)]} = \sqrt{\tfrac{1}{2}[1-\sin(\tfrac{4}{9}\pi)]}$$

And $\sin(\tfrac{4}{9}\pi)$ **solves the cubic** $4x^3 - 3x - \tfrac{1}{2}\sqrt{3} = 0$

<u>*Proof*</u>: **By $\mathbf{7_{sin}}$. based on** $\pi\dfrac{1}{\sin(\pi z)}\dfrac{1}{(z-\frac{1}{18})^3}$

$$\pi^3 = 18^3\frac{2\sin(\frac{1}{18}\pi)}{2\cot^2(\frac{1}{18}\pi)+1}\left(\frac{1}{1^3}+\frac{1}{(18-1)^3} - \frac{1}{(18+1)^3} - \frac{1}{(2(18)-1)^3} + \frac{1}{(2(18)+1)^3} + \frac{1}{(3(18)-1)^3} - ...\right).\square$$

$$\sin 3A = -4\sin^3 A + 3\sin A \Rightarrow$$

$$\Rightarrow \underbrace{\sin(3\tfrac{4}{9}\pi)}_{-\frac{1}{2}\sqrt{3}} = -4\sin^3(\tfrac{4}{9}\pi) + 3\sin(\tfrac{4}{9}\pi) \Rightarrow$$

$\Rightarrow \sin(\tfrac{4}{9}\pi)$ **solves the cubic** $4x^3 - 3x - \tfrac{1}{2}\sqrt{3} = 0.\square$

34.3

$$\boxed{\pi^3 = 9^3\frac{2\cos^3(\frac{1}{18}\pi)}{\sin^2(\frac{1}{18}\pi)+1}\left(\frac{1}{4^3}+\frac{1}{5^3} - \frac{1}{13^3} - \frac{1}{14^3} + \frac{1}{22^3} + \frac{1}{23^3} - ...\right)}$$

$$C_{1/18} = \frac{1}{4^3}+\frac{1}{5^3}-\frac{1}{13^2}-\frac{1}{14^2}+\frac{1}{22^2}+\frac{1}{23^2}-... = \frac{\pi^3}{9^3}\frac{\sin^2(\frac{1}{18}\pi)+1}{2\cos^3(\frac{1}{18}\pi)}$$

<u>*Proof*</u>: By $\mathbf{7_{cos}}$. based on $\pi \dfrac{1}{\cos(\pi z)} \dfrac{1}{(z - \frac{1}{18})^3}$

$$\pi^3 = 8(18^3)\frac{2\cos^3(\frac{1}{18}\pi)}{\sin^2(\frac{1}{18}\pi)+1}\Bigg(\frac{1}{(18-2)^3} + \frac{1}{(18+2)^3} - \frac{1}{(3(18)-2)^3} - \frac{1}{(3(18)+2)^3} + \frac{1}{(5(18)-2)^3} + \frac{1}{(5(18)+2)^3} - \dots\Bigg)$$

34.4

$$\boxed{\pi^3 = 9^3\frac{\cos^3(\frac{1}{18}\pi)}{\sin(\frac{1}{18}\pi)}\Bigg(\frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{13^3} - \frac{1}{14^3} + \frac{1}{22^3} - \frac{1}{23^3} + \dots\Bigg)}$$

$$T_{1/18} = \frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{13^3} - \frac{1}{14^3} + \frac{1}{22^3} - \frac{1}{23^3} + \dots = \frac{\pi^3}{9^3}\frac{\sin(\frac{1}{18}\pi)}{\cos^3(\frac{1}{18}\pi)}$$

<u>*Proof*</u>: By $\mathbf{7_{tan}}$. based on $\pi \tan(\pi z) \dfrac{1}{(z - \frac{1}{18})^3}$

$$\pi^3 = 8(18^3)\frac{\cos^3(\frac{1}{18}\pi)}{\sin(\frac{1}{18}\pi)}\Bigg(\frac{1}{(18-2)^3} - \frac{1}{(18+2)^3} + \frac{1}{(3(18)-2)^3} - \frac{1}{(3(18)+2)^3} + \frac{1}{(5(18)-2)^3} - \frac{1}{(5(18)+2)^3} - \dots\Bigg)$$

35.

35.1

$$\pi^3 = 18^3 \frac{\sin^2(\frac{5}{18}\pi)}{\cot(\frac{5}{18}\pi)}\left(\frac{1}{5^3} - \frac{1}{13^3} + \frac{1}{23^3} - \frac{1}{31^3} + \frac{1}{41^3} - \frac{1}{49^3} + \dots\right)$$

$$R_{5/18} = \frac{1}{5^3} - \frac{1}{13^3} + \frac{1}{23^3} - \frac{1}{31^3} + \frac{1}{41^3} - \frac{1}{49^3} + \dots = \frac{\pi^3}{18^3}\frac{\cot(\frac{5}{18}\pi)}{\sin^2(\frac{5}{18}\pi)}$$

Proof: By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\frac{1}{(z - \frac{5}{18})^3}$,

$$\pi^3 = 18^3 \frac{\sin^2(\frac{5}{18}\pi)}{\cot(\frac{5}{18}\pi)}\left(\frac{1}{5^3} - \frac{1}{(18-5)^3} + \frac{1}{(18+5)^3} - \frac{1}{(2(18)-5)^3} + \frac{1}{(2(18)+5)^3} - \frac{1}{(3(18)-5)^3} + \dots\right)$$

35.2

$$\pi^3 = 18^3 \frac{2\sin^3(\frac{5}{18}\pi)}{\cos^2(\frac{5}{18}\pi)+1}\left(\frac{1}{5^3} + \frac{1}{13^3} - \frac{1}{23^3} - \frac{1}{31^3} + \frac{1}{41^3} + \frac{1}{49^3} - \dots\right)$$

$$S_{5/18} = \frac{1}{5^3} + \frac{1}{13^3} - \frac{1}{23^3} - \frac{1}{31^3} + \frac{1}{41^3} + \frac{1}{49^3} - \dots = \frac{\pi^3}{18^3}\frac{\cos^2(\frac{5}{18}\pi)+1}{2\sin^3(\frac{5}{18}\pi)}$$

Proof: By $\mathbf{7_{sin}}$. Based on $\pi \frac{1}{\sin(\pi z)} \frac{1}{(z - \frac{5}{18})^3}$,

$$\pi^3 = 18^3 \frac{2\sin^3(\frac{5}{18}\pi)}{\cos^2(\frac{5}{18}\pi)+1}\left(\frac{1}{5^3} + \frac{1}{(18-5)^3} - \frac{1}{(18+5)^3} - \frac{1}{(2(18)-5)^3} + \frac{1}{(2(18)+5)^3} + \frac{1}{(3(18)-5)^3} - ...\right). \square$$

35.3

$$\boxed{\pi^3 = 9^3 \frac{2\cos^3(\frac{5}{18}\pi)}{\sin^2(\frac{5}{18}\pi)+1}\left(\frac{1}{2^3} + \frac{1}{7^3} - \frac{1}{11^3} - \frac{1}{16^3} + \frac{1}{20^3} + \frac{1}{25^3} - ...\right)}$$

$$C_{5/18} = \frac{1}{2^3} + \frac{1}{7^3} - \frac{1}{11^3} - \frac{1}{16^3} + \frac{1}{20^3} + \frac{1}{25^3} - ... = \frac{\pi^3}{9^3} \frac{\sin^2(\frac{5}{18}\pi)+1}{2\cos^3(\frac{5}{18}\pi)}$$

Proof: By $\mathbf{7_{cos}}$. Based on $\pi \frac{1}{\cos(\pi z)} \frac{1}{(z - \frac{5}{18})^3}$,

$$\pi^3 = 8(18^3) \frac{2\cos^3(\frac{5}{18}\pi)}{\sin^2(\frac{5}{18}\pi)+1}\left(\frac{1}{(18-2(5))^3} + \frac{1}{(18+2(5))^3} - \frac{1}{(3(18)-2(5))^3} - \frac{1}{(3(18)+2(5))^3} + \frac{1}{(5(18)-2(5))^3} + \frac{1}{(5(18)+2(5))^3} - ...\right)$$

35.4

$$\boxed{\pi^3 = 9^3 \frac{\cos^3(\frac{5}{18}\pi)}{\sin(\frac{5}{18}\pi)}\left(\frac{1}{2^3} - \frac{1}{7^3} + \frac{1}{11^3} - \frac{1}{16^3} + \frac{1}{20^3} - \frac{1}{25^3} - \ldots\right)}$$

$$T_{5/18} = \frac{1}{2^3} - \frac{1}{7^3} + \frac{1}{11^3} - \frac{1}{16^3} + \frac{1}{20^3} - \frac{1}{25^3} + \ldots = \frac{\pi^3}{9^3}\frac{\sin(\frac{5}{18}\pi)}{\cos^3(\frac{5}{18}\pi)}$$

<u>*Proof*</u>: By $\mathbf{7_{tan}}$. Based on $\pi\tan(\pi z)\dfrac{1}{(z-\frac{5}{18})^3}$,

$$\pi^3 = 8(18^3)\frac{\cos^3(\frac{5}{18}\pi)}{\sin(\frac{5}{18}\pi)}\left(\frac{1}{(18-2(5))^3} - \frac{1}{(18+2(5))^3} + \frac{1}{(3(18)-2(5))^3} - \frac{1}{(3(18)+2(5))^3} + \frac{1}{(5(18)-2(5))^3} - \frac{1}{(5(18)+2(5))^3} + \ldots\right)$$

36.

36.1

$$\pi^3 = 18^3 \frac{\sin^2(\frac{7}{18}\pi)}{\cot(\frac{7}{18}\pi)}\left(\frac{1}{7^3} - \frac{1}{11^3} + \frac{1}{25^3} - \frac{1}{29^3} + \frac{1}{43^3} - \frac{1}{47^3} + ...\right)$$

$$R_{7/18} = \frac{1}{7^3} - \frac{1}{11^3} + \frac{1}{25^3} - \frac{1}{29^3} + \frac{1}{43^3} - \frac{1}{47^3} + ... = \frac{\pi^2}{18^3}\frac{\cot(\frac{7}{18}\pi)}{\sin^2(\frac{7}{18}\pi)}$$

Proof: By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\dfrac{1}{(z-\frac{7}{18})^3}$,

$$\pi^3 = 18^3 \frac{\sin^2(\frac{7}{18}\pi)}{\cot(\frac{7}{18}\pi)}\left(\frac{1}{7^3} - \frac{1}{(18-7)^3} + \frac{1}{(18+7)^3} + \frac{1}{(2(18)-7)^3} + \frac{1}{(2(18)+7)^3} + \frac{1}{(3(18)-7)^3} + ...\right).\square$$

36.2

$$\pi^3 = 18^3 \frac{2\sin^3(\frac{7}{18}\pi)}{\cos^2(\frac{7}{18}\pi)+1}\left(\frac{1}{7^3} + \frac{1}{11^3} - \frac{1}{25^3} - \frac{1}{29^3} + \frac{1}{43^3} + \frac{1}{47^3} - ...\right)$$

$$S_{7/18} = \frac{1}{7^3} + \frac{1}{11^3} - \frac{1}{25^3} - \frac{1}{29^3} + \frac{1}{43^3} + \frac{1}{47^3} - ... = \frac{\pi^3}{18^3}\frac{2\cot^2(\frac{7}{18}\pi)+1}{2\sin(\frac{7}{18}\pi)}$$

Proof: By $\mathbf{7_{sin}}$. Based on $\pi\frac{1}{\sin(\pi z)}\frac{1}{(z-\frac{7}{18})^3}$,

$$\pi^3 = 18^3\frac{2\sin^3(\frac{7}{18}\pi)}{\cos^2(\frac{7}{18}\pi)+1}\left(\frac{1}{7^3}+\frac{1}{(18-7)^3} - \frac{1}{(18+7)^3}-\frac{1}{(2(18)-7)^3} + \frac{1}{(2(18)+7)^3}+\frac{1}{(3(18)-7)^3}-...\right).\square$$

36.3

$$\boxed{\pi^3 = 9^3\frac{2\cos^3(\frac{7}{18}\pi)}{\sin^2(\frac{7}{18}\pi)+1}\left(\frac{1}{1}+\frac{1}{8^3} - \frac{1}{10^3}-\frac{1}{17^3} + \frac{1}{19^3}+\frac{1}{26^3}-...\right)}$$

$$C_{7/18} = \frac{1}{1^3}+\frac{1}{8^3}-\frac{1}{10^3}-\frac{1}{17^3}+\frac{1}{19^3}+\frac{1}{26^3}-... = \frac{\pi^3}{9^3}\frac{\sin^2(\frac{7}{18}\pi)+1}{2\cos^3(\frac{7}{18}\pi)}$$

Proof: By $\mathbf{7_{cos}}$. Based on $\pi\frac{1}{\cos(\pi z)}\frac{1}{(z-\frac{7}{18})^3}$,

$$\pi^3 = 8(18^3)\frac{2\cos^3(\frac{7}{18}\pi)}{\sin^2(\frac{7}{18}\pi)+1}\left(\frac{1}{(18-2(7))^3}+\frac{1}{(18+2(7))^3} - \frac{1}{(3(18)-2(7))^3}-\frac{1}{(3(18)+2(7))^3} + \frac{1}{(5(18)-2(7))^3}+\frac{1}{(5(18)+2(7))^3}-...\right).\square$$

36.4
$$\boxed{\pi^3 = 9^3 \frac{\cos^3(\frac{7}{18}\pi)}{\sin(\frac{7}{18}\pi)}\left(\frac{1}{1} - \frac{1}{8^3} + \frac{1}{10^3} - \frac{1}{17^3} + \frac{1}{19^3} - \frac{1}{26^3} + \ldots\right)}$$

$$T_{7/18} = \frac{1}{1^3} - \frac{1}{8^3} + \frac{1}{10^3} - \frac{1}{17^3} + \frac{1}{19^3} - \frac{1}{26^3} + \ldots = \frac{\pi^3}{9^3}\frac{\sin(\frac{7}{18}\pi)}{\cos^3(\frac{7}{18}\pi)}$$

Proof: By $\mathbf{7_{tan}}$. Based on $\pi\tan(\pi z)\dfrac{1}{(z-\frac{7}{18})^3}$,

$$\pi^3 = 8(18^3)\frac{\cos^3(\frac{7}{18}\pi)}{\sin(\frac{7}{18}\pi)}\left(\frac{1}{(18-2(7))^3} - \frac{1}{(18+2(7))^3} + \frac{1}{(3(18)-2(7))^3} - \frac{1}{(3(18)+2(7))^3} + \frac{1}{(5(18)-2(7))^3} - \frac{1}{(5(18)+2(7))^3} + \ldots\right).\square$$

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