π Series

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Abstract: If $l < \frac{1}{2}m$, l and m are natural numbers with no common factor. Then,

$$\pi = m\tan(\tfrac{l}{m}\pi)\left\{\frac{1}{l} - \frac{1}{m-l} + \frac{1}{m+l} - \frac{1}{2m-l} + \frac{1}{2m+l} - \frac{1}{3m-l} + \frac{1}{3m+l} + \dots\right\}$$

The associated π series is

$$R_{l/m} = \frac{1}{l} - \frac{1}{m-l} + \frac{1}{m+l} - \frac{1}{2m-l} + \frac{1}{2m+l} - \frac{1}{3m-l} + \frac{1}{3m+l} + .. = \frac{\pi}{m}\cot(\tfrac{l}{m}\pi)$$

And

$$\pi = m\sin(\tfrac{l}{m}\pi)\left\{\frac{1}{l} + \frac{1}{m-l} - \frac{1}{m+l} - \frac{1}{2m-l} + \frac{1}{2m+l} + \frac{1}{3m-l} - \frac{1}{3m+l} + \dots\right\}$$

The associated π series is

$$S_{l/m} = \frac{1}{l} + \frac{1}{m-l} - \frac{1}{m+l} - \frac{1}{2m-l} + \frac{1}{2m+l} + \frac{1}{3m-l} - \frac{1}{3m+l} + \dots = \frac{\pi}{m\sin(\frac{l}{m}\pi)}$$

For instance,

$$\boxed{\pi = 4\underbrace{\cot(\tfrac{1}{4}\pi)}_{\tan(\frac{1}{4}\pi)}\left(\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-...\right)} \text{ (Leibniz)}$$

$$\boxed{\pi = 4\underbrace{\sin(\tfrac{1}{4}\pi)}_{\cos(\frac{1}{4}\pi)}\left(\frac{1}{1}+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}+...\right)}$$

$$R_{1/4} = \frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-... = \frac{\pi}{4} \text{ (Leibniz)}$$

$$S_{1/4} = \frac{1}{1}+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}+... = \frac{\pi}{4}\sqrt{2}$$

Contents

12)
$$\pi = 38\tan(\tfrac{15}{38}\pi)\left\{\frac{1}{15} - \frac{1}{38-15} + \frac{1}{38+15} - \frac{1}{2(38)-15} + \frac{1}{2(38)+15} - \frac{1}{3(38)-15} + \frac{1}{3(38)+15} + ...\right\}$$

$$\pi = 38\sin(\tfrac{15}{38}\pi)\left\{\frac{1}{15} + \frac{1}{38-15} - \frac{1}{38+15} - \frac{1}{2(38)-15} + \frac{1}{2(38)+15} + \frac{1}{3(38)-15} - \frac{1}{3(38)+15} - \frac{1}{4(38)-15}...\right\}$$

13)
$$\pi = 2^n\tan(\tfrac{2k+1}{2^n}\pi)\left\{\frac{1}{2k+1} - \frac{1}{2^n-2k-1} + \frac{1}{2^n+2k+1} - \frac{1}{2^{n+1}-2k-1} + \frac{1}{2^{n+1}+2k+1} - \frac{1}{2^{n+2}-2k-1} + \frac{1}{2^{n+2}+2k+1} + ...\right\}$$

$$\pi = 2^n\sin(\tfrac{2k+1}{2^n}\pi)\left\{\frac{1}{2k+1} + \frac{1}{2^n-2k-1} - \frac{1}{2^n+2k+1} - \frac{1}{2^{n+1}-2k-1} + \frac{1}{2^{n+1}+2k+1} + \frac{1}{2^{n+2}-2k-1} - \frac{1}{2^{n+2}+2k+1} - ...\right\}$$

14)
$$\pi = 4\underbrace{\cot(\tfrac{1}{4}\pi)}_{\tan(\frac{1}{4}\pi)}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - ...\right)$$ (Leibniz)

$$\pi = 4\underbrace{\sin(\tfrac{1}{4}\pi)}_{\cos(\frac{1}{4}\pi)}\left(\frac{1}{1} + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} + ...\right)$$

15)
$$\pi = 8\underbrace{\tan(\tfrac{1}{8}\pi)}_{\cot(\frac{3}{8}\pi)}\left(\frac{1}{1} - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \frac{1}{23} + \frac{1}{25} - ...\right)$$

$$\pi = 8\underbrace{\sin(\tfrac{1}{8}\pi)}_{\cos(\frac{3}{8}\pi)}\left(\frac{1}{1}+\frac{1}{7}-\frac{1}{9}-\frac{1}{15}+\frac{1}{17}+\frac{1}{23}-\frac{1}{25}+...\right)$$

16) $$\pi = 8\underbrace{\tan(\tfrac{3}{8}\pi)}_{\cot(\frac{1}{8}\pi)}\left(\frac{1}{3}-\frac{1}{5}+\frac{1}{11}-\frac{1}{13}+\frac{1}{19}-\frac{1}{21}+\frac{1}{27}-\frac{1}{29}+...\right)$$

$$\pi = 8\underbrace{\sin(\tfrac{3}{8}\pi)}_{\cos(\frac{1}{8}\pi)}\left(\frac{1}{3}+\frac{1}{5}-\frac{1}{11}-\frac{1}{13}+\frac{1}{19}+\frac{1}{21}-\frac{1}{27}-\frac{1}{29}+...\right)$$

17) $R_{1/8} + R_{3/8} = S_{1/4}$

18) $S_{1/8} - S_{3/8}$

19) $R_{1/8} - R_{3/8} = R_{1/4}$

20) $$\pi = 16\underbrace{\tan(\tfrac{1}{16}\pi)}_{\cot(\frac{7}{16}\pi)}\left(\frac{1}{1}-\frac{1}{15}+\frac{1}{17}-\frac{1}{31}+\frac{1}{33}-\frac{1}{47}+\frac{1}{49}-\frac{1}{51}+...\right)$$

$$\pi = 16\underbrace{\sin(\tfrac{1}{16}\pi)}_{\cos(\frac{7}{16}\pi)}\left(\frac{1}{1}+\frac{1}{15}-\frac{1}{17}-\frac{1}{31}+\frac{1}{33}+\frac{1}{47}-\frac{1}{49}-\frac{1}{51}+...\right)$$

21) $$\pi = 16\underbrace{\tan(\tfrac{3}{16}\pi)}_{\cot(\frac{5}{16}\pi)}\left(\frac{1}{3}-\frac{1}{13}+\frac{1}{19}-\frac{1}{29}+\frac{1}{35}-\frac{1}{45}+\frac{1}{51}-\frac{1}{61}+...\right)$$

$$\pi = 16\underbrace{\sin(\tfrac{3}{16}\pi)}_{\cos(\frac{5}{16}\pi)}\left(\frac{1}{3}+\frac{1}{13}-\frac{1}{19}-\frac{1}{29}+\frac{1}{35}+\frac{1}{45}-\frac{1}{51}-\frac{1}{61}+...\right)$$

22) $$\pi = 16\underbrace{\tan(\tfrac{5}{16}\pi)}_{\cot(\frac{3}{16}\pi)}\left(\frac{1}{5}-\frac{1}{11}+\frac{1}{21}-\frac{1}{27}+\frac{1}{37}-\frac{1}{43}+\frac{1}{53}-\frac{1}{59}+\frac{1}{69}-.\right)$$

$$\boxed{\pi = 16\underbrace{\sin(\tfrac{5}{16}\pi)}_{\cos(\frac{3}{16}\pi)}\left(\frac{1}{5}+\frac{1}{11}-\frac{1}{21}-\frac{1}{27}+\frac{1}{37}+\frac{1}{43}-\frac{1}{53}-\frac{1}{59}+\frac{1}{69}+...\right)}$$

23) $$\boxed{\pi = 16\underbrace{\tan(\tfrac{5}{16}\pi)}_{\cot(\frac{3}{16}\pi)}\left(\frac{1}{5}-\frac{1}{11}+\frac{1}{21}-\frac{1}{27}+\frac{1}{37}-\frac{1}{43}+\frac{1}{53}-\frac{1}{59}+\frac{1}{69}-.\right)}$$

$$\boxed{\pi = 16\underbrace{\sin(\tfrac{5}{16}\pi)}_{\cos(\frac{3}{16}\pi)}\left(\frac{1}{5}+\frac{1}{11}-\frac{1}{21}-\frac{1}{27}+\frac{1}{37}+\frac{1}{43}-\frac{1}{53}-\frac{1}{59}+\frac{1}{69}+...\right)}$$

24) $R_{1/16}+R_{3/16}+R_{5/16}+R_{7/16}$

25) $R_{1/16}-R_{3/16}$

26) $R_{5/16}-R_{7/16}$

27) $R_{3/16}-R_{5/16}$

28) $R_{1/16}-R_{5/16}$

29) $R_{1/16}-R_{7/16}$

30) $(R_{3/16}-R_{5/16})(R_{1/16}-R_{7/16})$

31) $R_{3/16}-R_{7/16}$

32) $$\boxed{\begin{aligned}\pi = p\tan(\tfrac{1}{p}\pi)\Big\{\frac{1}{1}&-\frac{1}{p-1}+\frac{1}{p+1}\\&-\frac{1}{2p-1}+\frac{1}{2p+1}\\&-\frac{1}{3p-1}+\frac{1}{3p+1}+...\Big\}\end{aligned}}$$

$$\pi = p\sin(\tfrac{1}{p}\pi)\left\{\frac{1}{1}+\frac{1}{p-1}-\frac{1}{p+1} -\frac{1}{2p-1}+\frac{1}{2p+1} +\frac{1}{3p-1}-\frac{1}{3p+1}+\dots\right\}$$

33)

$$\pi = p^n\tan(\tfrac{m}{p^n}\pi)\left\{\frac{1}{m}-\frac{1}{p^n-m}+\frac{1}{p^n+m} -\frac{1}{2p^n-m}+\frac{1}{2p^n+m} -\frac{1}{3p^n-m}+\frac{1}{3p^n+m}+\dots\right\}$$

$$\pi = p^n\tan(\tfrac{m}{p^n}\pi)\left\{\frac{1}{m}+\frac{1}{p^n-m}-\frac{1}{p^n+m} -\frac{1}{2p^n-m}+\frac{1}{2p^n+m} +\frac{1}{3p^n-m}-\frac{1}{3p^n+m}-\dots\right\}$$

34)

$$\pi = 3\underbrace{\tan(\tfrac{1}{3}\pi)}_{\cot(\frac{1}{6}\pi)}\left(\frac{1}{1}-\frac{1}{2}+\frac{1}{4}-\frac{1}{5}+\frac{1}{7}-\frac{1}{8}+\frac{1}{10}-\frac{1}{11}+\dots\right)$$

$$\pi = 3\underbrace{\sin(\tfrac{1}{3}\pi)}_{\cos(\frac{1}{6}\pi)}\left(\frac{1}{1}+\frac{1}{2}-\frac{1}{4}-\frac{1}{5}+\frac{1}{7}+\frac{1}{8}-\frac{1}{10}-\frac{1}{11}+\dots\right)$$

35)

$$\pi = 9\underbrace{\tan(\tfrac{1}{9}\pi)}_{\cot(\frac{7}{18}\pi)}\left(\frac{1}{1}-\frac{1}{8}+\frac{1}{10}-\frac{1}{17}+\frac{1}{19}-\frac{1}{26}+\frac{1}{28}-\dots\right)$$

$$\pi = 9\underbrace{\sin(\tfrac{1}{9}\pi)}_{\cos(\frac{7}{18}\pi)}\left(\frac{1}{1}+\frac{1}{8}-\frac{1}{10}-\frac{1}{17}+\frac{1}{19}+\frac{1}{26}-\frac{1}{28}-\frac{1}{35}+\dots\right)$$

36) $$\pi = 9\underbrace{\tan(\tfrac{2}{9}\pi)}_{\cot(\frac{5}{18}\pi)}\left(\frac{1}{2}-\frac{1}{7}+\frac{1}{11}-\frac{1}{16}+\frac{1}{20}-\frac{1}{25}+\frac{1}{29}-...\right)$$

$$\pi = 9\underbrace{\sin(\tfrac{2}{9}\pi)}_{\cos(\frac{5}{18}\pi)}\left(\frac{1}{2}+\frac{1}{7}-\frac{1}{11}-\frac{1}{16}+\frac{1}{20}+\frac{1}{25}-\frac{1}{29}-\frac{1}{34}+...\right)$$

37) $$\pi = 9\underbrace{\tan(\tfrac{4}{9}\pi)}_{\cot(\frac{1}{18}\pi)}\left(\frac{1}{4}-\frac{1}{5}+\frac{1}{13}-\frac{1}{14}+\frac{1}{22}-\frac{1}{23}+\frac{1}{31}-...\right)$$

$$\pi = 9\underbrace{\sin(\tfrac{4}{9}\pi)}_{\cos(\frac{1}{18}\pi)}\left(\frac{1}{4}+\frac{1}{5}-\frac{1}{13}-\frac{1}{14}+\frac{1}{22}+\frac{1}{23}-\frac{1}{31}-\frac{1}{32}+...\right)$$

38) $$\pi = 2\sqrt{3}\left(\frac{1}{1}-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\frac{1}{17}+\frac{1}{29}-\frac{1}{23}+...\right)$$

$$\pi = 3\left(\frac{1}{1}+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}-\frac{1}{29}-\frac{1}{23}+...\right)$$

39) $$\pi = 18\underbrace{\tan(\tfrac{1}{18}\pi)}_{\cot(\frac{8}{18}\pi)}\left(\frac{1}{1}+\frac{1}{17}-\frac{1}{19}-\frac{1}{35}+\frac{1}{37}+\frac{1}{53}-\frac{1}{55}-\frac{1}{71}...\right)$$

$$\pi = 18\underbrace{\sin(\tfrac{1}{18}\pi)}_{\cos(\frac{8}{18}\pi)}\left(\frac{1}{1}+\frac{1}{17}-\frac{1}{19}-\frac{1}{35}+\frac{1}{37}+\frac{1}{53}-\frac{1}{55}-\frac{1}{71}...\right)$$

40) $$\pi = 18\underbrace{\tan(\tfrac{5}{18}\pi)}_{\cot(\frac{4}{18}\pi)}\left(\frac{1}{5}-\frac{1}{13}+\frac{1}{23}-\frac{1}{31}+\frac{1}{41}-\frac{1}{49}+\frac{1}{59}-\frac{1}{67}...\right)$$

$$\pi = 18\underbrace{\sin(\tfrac{5}{18}\pi)}_{\cos(\frac{4}{18}\pi)}\left(\frac{1}{5}+\frac{1}{13}-\frac{1}{23}-\frac{1}{31}+\frac{1}{41}+\frac{1}{49}-\frac{1}{59}-\frac{1}{67}...\right)$$

41) $$\pi = 18 \underbrace{\tan(\tfrac{7}{18}\pi)}_{\cot(\frac{2}{18}\pi)} \left(\frac{1}{7} - \frac{1}{11} + \frac{1}{25} - \frac{1}{29} + \frac{1}{43} - \frac{1}{47} + \frac{1}{61} - \frac{1}{65} + \frac{1}{79} - . \right)$$

$$\pi = 18 \underbrace{\sin(\tfrac{7}{18}\pi)}_{\cos(\frac{2}{18}\pi)} \left(\frac{1}{7} + \frac{1}{11} - \frac{1}{25} - \frac{1}{29} + \frac{1}{43} + \frac{1}{47} - \frac{1}{61} - \frac{1}{65} + \frac{1}{79} + \dots \right)$$

1.

Residue of $f(z)$ Singular at z_0

$$\boxed{\text{Res}\left\{f(z)\right\}_{z=z_0} \equiv a_{-1} = \frac{1}{2\pi i}\oint_{\zeta=z_0+\varepsilon e^{i\phi}} f(\zeta)d\zeta}$$

Proof: $f(z) = .. + \dfrac{a_{-k}}{(z-z_0)^k} + .. + \dfrac{a_{-2}}{(z-z_0)^2} +$

$$+\frac{a_{-1}}{z-z_0}+$$

$$+a_0 + ... + a_n(z-z_0)^n + ..$$

$$\Rightarrow \oint_{\zeta=z_0+\rho e^{i\phi}} f(\zeta)d\zeta =$$

$$= ... + a_{-k}\underbrace{\oint_{\zeta=z_0+\varepsilon e^{i\phi}}\frac{1}{(\zeta-z_0)^k}d\zeta}_{\frac{1}{\varepsilon^k}\varepsilon i\underbrace{\oint\frac{e^{i\phi}}{e^{ki\phi}}d\phi}_{0}} + ... + a_{-2}\underbrace{\oint_{\zeta=z_0+\varepsilon e^{i\phi}}\frac{1}{(\zeta-z_0)^2}d\zeta}_{\frac{1}{\varepsilon^2}\varepsilon i\underbrace{\oint\frac{e^{i\phi}}{e^{2i\phi}}d\phi}_{0}}$$

$$+a_{-1}\underbrace{\oint_{\zeta=z_0+\varepsilon e^{i\phi}}\frac{1}{\zeta-z_0}d\zeta}_{\frac{1}{\varepsilon}\varepsilon i\oint\frac{e^{i\phi}}{e^{i\phi}}d\phi = i\underbrace{\oint d\phi}_{2\pi}}$$

$$+a_0 \underbrace{\oint_{\zeta = z_0 + \varepsilon e^{i\phi}} d\zeta}_{\varepsilon i \underbrace{\oint e^{i\phi} d\phi}_{0}} + ... + a_n \underbrace{\oint_{\zeta = z_0 + \varepsilon e^{i\phi}} (\zeta - z_0)^n d\zeta}_{\varepsilon^{n+1} i \underbrace{\oint e^{in\phi} e^{i\phi} d\phi}_{0}} + ...$$

$$\Rightarrow \text{Res}\left\{ f(z) \right\}_{z = z_0} \equiv a_{-1} = \frac{1}{2\pi i} \oint_{\zeta = z_0 + \varepsilon e^{i\phi}} f(\zeta) d\zeta . \square$$

2.

Residue at Pole of Order k

2.1 $f(z) = \dfrac{a_{-k}}{(z-z_0)^k} + ... + \dfrac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + ...$

$$\Rightarrow \boxed{\text{Res}_{-1,k}\left\{f(z)\right\}_{z=z_0} = a_{-1} = \left[\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left\{(z-z_0)^k f(z)\right\}\right]_{z=z_0}}$$

2.2 $f(z) = \dfrac{a_{-2}}{(z-z_0)^2} + \dfrac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + ...$

$$\Rightarrow \boxed{\text{Res}_{-1,2}\left\{f(z)\right\}_{z=z_0} = a_{-1} = \left[\frac{d}{dz}\left\{(z-z_0)^2 f(z)\right\}\right]_{z=z_0}}$$

2.3 $f(z) = \dfrac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + ...$

$$\Rightarrow \boxed{\text{Res}_{-1,1}\left\{f(z)\right\}_{z=z_0} = a_{-1} = \left[(z-z_0)f(z)\right]_{z=z_0}}$$

3.

Residue at Pole of Infinite order

3.1 $e^{-\frac{1}{z}} = 1 - \frac{1}{z} + \frac{1}{2!}\left(-\frac{1}{z}\right)^2 + \frac{1}{3!}\left(-\frac{1}{z}\right)^3 + \frac{1}{4!}\left(-\frac{1}{z}\right)^4 + ... \Rightarrow$

$$\Rightarrow \text{Res}_{-2}\{e^{-\frac{1}{z}}\}_{z=0} = \frac{1}{2}$$

$$\Rightarrow \text{Res}_{-1}\{e^{-\frac{1}{z}}\}_{z=0} = -1$$

$$\Rightarrow \text{Res}_{0}\{e^{-\frac{1}{z}}\}_{z=0} = 1$$

4.

$$\operatorname{Res}\left\{\frac{\cot z\coth z}{z^3}\right\}_{z=0} = -\frac{7}{45}$$

Proof: divide the series,

$$\frac{1}{z^3}\frac{\cos z}{\sin z}\frac{\cosh z}{\sinh z} = \frac{1}{z^3}\frac{1-\frac{1}{2!}z^2+\frac{1}{4!}z^4+..}{z-\frac{1}{3!}z^3+\frac{1}{5!}z^5+...}\times\frac{1+\frac{1}{2!}z^2+\frac{1}{4!}z^4+..}{z+\frac{1}{3!}z^3+\frac{1}{5!}z^5+...}$$

$$=\frac{1}{z^3}\frac{1+\frac{1}{4!}z^4-\frac{1}{2!}z^2+..}{z+\frac{1}{5!}z^5-\frac{1}{3!}z^3+...}\times\frac{1+\frac{1}{4!}z^4+\frac{1}{2!}z^2+..}{z+\frac{1}{5!}z^5+\frac{1}{3!}z^3+...}$$

$$\approx\frac{1}{z^5}\frac{\left(1+\frac{1}{4!}z^4\right)^2-\left(\frac{1}{2!}z^2\right)^2}{\left(1+\frac{1}{5!}z^4\right)^2-\left(\frac{1}{3!}z^2\right)^2}$$

$$\approx\frac{1}{z^5}\frac{1-\left(\frac{1}{4}-\frac{1}{12}\right)z^4+\frac{1}{(4!)^2}z^8}{1-\left(\frac{1}{36}-\frac{1}{60}\right)z^4+\frac{1}{(5!)^2}z^8}$$

$$\approx\frac{1}{z^5}\frac{1-\frac{1}{6}z^4}{1-\frac{14}{90}z^4}$$

$$\approx\frac{1}{z^5}\left(1-\frac{1}{6}z^4\right)\left(1+\frac{1}{90}z^4\right)$$

$$=\left(1-\left[\frac{1}{6}-\frac{1}{90}\right]z^4-\frac{1}{540}z^8\right)$$

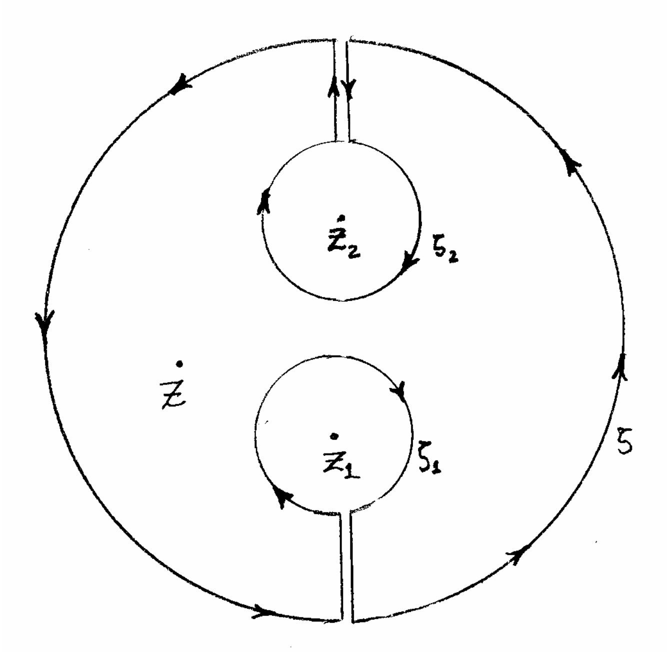
$$=\frac{1}{z^5}\left(1-\frac{14}{90}z^4-\frac{1}{540}z^8\right)$$

$$=\underbrace{1}_{a_{-5}}\frac{1}{z^5}-\underbrace{\frac{14}{90}}_{a_{-1}}\frac{1}{z}+\underbrace{\frac{1}{540}}_{a_3}z^3...$$

5.

Residue Theorem for a_{-1}

5.1



$$\frac{1}{2\pi i}\oint_C f(\zeta)d\zeta = \text{Res}_{-1}\{f(z)\}_{z=z_1} + \text{Res}_{-1}\{f(z)\}_{z=z_2}$$

Proof: $$\oint_{\zeta\in C} f(\zeta)d\zeta + \oint_{\zeta_1\in c_1} f(\zeta_1)d\zeta_1 + \oint_{\zeta_2\in c_2} f(\zeta_2)d\zeta_2 = 0$$

$$\frac{1}{2\pi i}\oint_{\zeta\in C} f(\zeta)d\zeta = \frac{1}{2\pi i}\oint_{\zeta_1=z_1+\rho e^{i\phi}} f(\zeta_1)d\zeta_1 + \frac{1}{2\pi i}\oint_{\zeta_2=z_2+\rho e^{i\phi}} f(\zeta_2)d\zeta_2,$$

For $\zeta_1 \in c_1$, $f(\zeta_1) = \dfrac{a_{-k_1,1}}{(\zeta_1 - z_1)^{k_1}} + ... + \dfrac{a_{-1,1}}{\zeta_1 - z_1} + a_{0,1} + a_{1,1}(\zeta_1 - z_1) + ...$

$$\Rightarrow \frac{1}{2\pi i} \oint_{\zeta_1 = z_1 + \rho e^{i\phi}} f(\zeta_1) d\zeta = a_{-1,1} = \text{Res}_{-1}\left\{f(z)\right\}_{z=z_1}$$

For $\zeta_2 \in c_2$,

$$f(\zeta_2) = \frac{a_{-k_2,2}}{(\zeta_2 - z_2)^{k_2}} + ... + \frac{a_{-1,2}}{\zeta_2 - z_2} + a_{0,1} + a_{1,2}(\zeta_2 - z_2) + ...$$

$$\Rightarrow \frac{1}{2\pi i} \oint_{\zeta_2 = z_2 + \rho e^{i\phi}} f(\zeta_2) d\zeta_2 = a_{-1,2} = \text{Res}_{-1}\left\{f(z)\right\}_{z=z_2}$$

$$\frac{1}{2\pi i} \oint_C f(\zeta) d\zeta = \text{Res}_{-1}\left\{f(z)\right\}_{z=z_1} + \text{Res}_{-1}\left\{f(z)\right\}_{z=z_2}. \square$$

5.2 $f(z)$ has poles at $z_1, z_2, ..., z_N \Rightarrow$

$$\frac{1}{2\pi i} \oint_C f(\zeta) d\zeta = \text{Res}_{-1}\left\{f(z)\right\}_{z=z_1} + ... + \text{Res}_{-1}\left\{f(z)\right\}_{z=z_N}$$

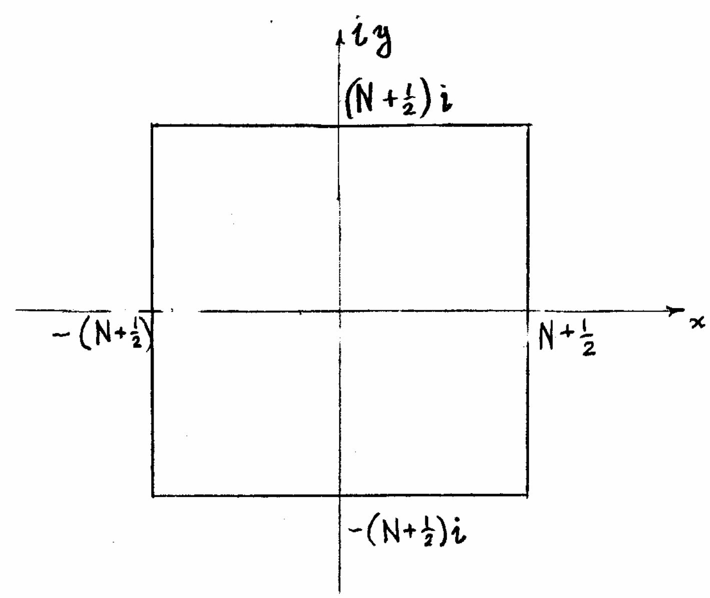
6.

$$\boxed{\pi \cot(\pi z) f(z)}$$

$$\boxed{\text{Res}\{\pi \cot(\pi z) f(z)\}_{z=n} = f(n)}$$

$$\boxed{\begin{array}{l} ...f(-3) + f(-2) + f(-1) + f(0) + f(1) + f(2) + f(3) + ... \\ \qquad + \sum \text{Res}_{-1}\left\{\pi \cot(\pi\sigma) f(\sigma)\right\}_{\sigma = \text{pole of } f(\sigma)} = 0 \end{array}}$$

6.1 $\left|\cot \pi z\right| \le A$ on $\square_{N+\frac{1}{2}}$ for any N



$$\underline{y = -N - \frac{1}{2},\ \text{and}\ -N - \frac{1}{2} \le x \le N + \frac{1}{2}} \Big| \Rightarrow$$

$$\left|\cot \pi z\right| = \frac{\left|e^{i\pi z} + e^{-i\pi z}\right|}{\left|e^{i\pi z} - e^{-i\pi z}\right|}$$

$$\leq \frac{\left|e^{i\pi z}\right| + \left|e^{-i\pi z}\right|}{\left|\left|e^{i\pi z}\right| - \left|e^{-i\pi z}\right|\right|}$$

$$= \frac{\left|e^{i\pi(x+i[-N-\frac{1}{2}])}\right| + \left|e^{-i\pi(x+i[-N-\frac{1}{2}])}\right|}{\left|\left|e^{i\pi(x+i[-N-\frac{1}{2}])}\right| - \left|e^{-i\pi(x+i[-N-\frac{1}{2}])}\right|\right|}$$

$$= \frac{\left|e^{i\pi x+\pi N+\pi/2}\right| + \left|e^{-i\pi x-\pi N-\pi/2}\right|}{\left|\left|e^{i\pi x+\pi N+\pi/2}\right| - \left|e^{-i\pi x-\pi N-\pi/2}\right|\right|}$$

$$= \frac{\left(e^{\pi N+\pi/2} + e^{-\pi N-\pi/2}\right)}{\left(e^{\pi N+\pi/2} - e^{-\pi N-\pi/2}\right)} \frac{e^{-\pi N+\pi/2}}{e^{-\pi N+\pi/2}}$$

$$= \frac{e^{\pi} + e^{-2\pi N}}{e^{\pi} - e^{-2\pi N}}$$

$$= \frac{e^{\pi} + \frac{1}{e^{2\pi N}}}{e^{\pi} - \frac{1}{e^{2\pi N}}}$$

$$< \frac{e^{\pi} + 1}{e^{\pi} - 1}$$

$$= 1 + \frac{2}{e^{\pi} - 1} < 1.1. \square$$

$$\underline{\left. y = N + \frac{1}{2}, \text{ and } -N - \frac{1}{2} \leq x \leq N + \frac{1}{2} \right|} \Rightarrow$$

$$\left|\cot \pi z\right| = \left|\frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}}\right|$$

$$\leq \frac{\left|e^{i\pi z}\right| + \left|e^{-i\pi z}\right|}{\left|\left|e^{i\pi z}\right| - \left|e^{-i\pi z}\right|\right|}$$

$$= \frac{\left|e^{i\pi(x+i[N+\frac{1}{2}])}\right| + \left|e^{-i\pi(x+i[N+\frac{1}{2}])}\right|}{\left|\left|e^{i\pi(x+i[N+\frac{1}{2}])}\right| - \left|e^{-i\pi(x+i[N+\frac{1}{2}])}\right|\right|}$$

$$= \frac{\left|e^{i\pi x-\pi N-\pi/2}\right| + \left|e^{-i\pi x+\pi N+\pi/2}\right|}{\left|\left|e^{i\pi x-\pi N-\pi/2}\right| - \left|e^{-i\pi x+\pi N+\pi/2}\right|\right|}$$

$$= \frac{(e^{-\pi N-\pi/2} + e^{\pi N+\pi/2})}{(e^{\pi N+\pi/2} - e^{-\pi N-\pi/2})} \frac{e^{-\pi N+\pi/2}}{e^{-\pi N+\pi/2}}$$

$$= \frac{e^{\pi} + e^{-2\pi N}}{e^{\pi} - e^{-2\pi N}}$$

$$= \frac{e^{\pi} + \frac{1}{e^{2\pi N}}}{e^{\pi} - \frac{1}{e^{2\pi N}}}$$

$$< \frac{e^{\pi} + 1}{e^{\pi} - 1}$$

$$= 1 + \frac{2}{e^{\pi} - 1} < 1.1. \square$$

$\underline{x = N + \frac{1}{2}, \text{ and } -N - \frac{1}{2} \le y \le N + \frac{1}{2}} \Rightarrow$

For $y = 0$, $\left|\cot \pi z\right| = \left|\cot \pi(N + \tfrac{1}{2})\right|$

$$= \left|\cot(\pi N + \tfrac{\pi}{2})\right|$$

$$= \left|\cot \frac{\pi}{2}\right|$$

$$= \tan 0 = 0.$$

For $y \neq 0$, $\left|\cot \pi z\right| = \left|\cot\left(\pi N + \tfrac{\pi}{2} + i\pi y\right)\right|$

$$= \left|\cot\left(\tfrac{\pi}{2} + i\pi y\right)\right|$$

$$= \left|\tan\left(i\pi y\right)\right|$$

$$= \left|\frac{\sin i\pi y}{\cos i\pi y}\right|$$

$$= \left|\frac{\dfrac{e^{i\pi i y} - e^{-i\pi i y}}{2i}}{\dfrac{e^{i\pi i y} + e^{-i\pi i y}}{2}}\right|$$

$$= \left|\frac{e^{-\pi y} - e^{\pi y}}{e^{-\pi y} + e^{\pi y}}\right|$$

$$= \left|\frac{1 - e^{2\pi y}}{1 + e^{2\pi y}}\right| = \left|\frac{e^{2\pi y} - 1}{e^{2\pi y} + 1}\right|$$

For $y > 0$, $\dfrac{e^{2\pi y} - 1}{e^{2\pi y} + 1} < 1.\square$

For $y < 0$, $\dfrac{1 - e^{2\pi y}}{1 + e^{2\pi y}} = \dfrac{1 - \dfrac{1}{e^{2\pi(-y)}}}{1 + \dfrac{1}{e^{2\pi(-y)}}} < 1.\square$

$\underline{x = -N - \frac{1}{2}, \text{ and } -N - \frac{1}{2} \le y \le N + \frac{1}{2}} \Rightarrow$

For $y = 0$, $\left|\cot \pi z\right| = \left|\cot \pi(-N - \tfrac{1}{2})\right|$

$$= \left|\cot(-\pi N - \tfrac{\pi}{2})\right|$$

$$= \left|\cot(-\tfrac{\pi}{2})\right|$$

$$= \tan 0 = 0.$$

For $y \neq 0$, $\left|\cot \pi z\right| = \left|\cot\left(-\pi N - \tfrac{\pi}{2} + i\pi y\right)\right|$

$$= \left|\cot\left(-\tfrac{\pi}{2} + i\pi y\right)\right|$$

$$= \left|\tan\left(i\pi y\right)\right|$$

$$= \left|\frac{\sin i\pi y}{\cos i\pi y}\right|$$

$$= \left|\frac{\dfrac{e^{i\pi i y} - e^{-i\pi i y}}{2i}}{\dfrac{e^{i\pi i y} + e^{-i\pi i y}}{2}}\right|$$

$$= \left|\frac{e^{-\pi y} - e^{\pi y}}{e^{-\pi y} + e^{\pi y}}\right|$$

$$= \left|\frac{1 - e^{2\pi y}}{1 + e^{2\pi y}}\right| = \left|\frac{e^{2\pi y} - 1}{e^{2\pi y} + 1}\right|$$

For $y > 0$, $\dfrac{e^{2\pi y} - 1}{e^{2\pi y} + 1} < 1.\square$

For $y < 0$, $\dfrac{1 - e^{2\pi y}}{1 + e^{2\pi y}} = \dfrac{1 - \dfrac{1}{e^{2\pi(-y)}}}{1 + \dfrac{1}{e^{2\pi(-y)}}} < 1.\square$

6.2 $\cot \pi z = \dfrac{\cos \pi z}{\sin \pi z}$ **has poles at** $z = n = ... - 2, -1, 0, 1, 2, ...$

6.3

$$\boxed{\operatorname{Res}\left\{\pi \cot(\pi z)\right\}_{z=n} = 1}$$

Proof: $\operatorname{Res}\left\{\pi \cot(\pi z)\right\}_{z=n} = \left[(z - n)\pi \dfrac{\cos \pi z}{\sin \pi z}\right]_{z=n}$

$$= \left[\frac{D_z(\pi z - \pi n)}{D_z \sin(\pi z)} \cos(\pi z) \right]_{z=n}$$

$$= \left[\frac{\pi}{\pi \cos(\pi z)} \cos(\pi z) \right]_{z=n} = 1. \Box$$

6.4 $$\boxed{\operatorname{Res}\left\{ \pi \cot(\pi z) f(z) \right\}_{z=n} = f(n)}$$

6.5 $\left| f(z) \right|_{\Box_{N+\frac{1}{2}}} \le \dfrac{M}{z^k} \Rightarrow$

$$\boxed{\begin{gathered} \ldots f(-3) + f(-2) + f(-1) + f(0) + f(1) + f(2) + f(3) + \ldots \\ + \sum \operatorname{Res}_{-1} \left\{ \pi \cot(\pi\sigma) f(\sigma) \right\}_{\sigma = \text{pole of } f(\sigma)} = 0 \end{gathered}}$$

<u>*Proof*</u>:

$$\oint_{\Box_{N+\frac{1}{2}}} \pi \cot(\pi\zeta) f(\zeta) d\zeta = \sum \operatorname{Res}_{-1} \left\{ \pi \cot(\pi z) f(z) \right\}_{z = \text{pole of } \cot \pi z \text{ in } \Box_{N+\frac{1}{2}}}$$

$$+ \sum \operatorname{Res}_{-1} \left\{ \pi \cot(\pi\sigma) f(\sigma) \right\}_{\sigma = \text{pole of } f(\sigma) \text{ in } \Box_{N+\frac{1}{2}}}$$

$$\left| \oint_{\Box_{N+\frac{1}{2}}} \pi \cot(\pi\zeta) f(\zeta) d\zeta \right| \le \pi \underbrace{\left| \cot \pi\zeta \right|}_{\le A} \frac{M}{N^k} \underbrace{(\text{length} \Box_{N+\frac{1}{2}})}_{8\left(N+\frac{1}{2}\right)} \xrightarrow[N \to \infty]{} 0$$

$$\operatorname{Res}\left\{ \pi \cot(\pi z) f(z) \right\}_{z=n} = f(n). \Box$$

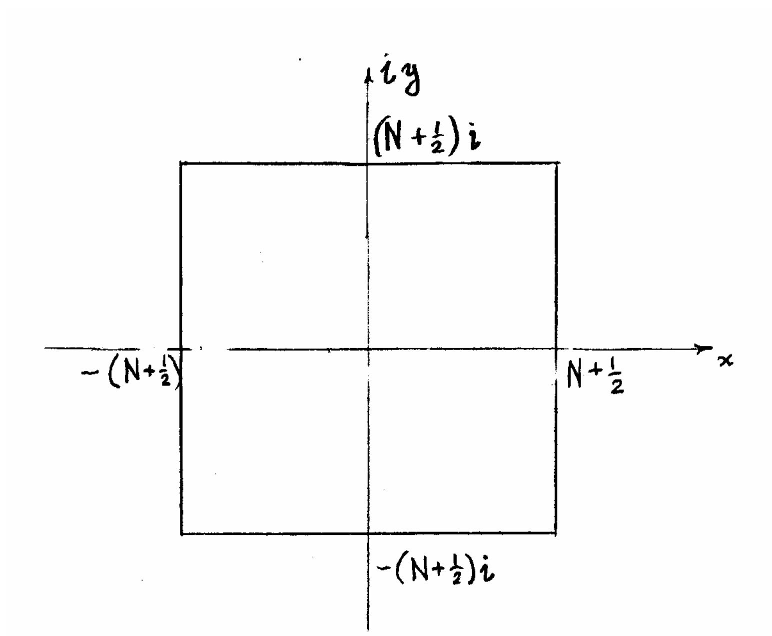
7.

$$\boxed{\frac{\pi}{\sin \pi z} f(z)}$$

$$\boxed{\text{Res}\left\{\pi \frac{1}{\sin \pi z} f(z)\right\}_{z=n} = (-1)^n f(n)}$$

$$\boxed{\begin{aligned}&... - f(-3) + f(-2) - f(-1) + f(0) - f(1) + f(2) - f(3) + ...\\ &+ \sum \text{Res}_{-1}\left\{\frac{\pi}{\sin(\pi\sigma)} f(\sigma)\right\}_{\sigma=\text{pole of } f(\sigma)} = 0\end{aligned}}$$

7.1 $\left|\frac{1}{\sin \pi z}\right| \leq A$ on $\square_{N+\frac{1}{2}}$ for any N



$$\underline{\left. y = -N - \frac{1}{2}, \text{ and } -N - \frac{1}{2} \leq x \leq N + \frac{1}{2} \right|} \Rightarrow$$

$$\left|\frac{1}{\sin \pi z}\right| = \frac{2}{\left|e^{i\pi z} - e^{-i\pi z}\right|}$$

$$= \frac{2}{\left|e^{i\pi(x+iy)} - e^{-i\pi(x+iy)}\right|}$$

$$\leq \frac{2}{\left|\left|e^{i\pi(x+iy)}\right| - \left|e^{-i\pi(x+iy)}\right|\right|}$$

$$= \frac{2}{\left|e^{-\pi y} - e^{\pi y}\right|}$$

$$= \frac{2}{e^{\pi N + \frac{\pi}{2}} - e^{-\pi N - \frac{\pi}{2}}} \frac{e^{-\pi N + \frac{\pi}{2}}}{e^{-\pi N + \frac{\pi}{2}}}$$

$$= \frac{2e^{-\pi N + \frac{\pi}{2}}}{e^{\pi} - e^{-2\pi N}}$$

$$= \frac{\dfrac{2e^{\frac{\pi}{2}}}{e^{\pi N}}}{e^{\pi} - \dfrac{1}{e^{2\pi N}}}$$

$$< \frac{2e^{\frac{\pi}{2}}}{e^{\pi} - 1} \equiv A. \square$$

$$\underline{\left. y = N + \frac{1}{2}, \text{ and } -N - \frac{1}{2} \leq x \leq N + \frac{1}{2} \right|} \Rightarrow$$

$$\left|\frac{1}{\sin \pi z}\right| = \frac{2}{\left|e^{i\pi z} - e^{-i\pi z}\right|}$$

$$= \frac{2}{\left|e^{i\pi(x+iy)} - e^{-i\pi(x+iy)}\right|}$$

$$\leq \frac{2}{\left|\left|e^{i\pi(x+iy)}\right| - \left|e^{-i\pi(x+iy)}\right|\right|}$$

$$= \frac{2}{\left|e^{-\pi y} - e^{\pi y}\right|}$$

$$= \frac{2}{e^{\pi N + \frac{\pi}{2}} - e^{-\pi N - \frac{\pi}{2}}} \frac{e^{-\pi N + \frac{\pi}{2}}}{e^{-\pi N + \frac{\pi}{2}}}$$

$$= \frac{2e^{-\pi N + \frac{\pi}{2}}}{e^{\pi} - e^{-2\pi N}}$$

$$= \frac{\frac{2e^{\frac{\pi}{2}}}{e^{\pi N}}}{e^{\pi} - \frac{1}{e^{2\pi N}}}$$

$$< \frac{2e^{\frac{\pi}{2}}}{e^{\pi} - 1} \equiv A. \square$$

$$\underline{x = N + \frac{1}{2}, \text{ and } -N - \frac{1}{2} \leq y \leq N + \frac{1}{2}} \Big| \Rightarrow$$

$$\left|\frac{1}{\sin \pi z}\right| = \frac{1}{\left|\sin(\pi N + \frac{\pi}{2} + i\pi y)\right|}$$

$$= \left|\csc(\pi N + \tfrac{\pi}{2} + i\pi y)\right|$$

$$= \left|\csc(\tfrac{\pi}{2} + i\pi y)\right|$$

$$= \frac{1}{\left|\sin(\frac{\pi}{2} + i\pi y)\right|}$$

$$= \frac{1}{\left|\cos(i\pi y)\right|}$$

$$= \frac{1}{\left|\frac{e^{i\pi(iy)} + e^{-i\pi(iy)}}{2}\right|}$$

$$= \frac{2}{e^{-\pi y} + e^{\pi y}}$$

For $y = 0$, $\quad \frac{2}{e^{-\pi y} + e^{\pi y}} = 1$

For $y > 0$, $\quad \frac{2}{e^{-\pi y} + e^{\pi y}} \frac{e^{\pi y}}{e^{\pi y}} = 2\frac{e^{\pi y}}{e^{2\pi y} + 1} < 2$

For $y < 0$, $\quad \frac{2}{e^{-\pi y} + e^{\pi y}} \frac{e^{-\pi y}}{e^{-\pi y}} = 2\frac{e^{\pi(-y)}}{e^{2\pi(-y)} + 1} < 2 . \square$

$\underline{x = -N - \frac{1}{2}, \text{ and } -N - \frac{1}{2} \le y \le N + \frac{1}{2}} \Rightarrow$

$$\left|\frac{1}{\sin \pi z}\right| = \frac{1}{\left|\sin(-\pi N - \frac{\pi}{2} + i\pi y)\right|}$$

$$= \left|\csc(-\pi N - \tfrac{\pi}{2} + i\pi y)\right|$$

$$= \left|\csc(-\tfrac{\pi}{2} + i\pi y)\right|$$

$$= \frac{1}{\left|\sin(\frac{\pi}{2} + i\pi y)\right|}$$

$$= \frac{1}{\left|\cos(i\pi y)\right|}$$

$$= \frac{1}{\left|\frac{e^{i\pi(iy)} + e^{-i\pi(iy)}}{2}\right|}$$

$$= \frac{2}{e^{-\pi y} + e^{\pi y}}$$

For $y = 0$, $\frac{2}{e^{-\pi y} + e^{\pi y}} = 1$

For $y > 0$, $\frac{2}{e^{-\pi y} + e^{\pi y}} \frac{e^{\pi y}}{e^{\pi y}} = 2 \frac{e^{\pi y}}{e^{2\pi y} + 1} < 2$

For $y < 0$, $\frac{2}{e^{-\pi y} + e^{\pi y}} \frac{e^{-\pi y}}{e^{-\pi y}} = 2 \frac{e^{\pi(-y)}}{e^{2\pi(-y)} + 1} < 2 . \square$

7.2 $\frac{1}{\sin \pi z}$ **has poles of order 1 at** $z = n = ... -2, -1, 0, 1, 2, ...$

7.3

$$\boxed{\text{Res}\left\{\pi \frac{1}{\sin \pi z}\right\}_{z=n} = (-1)^n}$$

Proof: $\text{Res}\left\{\pi \frac{1}{\sin \pi z}\right\}_{z=n} = \left[(z-n)\pi \frac{1}{\sin \pi z}\right]_{z=n}$

$$= \left[\frac{D_z(\pi z - \pi n)}{D_z \sin(\pi z)}\right]_{z=n}$$

$$= \left[\frac{\pi}{\pi \cos(\pi z)}\right]_{z=n} = (-1)^n . \square$$

7.4

$$\boxed{\text{Res}\left\{\pi \frac{1}{\sin \pi z} f(z)\right\}_{z=n} = (-1)^n f(n)}$$

7.5 $\left|f(z)\right|_{\square_{N+\frac{1}{2}}} \leq \dfrac{M}{z^k} \Rightarrow$

$$\boxed{\begin{aligned}&... - f(-3) + f(-2) - f(-1) + f(0) - f(1) + f(2) - f(3) + ...\\ &\qquad + \sum \mathrm{Res}_{-1}\left\{\frac{\pi}{\sin(\pi\sigma)}f(\sigma)\right\}_{\sigma=\text{pole of } f(\sigma)} = 0\end{aligned}}$$

Proof:

$$\oint_{\square_{N+\frac{1}{2}}} \frac{\pi}{\sin(\pi\zeta)}f(\zeta)d\zeta = \sum \mathrm{Res}_{-1}\left\{\frac{\pi}{\sin(\pi z)}f(z)\right\}_{z=\text{pole of } \frac{\pi}{\sin(\pi\zeta)} \text{ in } \square_{N+\frac{1}{2}}}$$

$$+\sum \mathrm{Res}_{-1}\left\{\frac{\pi}{\sin(\pi\sigma)}f(\sigma)\right\}_{\sigma=\text{pole of } f(\sigma) \text{ in } \square_{N+\frac{1}{2}}}$$

$$\left|\oint_{\square_{N+\frac{1}{2}}} \frac{\pi}{\sin(\pi\zeta)}f(\zeta)d\zeta\right| \leq \pi\underbrace{\left|\frac{1}{\sin(\pi\zeta)}\right|}_{\leq A}\frac{M}{N^k}\underbrace{(\text{length}\square_{N+\frac{1}{2}})}_{8\left(N+\frac{1}{2}\right)} \xrightarrow[N\to\infty]{} 0$$

$\dfrac{1}{\sin \pi z}$ **has poles at** $z = n$

$$\mathrm{Res}_{-1}\left\{\frac{\pi}{\sin(\pi z)}f(z)\right\}_{z=\text{pole of } \frac{\pi}{\sin(\pi z)} \text{ in } \square_{N+\frac{1}{2}}} = (-1)^n f(n).\square$$

$\Rightarrow$

$$... - f(-3) + f(-2) - f(-1) + f(0) - f(1) + f(2) - f(3) + ...$$

$$+ \sum \mathrm{Res}_{-1}\left\{\frac{\pi}{\sin(\pi\sigma)}f(\sigma)\right\}_{\sigma=pole\ \ of\ \ f(\sigma)} = 0$$

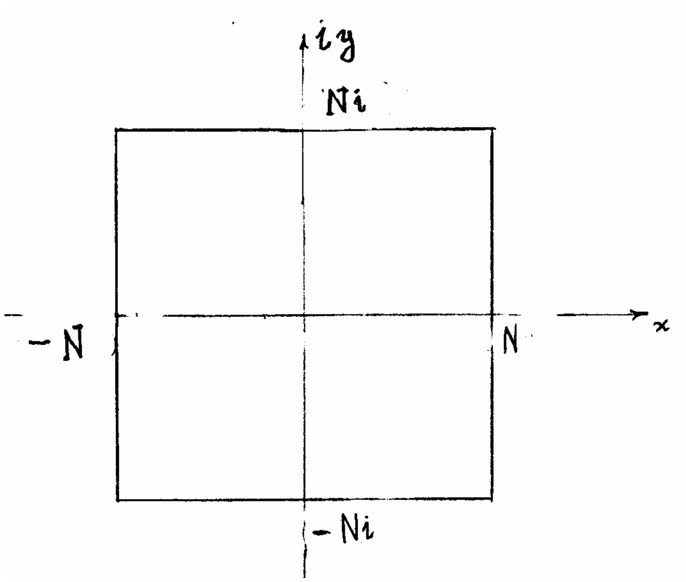
8.

$$\boxed{\frac{\pi}{\cos \pi z} f(z)}$$

$$\boxed{\text{Res}\left\{\frac{\pi}{\cos(\pi z)} f(z)\right\}_{z=n+\frac{1}{2}} = -(-1)^n f(n+\tfrac{1}{2})}$$

$$\boxed{\begin{aligned}&... - f(-\tfrac{5}{2}) + f(-\tfrac{3}{2}) - f(-\tfrac{1}{2}) + f(\tfrac{1}{2}) - f(\tfrac{3}{2}) + f(\tfrac{5}{2}) - ... = \\ &= \sum \text{Res}_{-1}\left\{\frac{\pi}{\cos(\pi\sigma)} f(\sigma)\right\}_{\sigma=\text{pole of } f(\sigma)}\end{aligned}}$$

8.1 $\left|\frac{1}{\cos \pi z}\right| \le A$ **on** $\Box_N$ **for any** N



$\underline{y = -N, \text{ and } -N \le x \le N}| \Rightarrow$

$$\frac{1}{|\cos \pi z|} = \frac{2}{|e^{i\pi z} + e^{-i\pi z}|}$$

$$= \frac{2}{\left|e^{i\pi(x+iy)} + e^{-i\pi(x+iy)}\right|}$$

$$\leq \frac{2}{\left|\left|e^{i\pi(x+iy)}\right| - \left|e^{-i\pi(x+iy)}\right|\right|}$$

$$= \frac{2}{\left|e^{-\pi y} - e^{\pi y}\right|}$$

$$= \frac{2}{\left|e^{-\pi(-N)} - e^{\pi(-N)}\right|}$$

$$= \frac{2}{e^{\pi N} - e^{-\pi N}}$$

$$\leq 2\frac{1}{e^{\pi} - 1}.\square$$

$\underline{y = N, \text{ and } -N \leq x \leq N}\Big| \Rightarrow$

$$\left|\frac{1}{\cos \pi z}\right| = \frac{2}{\left|e^{-i\pi z} + e^{i\pi z}\right|}$$

$$= \frac{2}{\left|e^{i\pi(x+iy)} + e^{-i\pi(x+iy)}\right|}$$

$$\leq \frac{2}{\left|\left|e^{i\pi(x+iy)}\right| - \left|e^{-i\pi(x+iy)}\right|\right|}$$

$$= \frac{2}{\left|e^{-\pi y} - e^{\pi y}\right|}$$

$$= \frac{2}{\left|e^{-\pi N} - e^{\pi N}\right|}$$

$$\leq 2\frac{1}{e^{\pi}-1}.\square$$

$\underline{x = N,\ \text{and}\ -N \leq y \leq N}| \Rightarrow$

$$\left|\frac{1}{\cos \pi z}\right| = \frac{1}{\left|\cos(\pi N + i\pi y)\right|}$$

$$= \left|\sec(\pi N + i\pi y)\right|$$

$$= \left|\sec(i\pi y)\right|$$

$$= \frac{1}{\left|\cos(i\pi y)\right|}$$

$$= \frac{2}{\left|e^{i(i\pi y)} + e^{-i(i\pi y)}\right|}$$

$$= \frac{2}{\left|e^{-\pi y} + e^{\pi y}\right|}$$

$$= \frac{2}{e^{-\pi y} + e^{\pi y}}$$

For $y = 0$, $\quad \frac{2}{e^{-\pi y} + e^{\pi y}} = 1$

For $y > 0$, $\quad \frac{2}{e^{-\pi y} + e^{\pi y}}\frac{e^{\pi y}}{e^{\pi y}} = 2\frac{e^{\pi y}}{e^{2\pi y}+1} < 2$

For $y < 0$, $\quad \frac{2}{e^{-\pi y} + e^{\pi y}}\frac{e^{-\pi y}}{e^{-\pi y}} = 2\frac{e^{\pi(-y)}}{e^{2\pi(-y)}+1} < 2.\square$

$\underline{x = -N,\ \text{and}\ -N \leq y \leq N}| \Rightarrow$

$$\left|\frac{1}{\cos \pi z}\right| = \frac{1}{\left|\cos(-\pi N + i\pi y)\right|}$$

$$= \left|\sec(-\pi N + i\pi y)\right|$$

$$= \left|\sec(i\pi y)\right|$$

$$= \frac{1}{\left|\cos(i\pi y)\right|}$$

$$= \frac{2}{\left|e^{i(i\pi y)} + e^{-i(i\pi y)}\right|}$$

$$= \frac{2}{\left|e^{-\pi y} + e^{\pi y}\right|}$$

$$= \frac{2}{e^{-\pi y} + e^{\pi y}}$$

For $y = 0\,,\quad \dfrac{2}{e^{-\pi y} + e^{\pi y}} = 1$

For $y > 0\,,\quad \dfrac{2}{e^{-\pi y} + e^{\pi y}}\dfrac{e^{\pi y}}{e^{\pi y}} = 2\dfrac{e^{\pi y}}{e^{2\pi y} + 1} < 2$

For $y < 0\,,\quad \dfrac{2}{e^{-\pi y} + e^{\pi y}}\dfrac{e^{-\pi y}}{e^{-\pi y}} = 2\dfrac{e^{\pi(-y)}}{e^{2\pi(-y)} + 1} < 2\,.\square$

8.2 $\quad \dfrac{1}{\cos \pi z}$ **has poles at** $z = n + \frac{1}{2} = ...,-\frac{5}{2},-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{5}{2},...$

8.3

$$\boxed{\operatorname{Res}\left\{\frac{\pi}{\cos(\pi z)}\right\}_{z=n+\frac{1}{2}} = -(-1)^n}$$

Proof: $\quad \operatorname{Res}\left\{\dfrac{\pi}{\cos(\pi z)}\right\}_{z=n+\frac{1}{2}} = \left[(z - [n + \tfrac{1}{2}])\dfrac{\pi}{\cos(\pi z)}\right]_{z=n+\frac{1}{2}}$

$$= \pi \left[\frac{D_z(z - n - \frac{1}{2})}{D_z \cos(\pi z)} \right]_{z=n+\frac{1}{2}}$$

$$= \left[\frac{1}{-\sin(\pi z)} \right]_{z=n+\frac{1}{2}}$$

$$= \frac{1}{-\sin(\pi n + \frac{\pi}{2})}$$

$$= -(-1)^n . \square$$

8.4
$$\boxed{\text{Res}\left\{ \frac{\pi}{\cos(\pi z)} f(z) \right\}_{z=n+\frac{1}{2}} = -(-1)^n f(n + \tfrac{1}{2})}$$

8.5 $\left| f(z) \right|_{\square_N} \leq \dfrac{M}{z^k} \Rightarrow$

$$\boxed{\begin{aligned} &... - f(-\tfrac{5}{2}) + f(-\tfrac{3}{2}) - f(-\tfrac{1}{2}) + f(\tfrac{1}{2}) - f(\tfrac{3}{2}) + f(\tfrac{5}{2}) - ... = \\ &= \sum \text{Res}_{-1} \left\{ \frac{\pi}{\cos(\pi\sigma)} f(\sigma) \right\}_{\sigma = \text{pole of } f(\sigma)} \end{aligned}}$$

Proof:

$$\oint_{\square_N} \frac{\pi}{\cos(\pi\zeta)} f(\zeta) d\zeta = \sum \text{Res}_{-1} \left\{ \frac{\pi}{\cos(\pi z)} f(z) \right\}_{z = \text{pole of } \frac{\pi}{\cos(\pi z)} \text{ in } \square_N}$$

$$+ \sum \text{Res}_{-1} \left\{ \frac{\pi}{\cos(\pi\sigma)} f(\sigma) \right\}_{\sigma = \text{pole of } f(\sigma) \text{ in } \square_N}$$

$$\left|\oint_{\square_N} \frac{\pi}{\cos(\pi\zeta)} f(\zeta)d\zeta\right| \leq \underbrace{\left|\frac{\pi}{\cos(\pi\zeta)}\right|}_{\leq A} \frac{M}{N^k} \underbrace{(\text{length}\square_N)}_{8(N)} \xrightarrow[N\to\infty]{} 0$$

$\dfrac{1}{\cos \pi z}$ **has poles at** $z = n + \frac{1}{2}$

$$\operatorname{Res}_{-1}\left\{\frac{\pi}{\cos(\pi z)} f(z)\right\}_{z=\text{pole of } \frac{\pi}{\cos(\pi z)} \text{ in } \square_N} = \operatorname{Res}_{-1}\left\{\frac{\pi}{\cos(\pi z)} f(z)\right\}_{z=n+\frac{1}{2}}$$

$$= -(-1)^n f(n+\tfrac{1}{2})$$

$$\Rightarrow \quad \sum(-1)^n f(n+\tfrac{1}{2}) = \sum \operatorname{Res}_{-1}\left\{\frac{\pi}{\cos(\pi\sigma)} f(\sigma)\right\}_{\sigma=\text{pole of } f(z)}$$

$$\Rightarrow \quad ... + f(-\tfrac{7}{2}) - f(-\tfrac{5}{2}) + f(-\tfrac{3}{2}) - f(-\tfrac{1}{2}) + f(\tfrac{1}{2}) - f(\tfrac{3}{2}) + f(\tfrac{5}{2}) + ... =$$

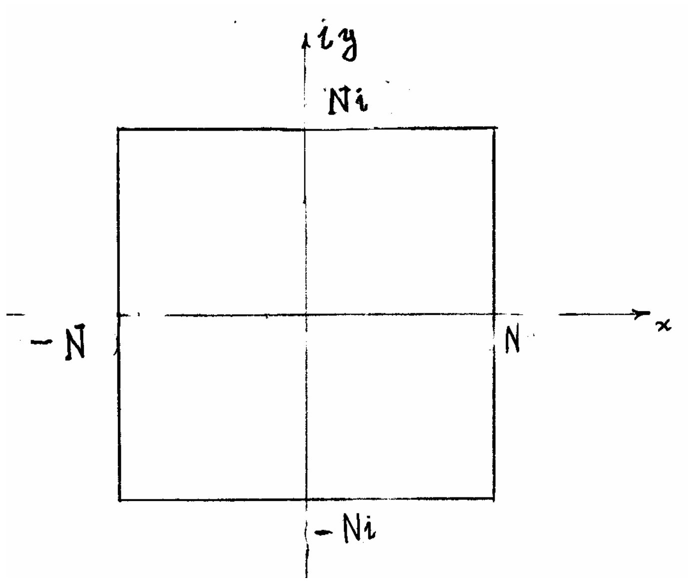
$$= \sum \operatorname{Re}s_{-1}\left\{\frac{\pi}{\cos(\pi\sigma)} f(\sigma)\right\}_{\sigma=pole\ \ of\ \ f(z)} . \square$$

9.

$$\boxed{\pi\tan(\pi z)f(z)}$$

$$\boxed{\begin{array}{c} ..+f(-\tfrac{5}{2})+f(-\tfrac{3}{2})+f(-\tfrac{1}{2})\ +f(\tfrac{1}{2})+f(\tfrac{3}{2})+f(\tfrac{5}{2})+..= \\ \\ =\sum \text{Res}_{-1}\left\{\pi\tan(\pi\sigma)f(\sigma)\right\}_{\sigma=pole\ \ of\ \ f(\sigma)} \end{array}}$$

9.1 $\left|\tan\pi z\right| \le A$ **on** $\square_N$ **for any** N



$\underline{y=-N,\ \text{and}\ \ -N\le x\le N}\Big| \Rightarrow$

$$\left|\tan\pi z\right| = \left|\frac{e^{i\pi z}-e^{-i\pi z}}{e^{i\pi z}+e^{-i\pi z}}\right|$$

$$= \left|\frac{1-e^{-2i\pi z}}{1+e^{-2i\pi z}}\right|$$

$$= \left|\frac{1-e^{-2i\pi x-2\pi N}}{1+e^{-2i\pi x-2\pi N}}\right|$$

$$< \frac{1+e^{-2\pi N}}{1-e^{-2\pi N}}$$

$$< \frac{1+e^{-\pi}}{1-e^{-\pi}}.\square$$

$\underline{y = N, \text{ and } -N \le x \le N}| \Rightarrow$

$$\left|\tan \pi z\right| = \left|\frac{e^{i\pi z} - e^{-i\pi z}}{e^{i\pi z} + e^{-i\pi z}}\right|$$

$$= \left|\frac{e^{2i\pi z} - 1}{e^{2i\pi z} + 1}\right|$$

$$= \left|\frac{e^{-2i\pi x + 2\pi N} - 1}{e^{-2i\pi x + 2\pi N} + 1}\right|$$

$$< \frac{e^{2\pi N} + 1}{e^{2\pi N} - 1}\frac{e^{-2\pi N}}{e^{-2\pi N}}$$

$$< \frac{1 + \frac{1}{e^{2\pi N}}}{1 - \frac{1}{e^{2\pi N}}}$$

$$< \frac{1 + \frac{1}{e^{\pi}}}{1 - \frac{1}{e^{\pi}}}$$

$$< \frac{e^{\pi} + 1}{e^{\pi} - 1}$$

$$< 1 + 2\frac{1}{e^{\pi} - 1}.\square$$

$\underline{x = N, \text{ and } -N \le y \le N}| \Rightarrow$

For $y = 0$, $\left|\tan \pi z\right| = \left|\tan \pi N\right|$

$$= \tan 0 = 0.$$

For $y \neq 0$, $\left|\tan \pi z\right| = \left|\tan\left(\pi N + i\pi y\right)\right|$

$$= \left|\tan\left(i\pi y\right)\right|$$

$$= \left|\frac{\sin i\pi y}{\cos i\pi y}\right|$$

$$= \left|\frac{\dfrac{e^{i\pi i y} - e^{-i\pi i y}}{2i}}{\dfrac{e^{i\pi i y} + e^{-i\pi i y}}{2}}\right|$$

$$= \left|\frac{e^{-\pi y} - e^{\pi y}}{e^{-\pi y} + e^{\pi y}}\right|$$

$$= \left|\frac{1 - e^{2\pi y}}{1 + e^{2\pi y}}\right| = \left|\frac{e^{2\pi y} - 1}{e^{2\pi y} + 1}\right|$$

For $y > 0$, $\dfrac{e^{2\pi y} - 1}{e^{2\pi y} + 1} < 1.\square$

For $y < 0$, $\dfrac{1 - e^{2\pi y}}{1 + e^{2\pi y}} = \dfrac{1 - \dfrac{1}{e^{2\pi(-y)}}}{1 + \dfrac{1}{e^{2\pi(-y)}}} < 1.\square$

$\underline{x = -N, \text{ and } -N \leq y \leq N}\Rightarrow$

For $y = 0$, $\left|\tan \pi z\right| = \left|\tan \pi(-N)\right| = \tan 0 = 0$

For $y \neq 0$, $\left|\tan \pi z\right| = \left|\tan\left(-\pi N + i\pi y\right)\right|$

$$= \left|\tan\left(i\pi y\right)\right|$$

$$= \left| \frac{\sin i\pi y}{\cos i\pi y} \right|$$

$$= \left| \frac{\frac{e^{i\pi i y} - e^{-i\pi i y}}{2i}}{\frac{e^{i\pi i y} + e^{-i\pi i y}}{2}} \right|$$

$$= \left| \frac{e^{-\pi y} - e^{\pi y}}{e^{-\pi y} + e^{\pi y}} \right|$$

$$= \left| \frac{1 - e^{2\pi y}}{1 + e^{2\pi y}} \right| = \left| \frac{e^{2\pi y} - 1}{e^{2\pi y} + 1} \right|$$

For $y > 0$, $\frac{e^{2\pi y} - 1}{e^{2\pi y} + 1} < 1. \square$

For $y < 0$, $\frac{1 - e^{2\pi y}}{1 + e^{2\pi y}} = \frac{1 - \frac{1}{e^{2\pi(-y)}}}{1 + \frac{1}{e^{2\pi(-y)}}} < 1. \square$

9.2 $\pi \tan(\pi\zeta)$ **has poles at** $z = n + \frac{1}{2}$

9.3
$$\boxed{\text{Res}\left\{\pi \tan(\pi z)\right\}_{z = n+\frac{1}{2}} = -1}$$

Proof: $\text{Res}\left\{\pi \tan(\pi z)\right\}_{z = n+\frac{1}{2}} = \left[(z - [n + \tfrac{1}{2}]) \frac{\pi \sin(\pi z)}{\cos(\pi z)} \right]_{z = n+\frac{1}{2}}$

$$= \pi \left[\frac{D_z(z - n)}{D_z \cos(\pi z)} \sin(\pi z) \right]_{z = n+\frac{1}{2}}$$

$$= \left[\frac{1}{-\sin(\pi z)} \sin(\pi z) \right]_{z = n+\frac{1}{2}} = -1$$

9.4 $$\boxed{\text{Res}\left\{\pi\tan(\pi z)f(z)\right\}_{z=n+\frac{1}{2}} = -f(n+\tfrac{1}{2})}$$

9.5 $\left|f(z)\right|_{\square_N} \le \dfrac{M}{z^k} \Rightarrow$

$$\boxed{\begin{aligned}&..+f(-\tfrac{5}{2})+f(-\tfrac{3}{2})+f(-\tfrac{1}{2})\ +f(\tfrac{1}{2})+f(\tfrac{3}{2})+f(\tfrac{5}{2})+..=\\ &=\sum \text{Res}_{-1}\left\{\pi\tan(\pi\sigma)f(\sigma)\right\}_{\sigma=\text{pole of } f(\sigma)}\end{aligned}}$$

Proof:

$$\oint_{\square_N} \pi\tan(\pi\zeta)f(\zeta)d\zeta = \sum \text{Res}_{-1}\left\{\pi\tan(\pi\zeta)f(z)\right\}_{z=\text{pole of } \pi\tan(\pi\zeta) \text{ in } \square_N}$$

$$+\sum \text{Res}_{-1}\left\{\pi\tan(\pi\zeta)f(\sigma)\right\}_{\sigma=\text{pole of } f(\sigma) \text{ in } \square_N}$$

$$\left|\oint_{\square_N} \pi\tan(\pi\zeta)f(\zeta)d\zeta\right| \le \underbrace{\left|\pi\tan(\pi\zeta)\right|}_{\le A}\frac{M}{N^k}\underbrace{(\text{length}\square_N)}_{8N} \xrightarrow[N\to\infty]{} 0$$

$\pi\tan(\pi\zeta)$ **has poles at** $z = n+\frac{1}{2}$

$$\text{Res}_{-1}\left\{\pi\tan(\pi z)f(z)\right\}_{z=n+\frac{1}{2}} = -f(n+\tfrac{1}{2})$$

$$\Rightarrow\ ..+f(-\tfrac{5}{2})+f(-\tfrac{3}{2})+f(-\tfrac{1}{2})\ +f(\tfrac{1}{2})+f(\tfrac{3}{2})+f(\tfrac{5}{2})+...=$$

$$=\sum \text{Res}_{-1}\left\{\pi\tan(\pi\sigma)f(\sigma)\right\}_{\sigma=pole\ \ of\ \ f(\sigma)}$$

10.

$$\pi = m\tan(\tfrac{l}{m}\pi)\left\{\frac{1}{l}-\frac{1}{m-l}+\frac{1}{m+l}-\frac{1}{2m-l}+\frac{1}{2m+l}-\frac{1}{3m-l}+\frac{1}{3m+l}+\ldots\right\}$$

$l < \frac{1}{2}m$, l and m have no common factor

$$R_{l/m} = \frac{1}{l}-\frac{1}{m-l}+\frac{1}{m+l}-\frac{1}{2m-l}+\frac{1}{2m+l}-\frac{1}{3m-l}+\frac{1}{3m+l}+.. = \frac{\pi}{m}\cot(\tfrac{l}{m}\pi)$$

Proof: $\pi\cot(\pi z)\dfrac{1}{z+\frac{l}{m}}$ has poles of order 1 at $z = n$,

and a pole at $z = -\frac{l}{m}$

$$\oint_{\square_{N+\frac{1}{2}}}\pi\cot(\pi\zeta)\frac{1}{\zeta+\frac{l}{m}}d\zeta = \sum \mathrm{Res}_{-1}\left\{\pi\cot(\pi z)\frac{1}{z+\frac{l}{m}}\right\}_{z=n} + \mathrm{Res}_{-1}\left\{\pi\cot(\pi z)\frac{1}{z+\frac{l}{m}}\right\}_{z=-\frac{l}{m}}$$

$$\left|\oint_{\square_{N+\frac{1}{2}}}\pi\cot(\pi\zeta)\frac{1}{\zeta+\frac{l}{m}}d\zeta\right| \le \underbrace{\left|\pi\cot\pi\zeta\right|}_{\le A}\underbrace{\oint_{\square_{N+\frac{1}{2}}}\frac{1}{\zeta+\frac{l}{m}}d\zeta}_{\left[\log(\zeta+\frac{l}{m})\right]_{\square_{N+\frac{1}{2}}}=0} = 0.\square$$

$$\mathrm{Res}_{-1}\left\{\pi\cot(\pi z)\frac{1}{z+\frac{l}{m}}\right\}_{z=n} = \left[(z-n)\pi\frac{\cos\pi z}{\sin\pi z}\frac{1}{z+\frac{l}{m}}\right]_{z=n}$$

$$= \left[\frac{\pi D_z(z-n)}{D_z \sin(\pi z)}\right]_{z=n} \left(\cos(\pi n)\frac{1}{n+\frac{l}{m}}\right)$$

$$= \left[\frac{\pi}{\pi \cos(\pi z)}\right]_{z=n} \left(\cos(\pi n)\frac{1}{n+\frac{l}{m}}\right)$$

$$= \frac{1}{n+\frac{l}{m}}.\square$$

To find $\mathrm{Res}_{-1}\left\{\pi\cot(\pi z)\frac{1}{z+\frac{l}{m}}\right\}_{z=-\frac{l}{m}}$**, divide the series**

$$\pi\frac{\cos(\pi z)}{\sin(\pi z)}\frac{1}{z+\frac{l}{m}} = \pi\frac{\sin(\pi z+\frac{\pi}{2})}{-\cos(\pi z+\frac{\pi}{2})}\frac{1}{z+\frac{l}{m}}$$

$$= -\pi\frac{\sin\left[\pi(z+\frac{l}{m})+(\frac{1}{2}-\frac{l}{m})\pi\right]}{\cos\left[\pi(z+\frac{l}{m})+(\frac{1}{2}-\frac{l}{m})\pi\right]}\frac{1}{z+\frac{l}{m}}$$

$$= -\pi\frac{\sin\left[\pi u+(\frac{1}{2}-\frac{l}{m})\pi\right]}{\cos\left[\pi u+(\frac{1}{2}-\frac{l}{m})\pi\right]}\frac{1}{u}$$

$$= -\pi\frac{\sin(\pi u)\cos(\frac{1}{2}-\frac{l}{m})\pi+\cos(\pi u)\sin(\frac{1}{2}-\frac{l}{m})\pi}{\cos(\pi u)\cos(\frac{1}{2}-\frac{l}{m})\pi-\sin(\pi u)\sin(\frac{1}{2}-\frac{l}{m})\pi}\frac{1}{u}$$

$$= -\pi\frac{[\pi u-\frac{1}{3!}\pi^3u^3+...]\sin(\frac{l}{m}\pi)+[1-\frac{1}{2}\pi^2u^2+...]\cos(\frac{l}{m}\pi)}{[1-\frac{1}{2}\pi^2u^2+...]\sin(\frac{l}{m}\pi)-[\pi u-\frac{1}{3!}\pi^3u^3+...]\cos(\frac{l}{m}\pi)}\frac{1}{u}$$

$$= -\pi\frac{\cos(\frac{l}{m}\pi)+u\pi\sin(\frac{l}{m}\pi)+...}{\sin(\frac{l}{m}\pi)-\pi u\cos(\frac{l}{m}\pi)+...}\frac{1}{u}$$

$$= -\pi\cot(\tfrac{l}{m}\pi)\frac{1 + u\pi\tan(\frac{l}{m}\pi) + ...}{1 - \pi u\cot(\frac{l}{m}\pi) + ...}\frac{1}{u}$$

$$\approx -\pi\cot(\tfrac{l}{m}\pi)[1 + u\pi\tan(\tfrac{l}{m}\pi)][1 + \pi u\cot(\tfrac{l}{m}\pi)]\frac{1}{u}$$

$$= -\pi\cot(\tfrac{l}{m}\pi)\frac{1}{u} - \pi^2\cot(\tfrac{l}{m}\pi)[\cot(\tfrac{l}{m}\pi) + \tan(\tfrac{l}{m}\pi)] - \pi^3 u\cot(\tfrac{l}{m}\pi) + ...$$

$$= -\pi\cot(\tfrac{l}{m}\pi)\frac{1}{u} - \pi^2[\cot^2(\tfrac{l}{m}\pi) + 1] - \pi^3 u\cot(\tfrac{l}{m}\pi) + ...$$

$$\text{Res}_{-1}\left\{\pi\cot(\pi z)\frac{1}{z + \frac{l}{m}}\right\}_{z=-\frac{l}{m}} = \boxed{-\pi\cot(\tfrac{l}{m}\pi)}$$

Therefore,

$$... + \frac{1}{-3 + \frac{l}{m}} + \frac{1}{-2 + \frac{l}{m}} + \frac{1}{-1 + \frac{l}{m}} + \frac{1}{0 + \frac{l}{m}} +$$

$$+\frac{1}{1 + \frac{l}{m}} + \frac{1}{2 + \frac{l}{m}} + \frac{1}{3 + \frac{l}{m}} + ... - \pi\cot(\tfrac{l}{m}\pi) = 0$$

$$\begin{aligned}\pi = m\tan(\tfrac{l}{m}\pi)\Big\{\frac{1}{l} &- \frac{1}{m-l} + \frac{1}{m+l} \\ &- \frac{1}{2m-l} + \frac{1}{2m+l} \\ &- \frac{1}{3m-l} + \frac{1}{3m+l} + ...\Big\}\end{aligned}$$

$$\begin{aligned}R_{l/m} = \frac{1}{l} &- \frac{1}{m-l} + \frac{1}{m+l} \\ &- \frac{1}{2m-l} + \frac{1}{2m+l} \\ &- \frac{1}{3m-l} + \frac{1}{3m+l} + ... = \frac{\pi}{m}\cot(\tfrac{l}{m}\pi)\end{aligned}$$

11.

$$\pi = m\sin(\tfrac{l}{m}\pi)\left\{\frac{1}{l}+\frac{1}{m-l}-\frac{1}{m+l} -\frac{1}{2m-l}+\frac{1}{2m+l} +\frac{1}{3m-l}-\frac{1}{3m+l}+...\right\}$$

$l < \frac{1}{2}m$, l and m have no common factor.

$$S_{l/m} = \frac{1}{l}+\frac{1}{m-l}-\frac{1}{m+l} -\frac{1}{2m-l}+\frac{1}{2m+l} +\frac{1}{3m-l}-\frac{1}{3m+l}+... = \frac{\pi}{m\sin(\frac{l}{m}\pi)}$$

Proof: $\pi\frac{1}{\sin(\pi z)}\frac{1}{z+\frac{l}{m}}$ has poles of order 1 at $z = n$,

and a pole at $z = -\frac{l}{m}$

$$\oint_{\square_{N+\frac{1}{2}}} \pi\frac{1}{\sin(\pi\zeta)}\frac{1}{\zeta+\frac{l}{m}}d\zeta = \sum \mathrm{Res}_{-1}\left\{\pi\frac{1}{\sin(\pi z)}\frac{1}{z+\frac{l}{m}}\right\}_{z=n}$$

$$+\mathrm{Res}_{-1}\left\{\pi\frac{1}{\sin(\pi z)}\frac{1}{z+\frac{l}{m}}\right\}_{z=-\frac{4}{9}}$$

$$\left|\oint_{\square_{N+\frac{1}{2}}} \frac{\pi}{\sin(\pi\zeta)}\frac{1}{\zeta+\frac{l}{m}}d\zeta\right| \le \underbrace{\left|\frac{\pi}{\sin(\pi\zeta)}\right|}_{\le A}\underbrace{\oint_{\square_{N+\frac{1}{2}}}\frac{1}{\zeta+\frac{l}{m}}d\zeta}_{\left[\log(\zeta+\frac{l}{m})\right]_{\square_{N+\frac{1}{2}}}=0} = 0.\square$$

$$\text{Res}_{-1}\left\{\pi\frac{1}{\sin(\pi z)}\frac{1}{z+\frac{l}{m}}\right\}_{z=n} = \left[(z-n)\pi\frac{1}{\sin\pi z}\frac{1}{z+\frac{l}{m}}\right]_{z=n}$$

$$= \left[\frac{\pi D_z(z-n)}{D_z\sin(\pi z)}\right]_{z=n}\left(\frac{1}{n+\frac{l}{m}}\right)$$

$$= \left[\frac{\pi}{\pi\cos(\pi z)}\right]_{z=n}\left(\frac{1}{n+\frac{l}{m}}\right)$$

$$= \frac{(-1)^n}{n+\frac{l}{m}}.\square$$

To find $\text{Res}_{-1}\left\{\pi\frac{1}{\sin(\pi z)}\frac{1}{z+\frac{l}{m}}\right\}_{z=-\frac{l}{m}}$**, divide the series**

$$\pi\frac{1}{\sin(\pi z)}\frac{1}{z+\frac{l}{m}} = \pi\frac{1}{-\cos(\pi z+\frac{\pi}{2})}\frac{1}{z+\frac{l}{m}}$$

$$= -\pi\frac{1}{\cos\left[\pi(z+\frac{l}{m})+(\frac{1}{2}-\frac{l}{m})\pi\right]}\frac{1}{z+\frac{l}{m}}$$

$$= -\pi\frac{1}{\cos\left[\pi u+(\frac{1}{2}-\frac{l}{m})\pi\right]}\frac{1}{u}$$

$$= -\pi\frac{1}{\cos(\pi u)\cos(\frac{1}{2}-\frac{l}{m})\pi-\sin(\pi u)\sin(\frac{1}{2}-\frac{l}{m})\pi}\frac{1}{u}$$

$$= -\pi\frac{1}{[1-\frac{1}{2}\pi^2u^2+\ldots]\sin(\frac{l}{m}\pi)-[\pi u-\frac{1}{3!}\pi^3u^3+\ldots]\cos(\frac{l}{m}\pi)}\frac{1}{u}$$

$$= -\pi\frac{1}{\sin(\frac{l}{m}\pi)-\pi u\cos(\frac{l}{m}\pi)+\ldots}\frac{1}{u}$$

$$= -\pi \frac{1}{\sin(\frac{l}{m}\pi)} \frac{1}{1 - \pi u \cot(\frac{l}{m}\pi) + ...} \frac{1}{u}$$

$$\approx -\pi \frac{1}{\sin(\frac{l}{m}\pi)} [1 + \pi u \cot(\frac{l}{m}\pi) + ...] \frac{1}{u}$$

$$= -\pi \frac{1}{\sin(\frac{l}{m}\pi)} \frac{1}{u} - \pi^2 \frac{1}{\sin(\frac{l}{m}\pi)} \cot(\frac{l}{m}\pi) + ...$$

$$\text{Res}_{-1} \left\{ \pi \frac{1}{\sin(\pi z)} \frac{1}{z + \frac{l}{m}} \right\}_{z = -\frac{l}{m}} = -\pi \frac{1}{\sin(\frac{l}{m}\pi)}$$

Therefore,

$$... + \frac{1}{-4 + \frac{l}{m}} - \frac{1}{-3 + \frac{l}{m}} + \frac{1}{-2 + \frac{l}{m}} - \frac{1}{-1 + \frac{l}{m}} + \frac{1}{0 + \frac{l}{m}} +$$

$$- \frac{1}{1 + \frac{l}{m}} + \frac{1}{2 + \frac{l}{m}} - \frac{1}{3 + \frac{l}{m}} + ... - \pi \frac{1}{\sin(\frac{l}{m}\pi)} = 0$$

$$\pi = m \sin(\tfrac{l}{m}\pi) \left\{ \frac{1}{l} + \frac{1}{m - l} - \frac{1}{m + l} - \frac{1}{2m - l} + \frac{1}{2m + l} + \frac{1}{3m - l} - \frac{1}{3m + l} - \frac{1}{4m - l} .... \right\} . \square$$

12.

$$\pi = 38\tan(\tfrac{15}{38}\pi)\left\{\frac{1}{15} - \frac{1}{38-15} + \frac{1}{38+15} - \frac{1}{2(38)-15} + \frac{1}{2(38)+15} - \frac{1}{3(38)-15} + \frac{1}{3(38)+15} + \dots\right\}$$

$$\pi = 38\sin(\tfrac{15}{38}\pi)\left\{\frac{1}{15} + \frac{1}{38-15} - \frac{1}{38+15} - \frac{1}{2(38)-15} + \frac{1}{2(38)+15} + \frac{1}{3(38)-15} - \frac{1}{3(38)+15} - \frac{1}{4(38)-15}\dots\right\}$$

$$R_{15/38} = \frac{1}{15} - \frac{1}{38-15} + \frac{1}{38+15} - \frac{1}{2(38)-15} + \frac{1}{2(38)+15} - \frac{1}{3(38)-15} + \frac{1}{3(38)+15} + .. = \frac{\pi}{38}\cot(\tfrac{15}{38}\pi)$$

$$S_{15/38} = \frac{1}{15} + \frac{1}{38-15} - \frac{1}{38+15} - \frac{1}{2(38)-15} + \frac{1}{2(38)+15} + \frac{1}{3(38)-15} - \frac{1}{3(38)+15} - \frac{1}{4(38)-15} + .. = \frac{\pi}{38\sin(\tfrac{15}{38}\pi)}$$

Proof: $15 < \tfrac{1}{2}38$, 15 and 38 have no common factor

$$\dots + \frac{1}{-3+\tfrac{15}{38}} + \frac{1}{-2+\tfrac{15}{38}} + \frac{1}{-1+\tfrac{15}{38}} + \frac{1}{0+\tfrac{15}{38}} + \frac{1}{1+\tfrac{15}{38}} + \frac{1}{2+\tfrac{15}{38}} + \frac{1}{3+\tfrac{15}{38}} + \dots - \pi\cot(\tfrac{15}{38}\pi) = 0.\square$$

$$\dots - \frac{1}{-3+\tfrac{15}{38}} + \frac{1}{-2+\tfrac{15}{38}} - \frac{1}{-1+\tfrac{15}{38}} + \frac{1}{0+\tfrac{15}{38}} - \frac{1}{1+\tfrac{15}{38}} + \frac{1}{2+\tfrac{15}{38}} - \frac{1}{3+\tfrac{15}{38}} + \dots - \frac{\pi}{\sin(\tfrac{15}{38}\pi)} = 0.\square$$

13.

$$\pi = 2^n \tan(\tfrac{2k+1}{2^n}\pi)\Big\{\frac{1}{2k+1} - \frac{1}{2^n - 2k - 1} + \frac{1}{2^n + 2k + 1} - \frac{1}{2^{n+1} - 2k - 1} + \frac{1}{2^{n+1} + 2k + 1} - \frac{1}{2^{n+2} - 2k - 1} + \frac{1}{2^{n+2} + 2k + 1} + \dots\Big\}$$

$$\pi = 2^n \sin(\tfrac{2k+1}{2^n}\pi)\Big\{\frac{1}{2k+1} + \frac{1}{2^n - 2k - 1} - \frac{1}{2^n + 2k + 1} - \frac{1}{2^{n+1} - 2k - 1} + \frac{1}{2^{n+1} + 2k + 1} + \frac{1}{2^{n+2} - 2k - 1} - \frac{1}{2^{n+2} + 2k + 1} - \dots\Big\}$$

$$2k + 1 < 2^{n-1}$$

$$R_{(2k+1)/2^n} = \frac{1}{2k+1} - \frac{1}{2^n - 2k - 1} + \frac{1}{2^n + 2k + 1} - \frac{1}{2^{n+1} - 2k - 1} + \frac{1}{2^{n+1} + 2k + 1} - \frac{1}{2^{n+2} - 2k - 1} + \frac{1}{2^{n+1} + 2k + 1} + \dots = \frac{\pi}{2^n}\cot(\tfrac{2k+1}{2^n}\pi)$$

$$S_{(2k+1)/2^n} = \frac{1}{2k+1} + \frac{1}{2^n - 2k - 1} - \frac{1}{2^n + 2k + 1} - \frac{1}{2^{n+1} - 2k - 1} + \frac{1}{2^{n+1} + 2k + 1} + \frac{1}{2^{n+2} - 2k - 1} - \frac{1}{2^{n+1} + 2k + 1} + \dots = \frac{\pi}{2^n \sin(\tfrac{2k+1}{2^n}\pi)}$$

$$\tan(\tfrac{2k+1}{2^n}\pi) = \frac{\sqrt{\tfrac{1}{2}[1 - \cos(\tfrac{2k+1}{2^{n-1}}\pi)]}}{\sqrt{\tfrac{1}{2}[1 + \cos(\tfrac{2k+1}{2^{n-1}}\pi)]}}$$

$$= \frac{\sqrt{1-\frac{1}{\sqrt{2}}\sqrt{1+\cos(\frac{2k+1}{2^{n-2}}\pi)}}}{\sqrt{1+\frac{1}{\sqrt{2}}\sqrt{1+\cos(\frac{2k+1}{2^{n-2}}\pi)}}} = ...$$

$$= \text{a number composed of } \sqrt{2}\text{'s}$$

$$\sin(\tfrac{2k+1}{2^n}\pi) = \sqrt{\tfrac{1}{2}[1-\cos(\tfrac{2k+1}{2^{n-1}}\pi)]}$$

$$= \tfrac{1}{\sqrt{2}}\sqrt{1-\tfrac{1}{\sqrt{2}}\sqrt{1+\cos(\tfrac{2k+1}{2^{n-2}}\pi)}} = ...$$

$$= \text{a number composed of } \sqrt{2}\text{'s}$$

Proof. Based on $\pi\cot(\pi z)\dfrac{1}{z+\frac{2k+1}{2^n}}$,

$$... + \frac{1}{-3+\frac{2k+1}{2^n}} + \frac{1}{-2+\frac{2k+1}{2^n}} + \frac{1}{-1+\frac{2k+1}{2^n}} + \frac{1}{0+\frac{2k+1}{2^n}} +$$

$$+\frac{1}{1+\frac{2k+1}{2^n}} + \frac{1}{2+\frac{2k+1}{2^n}} + \frac{1}{3+\frac{2k+1}{2^n}} + ... - \pi\cot(\tfrac{2k+1}{2^n}\pi) = 0. \square$$

Based on $\pi\dfrac{1}{\sin(\pi z)}\dfrac{1}{z+\frac{2k+1}{2^n}}$,

$$... - \frac{1}{-3+\frac{2k+1}{2^n}} + \frac{1}{-2+\frac{2k+1}{2^n}} - \frac{1}{-1+\frac{2k+1}{2^n}} + \frac{1}{0+\frac{2k+1}{2^n}}$$

$$-\frac{1}{1+\frac{2k+1}{2^n}} + \frac{1}{2+\frac{2k+1}{2^n}} - \frac{1}{3+\frac{2k+1}{2^n}} + ... - \frac{\pi}{\sin(\frac{2k+1}{2^n}\pi)} = 0. \square$$

14.

$$\boxed{\pi = 4\underbrace{\cot(\tfrac{1}{4}\pi)}_{\tan(\frac{1}{4}\pi)}\left(\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-...\right)}$$ **(Leibniz)**

$$\boxed{\pi = 4\underbrace{\sin(\tfrac{1}{4}\pi)}_{\cos(\frac{1}{4}\pi)}\left(\frac{1}{1}+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}+...\right)}$$

$$R_{1/4} = \frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-... = \frac{\pi}{4}$$ **(Leibniz)**

$$S_{1/4} = \frac{1}{1}+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}+... = \frac{\pi}{4}\sqrt{2}$$

Proof: **Based on** $\pi\cot(\pi z)\dfrac{1}{z+\frac{1}{4}}$,

$$...+\frac{1}{-3+\frac{1}{4}}+\frac{1}{-2+\frac{1}{4}}+\frac{1}{-1+\frac{1}{4}}+$$

$$+\frac{1}{0+\frac{1}{4}}+\frac{1}{1+\frac{1}{4}}+\frac{1}{2+\frac{1}{4}}+\frac{1}{3+\frac{1}{4}}+...-\pi = 0.\square$$

Based on $\dfrac{\pi}{\sin(\pi z)}\dfrac{1}{z+\frac{1}{4}}$,

$$...+\frac{(-1)^{-3}}{-3+\frac{1}{4}}+\frac{(-1)^{-2}}{-2+\frac{1}{4}}+\frac{(-1)^{-1}}{-1+\frac{1}{4}}+$$

$$+\frac{(-1)^{0}}{0+\frac{1}{4}}+\frac{(-1)^{1}}{1+\frac{1}{4}}+\frac{(-1)^{2}}{2+\frac{1}{4}}+\frac{(-1)^{3}}{3+\frac{1}{4}}+...-\pi\sqrt{2} = 0$$

$$4\left(\frac{1}{1}+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}+...\right) = \pi\sqrt{2}\,.\square$$

15.

$$\pi = 8\underbrace{\tan(\tfrac{1}{8}\pi)}_{\cot(\frac{3}{8}\pi)}\left(\frac{1}{1} - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \frac{1}{23} + \frac{1}{25} - ...\right)$$

$$\pi = 8\underbrace{\sin(\tfrac{1}{8}\pi)}_{\cos(\frac{3}{8}\pi)}\left(\frac{1}{1} + \frac{1}{7} - \frac{1}{9} - \frac{1}{15} + \frac{1}{17} + \frac{1}{23} - \frac{1}{25} + ...\right)$$

$$R_{1/8} = \frac{1}{1} - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \frac{1}{23} + \frac{1}{25} - ... = \frac{\pi}{8}\underbrace{\cot(\tfrac{1}{8}\pi)}_{\tan(\frac{3}{8}\pi)}$$

$$S_{1/8} = \frac{1}{1} + \frac{1}{7} - \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \frac{1}{23} - \frac{1}{25} + ... = \frac{\pi}{8}\frac{1}{\sin(\frac{1}{8}\pi)}$$

$$\tan(\tfrac{1}{8}\pi) = \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}} = \textbf{depends on } \sqrt{2}\text{'s}$$

$$\sin(\tfrac{1}{8}\pi) = \frac{\sqrt{2-\sqrt{2}}}{2} \quad \textbf{depends on } \sqrt{2}\text{'s}$$

Proof: Based on $\pi\cot(\pi z)\dfrac{1}{z+\frac{1}{8}}$,

$$... + \frac{1}{-4+\frac{1}{8}} + \frac{1}{-3+\frac{1}{8}} + \frac{1}{-2+\frac{1}{8}} + \frac{1}{-1+\frac{1}{8}} + \frac{1}{0+\frac{1}{8}} +$$

$$+\frac{1}{1+\frac{1}{8}} + \frac{1}{2+\frac{1}{8}} + \frac{1}{3+\frac{1}{8}} + ... - \pi\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}} = 0$$

$$\left(\frac{1}{1} - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \frac{1}{23} + \frac{1}{25} - ...\right) = \frac{\pi}{8}\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}}$$

$$\cot(\tfrac{1}{8}\pi) = \frac{\cos(\tfrac{1}{8}\pi)}{\sin(\tfrac{1}{8}\pi)}$$

$$= \frac{\sqrt{\tfrac{1}{2}\left[1+\cos(\tfrac{1}{4}\pi)\right]}}{\sqrt{\tfrac{1}{2}\left[1-\cos(\tfrac{1}{4}\pi)\right]}}$$

$$= \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}}.\square$$

Based on $\pi\dfrac{1}{\sin(\pi z)}\dfrac{1}{z+\frac{1}{8}}$,

$$... + \frac{1}{-4+\frac{1}{8}} - \frac{1}{-3+\frac{1}{8}} + \frac{1}{-2+\frac{1}{8}} - \frac{1}{-1+\frac{1}{8}} + \frac{1}{0+\frac{1}{8}} +$$

$$-\frac{1}{1+\frac{1}{8}} + \frac{1}{2+\frac{1}{8}} - \frac{1}{3+\frac{1}{8}} + ... - \frac{\pi}{\sin(\frac{1}{8}\pi)} = 0$$

$$\pi = 8\sin(\tfrac{1}{8}\pi)\left(\frac{1}{1}+\frac{1}{7}-\frac{1}{9}-\frac{1}{15}+\frac{1}{17}+\frac{1}{23}-\frac{1}{25}-\frac{1}{31}+...\right)$$

$$\sin(\tfrac{1}{8}\pi) = \sqrt{\tfrac{1}{2}\left[1-\cos(\tfrac{1}{4}\pi)\right]}$$

$$= \sqrt{\tfrac{1}{2}\left[1-\tfrac{\sqrt{2}}{2}\right]}$$

$$= \frac{\sqrt{2-\sqrt{2}}}{2}.\square$$

16.

$$\pi = 8\underbrace{\tan(\tfrac{3}{8}\pi)}_{\cot(\frac{1}{8}\pi)}\left(\frac{1}{3}-\frac{1}{5}+\frac{1}{11}-\frac{1}{13}+\frac{1}{19}-\frac{1}{21}+\frac{1}{27}-\frac{1}{29}+...\right)$$

$$\pi = 8\underbrace{\sin(\tfrac{3}{8}\pi)}_{\cos(\frac{1}{8}\pi)}\left(\frac{1}{3}+\frac{1}{5}-\frac{1}{11}-\frac{1}{13}+\frac{1}{19}+\frac{1}{21}-\frac{1}{27}-\frac{1}{29}+...\right)$$

$$R_{3/8} = \frac{1}{3}-\frac{1}{5}+\frac{1}{11}-\frac{1}{13}+\frac{1}{19}-\frac{1}{21}+\frac{1}{27}-\frac{1}{29}+... = \frac{\pi}{8}\underbrace{\cot(\tfrac{3}{8}\pi)}_{\tan(\frac{1}{8}\pi)}$$

$$S_{3/8} = \frac{1}{3}+\frac{1}{5}-\frac{1}{11}-\frac{1}{13}+\frac{1}{19}+\frac{1}{21}-\frac{1}{27}-\frac{1}{29}+... = \frac{\pi}{8\sin(\frac{3}{8}\pi)}$$

$$\tan(\tfrac{3}{8}\pi) = \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}} = \textbf{depends on } \sqrt{2}\text{'s}$$

$$\sin(\tfrac{3}{8}\pi) = \frac{\sqrt{2+\sqrt{2}}}{2} \ \textbf{depends on } \sqrt{2}\text{'s}$$

Proof **Based on** $\pi\cot(\pi z)\dfrac{1}{z+\frac{3}{8}}$,

$$... + \frac{1}{-4+\frac{3}{8}} + \frac{1}{-3+\frac{3}{8}} + \frac{1}{-2+\frac{3}{8}} + \frac{1}{-1+\frac{3}{8}} + \frac{1}{0+\frac{3}{8}} +$$

$$+\frac{1}{1+\frac{3}{8}} + \frac{1}{2+\frac{3}{8}} + \frac{1}{3+\frac{3}{8}} + ... - \pi\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}} = 0$$

$$\left(\frac{1}{3}-\frac{1}{5}+\frac{1}{11}-\frac{1}{13}+\frac{1}{19}-\frac{1}{21}+\frac{1}{27}+...\right) = \frac{\pi}{8}\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}$$

$$\cot(\tfrac{3}{8}\pi) = \frac{\cos(\frac{3}{8}\pi)}{\sin(\frac{3}{8}\pi)}$$

$$= \frac{\sqrt{\frac{1}{2}\left[1+\cos(\frac{3}{4}\pi)\right]}}{\sqrt{\frac{1}{2}\left[1-\cos(\frac{3}{4}\pi)\right]}}$$

$$= \frac{\sqrt{1-\frac{\sqrt{2}}{2}}}{\sqrt{1+\frac{\sqrt{2}}{2}}}$$

$$= \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}.\square$$

Based on $\pi \frac{1}{\sin(\pi z)} \frac{1}{z+\frac{3}{8}}$,

$$... + \frac{1}{-4+\frac{3}{8}} - \frac{1}{-3+\frac{3}{8}} + \frac{1}{-2+\frac{3}{8}} - \frac{1}{-1+\frac{3}{8}} + \frac{1}{0+\frac{3}{8}} +$$

$$-\frac{1}{1+\frac{3}{8}} + \frac{1}{2+\frac{3}{8}} - \frac{1}{3+\frac{3}{8}} + ... - \pi \frac{2}{\sqrt{2+\sqrt{2}}} = 0$$

$$\pi = 8\sin(\tfrac{3}{8}\pi)\left(\frac{1}{3} + \frac{1}{5} - \frac{1}{11} - \frac{1}{13} + \frac{1}{19} + \frac{1}{21} - \frac{1}{27} - \frac{1}{29} + ...\right)$$

$$\sin(\tfrac{3}{8}\pi) = \sqrt{\tfrac{1}{2}\left[1-\cos(\tfrac{3}{4}\pi)\right]}$$

$$= \sqrt{\tfrac{1}{2}\left[1+\tfrac{\sqrt{2}}{2}\right]}$$

$$= \frac{\sqrt{2+\sqrt{2}}}{2}.\square$$

17.

$R_{1/8} + R_{3/8} = S_{1/4}$

$$\boxed{\begin{aligned}&\frac{1}{1}+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}-\frac{1}{15}+\frac{1}{17}+\frac{1}{19}-\frac{1}{21}-\frac{1}{23}+\ldots=\\ &\qquad=\frac{\pi}{8}[\cot(\tfrac{1}{8}\pi)+\cot(\tfrac{3}{8}\pi)]\\ &\qquad=\frac{\pi}{4}\sqrt{2}\end{aligned}}$$

Proof: **Based on** $\pi\cot(\pi z)\dfrac{1}{z+\frac{1}{8}}$,

$$R_{1/8} = \frac{1}{1}-\frac{1}{7}+\frac{1}{9}-\frac{1}{15}+\frac{1}{17}-\frac{1}{23}+\frac{1}{25}-\ldots=\frac{\pi}{8}\sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}}$$

Based on $\pi\cot(\pi z)\dfrac{1}{z+\frac{3}{8}}$,

$$R_{3/8} = \frac{1}{3}-\frac{1}{5}+\frac{1}{11}-\frac{1}{13}+\frac{1}{19}-\frac{1}{21}+\frac{1}{27}+\ldots=\frac{\pi}{8}\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}$$

$$R_{1/8}+R_{3/8}=$$

$$\frac{1}{1}+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}-\frac{1}{15}+\frac{1}{17}+\frac{1}{19}-\frac{1}{21}-\frac{1}{23}+\ldots=$$

$$=\frac{\pi}{8}\left(\sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}}+\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}\right)$$

$$=\frac{\pi}{8}\frac{[2+\sqrt{2}]+[2-\sqrt{2}]}{\sqrt{2^2-2}}$$

$$=\frac{\pi}{8}\frac{4}{\sqrt{2}}=\frac{\pi}{4}\sqrt{2}=S_{1/4}$$

18.

$S_{1/8} - S_{3/8}$

$$1-\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{9}+\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\frac{1}{17}-\frac{1}{19}-\frac{1}{21}-\frac{1}{23}-\frac{1}{25}+\frac{1}{27}+\frac{1}{29}+...$$
$$=\frac{\pi}{8}\left\{\frac{1}{\sin(\frac{1}{8}\pi)}-\frac{1}{\sin(\frac{3}{8}\pi)}\right\}$$

Proof: Based on $\pi\frac{1}{\sin(\pi z)}\frac{1}{z+\frac{1}{8}}$,

$$S_{1/8}=\frac{1}{1}+\frac{1}{7}-\frac{1}{9}-\frac{1}{15}+\frac{1}{17}-\frac{1}{23}-\frac{1}{25}+...=\frac{\pi}{8}\frac{1}{\sin(\frac{1}{8}\pi)}$$

Based on $\pi\frac{1}{\sin(\pi z)}\frac{1}{z+\frac{3}{8}}$,

$$S_{3/8}=\frac{1}{3}+\frac{1}{5}-\frac{1}{11}-\frac{1}{13}+\frac{1}{19}+\frac{1}{21}-\frac{1}{27}-\frac{1}{29}+...=\frac{\pi}{8\sin(\frac{3}{8}\pi)}$$

$S_{1/8} - S_{3/8} =$

$$1-\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{9}+\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\frac{1}{17}-\frac{1}{19}-\frac{1}{21}-\frac{1}{23}-\frac{1}{25}+\frac{1}{27}+\frac{1}{29}...$$

$$=\frac{\pi}{8}\left\{\frac{1}{\sin(\frac{1}{8}\pi)}-\frac{1}{\sin(\frac{3}{8}\pi)}\right\}$$

19.

$R_{1/8} - R_{3/8} = R_{1/4}$

$$\boxed{\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\frac{1}{17}-\frac{1}{19}+\frac{1}{21}-\frac{1}{23}+\frac{1}{25}-..=\frac{\pi}{4}}$$

Proof:

based on $\pi\cot(\pi z)\dfrac{1}{z+\frac{1}{8}}$,

$$R_{1/8} = \frac{1}{1}-\frac{1}{7}+\frac{1}{9}-\frac{1}{15}+\frac{1}{17}-\frac{1}{23}+\frac{1}{25}-... = \frac{\pi}{8}\sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}}$$

Based on $\pi\cot(\pi z)\dfrac{1}{z+\frac{3}{8}}$,

$$R_{3/8} = \frac{1}{3}-\frac{1}{5}+\frac{1}{11}-\frac{1}{13}+\frac{1}{19}-\frac{1}{21}+\frac{1}{27}+... = \frac{\pi}{8}\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}$$

$$R_{1/8} - R_{3/8} =$$

$$\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\frac{1}{17}-\frac{1}{19}+\frac{1}{21}-\frac{1}{23}+\frac{1}{25}-... =$$

$$= \frac{\pi}{8}\left(\sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}} - \sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}\right)$$

$$= \frac{\pi}{8}\frac{[2+\sqrt{2}]-[2-\sqrt{2}]}{\sqrt{2^2-2}}$$

$$= \frac{\pi}{8}\frac{2\sqrt{2}}{\sqrt{2}} = \frac{\pi}{4} = R_{1/4}$$

20.

$$\boxed{\pi = 16\underbrace{\tan(\tfrac{1}{16}\pi)}_{\cot(\frac{7}{16}\pi)}\left(\frac{1}{1}-\frac{1}{15}+\frac{1}{17}-\frac{1}{31}+\frac{1}{33}-\frac{1}{47}+\frac{1}{49}-\frac{1}{51}+...\right)}$$

$$\boxed{\pi = 16\underbrace{\sin(\tfrac{1}{16}\pi)}_{\cos(\frac{7}{16}\pi)}\left(\frac{1}{1}+\frac{1}{15}-\frac{1}{17}-\frac{1}{31}+\frac{1}{33}+\frac{1}{47}-\frac{1}{49}-\frac{1}{51}+...\right)}$$

$$R_{1/16} = \frac{1}{1}-\frac{1}{15}+\frac{1}{17}-\frac{1}{31}+\frac{1}{33}-\frac{1}{47}+\frac{1}{49}-...=\frac{\pi}{16}\underbrace{\cot(\tfrac{1}{16}\pi)}_{\tan(\frac{7}{16}\pi)}$$

$$S_{1/16} = \frac{1}{1}+\frac{1}{15}-\frac{1}{17}-\frac{1}{31}+\frac{1}{33}+\frac{1}{47}-\frac{1}{49}-\frac{1}{51}+... = \frac{\pi}{16\sin(\frac{1}{16}\pi)}$$

$$\tan(\tfrac{1}{16}\pi) = \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}} = \textbf{depends on } \sqrt{2}\text{'s}$$

$$\sin(\tfrac{1}{16}\pi) = \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2} = \textbf{depends on } \sqrt{2}$$

<u>Proof</u>: **Based on** $\pi\cot(\pi z)\dfrac{1}{z+\frac{1}{16}}$,

$$... + \frac{1}{-3+\frac{1}{16}} + \frac{1}{-2+\frac{1}{16}} + \frac{1}{-1+\frac{1}{16}} + \frac{1}{0+\frac{1}{16}} +$$

$$+\frac{1}{1+\frac{1}{16}} + \frac{1}{2+\frac{1}{16}} + \frac{1}{3+\frac{1}{16}} + ... - \pi\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{\sqrt{2-\sqrt{2+\sqrt{2}}}} = 0$$

$$16\left(\frac{1}{1}-\frac{1}{15}+\frac{1}{17}-\frac{1}{31}+\frac{1}{33}-\frac{1}{47}+\frac{1}{49}-...\right) = \pi\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{\sqrt{2-\sqrt{2+\sqrt{2}}}}.\square$$

$$\cot(\tfrac{1}{16}\pi) = \frac{\cos(\tfrac{1}{16}\pi)}{\sin(\tfrac{1}{16}\pi)}$$

$$= \frac{\sqrt{\tfrac{1}{2}\left[1+\cos(\tfrac{1}{8}\pi)\right]}}{\sqrt{\tfrac{1}{2}\left[1-\cos(\tfrac{1}{8}\pi)\right]}}$$

$$= \frac{\sqrt{1+\sqrt{\tfrac{1}{2}[1+\cos(\tfrac{1}{4}\pi)]}}}{\sqrt{1-\sqrt{\tfrac{1}{2}[1+\cos(\tfrac{1}{4}\pi)]}}}$$

$$= \frac{\sqrt{1+\sqrt{\tfrac{1}{2}[1+\tfrac{\sqrt{2}}{2}]}}}{\sqrt{1-\sqrt{\tfrac{1}{2}[1+\tfrac{\sqrt{2}}{2}]}}}$$

$$= \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{\sqrt{2-\sqrt{2+\sqrt{2}}}}$$

Based on $\pi\dfrac{1}{\sin(\pi z)}\dfrac{1}{z+\frac{1}{16}}$,

$$...+\frac{1}{-4+\frac{1}{16}}-\frac{1}{-3+\frac{1}{16}}+\frac{1}{-2+\frac{1}{16}}-\frac{1}{-1+\frac{1}{16}}+\frac{1}{0+\frac{1}{16}}+$$

$$-\frac{1}{1+\frac{1}{16}}+\frac{1}{2+\frac{1}{16}}-\frac{1}{3+\frac{1}{16}}+...-\pi\frac{1}{\sin(\frac{1}{16}\pi)}=0$$

$$\pi = 16\sin(\tfrac{1}{16}\pi)\left(\frac{1}{1}+\frac{1}{15}-\frac{1}{17}-\frac{1}{31}+\frac{1}{33}+\frac{1}{47}-\frac{1}{49}-\frac{1}{51}...\right).\square$$

$$\sin(\tfrac{1}{16}\pi) = \sqrt{\tfrac{1}{2}\left[1-\cos(\tfrac{1}{8}\pi)\right]}$$

$$= \sqrt{\tfrac{1}{2}\left[1-\sqrt{\tfrac{1}{2}\left[1+\cos(\tfrac{1}{4}\pi)\right]}\right]}$$

$$= \sqrt{\tfrac{1}{2}\left[1-\sqrt{\tfrac{1}{2}\left[1+\tfrac{\sqrt{2}}{2}\right]}\right]} = \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}$$

21.

$$\pi = 16\underbrace{\tan(\tfrac{3}{16}\pi)}_{\cot(\frac{5}{16}\pi)}\left(\frac{1}{3}-\frac{1}{13}+\frac{1}{19}-\frac{1}{29}+\frac{1}{35}-\frac{1}{45}+\frac{1}{51}-\frac{1}{61}+\ldots\right)$$

$$\pi = 16\underbrace{\sin(\tfrac{3}{16}\pi)}_{\cos(\frac{5}{16}\pi)}\left(\frac{1}{3}+\frac{1}{13}-\frac{1}{19}-\frac{1}{29}+\frac{1}{35}+\frac{1}{45}-\frac{1}{51}-\frac{1}{61}+\ldots\right)$$

$$R_{3/16} = \frac{1}{3}-\frac{1}{13}+\frac{1}{19}-\frac{1}{29}+\frac{1}{35}-\frac{1}{45}+\frac{1}{51}-\ldots=\frac{\pi}{16}\underbrace{\cot(\tfrac{3}{16}\pi)}_{\tan(\frac{5}{16}\pi)}$$

$$S_{3/16} = \frac{1}{3}+\frac{1}{13}-\frac{1}{19}-\frac{1}{29}+\frac{1}{35}+\frac{1}{45}-\frac{1}{51}-\frac{1}{61}+\ldots = \frac{\pi}{16\sin(\frac{3}{16}\pi)}$$

$$\tan(\tfrac{3}{16}\pi) = \frac{\sqrt{2-\sqrt{2-\sqrt{2}}}}{\sqrt{2+\sqrt{2-\sqrt{2}}}} = \textbf{depends on } \sqrt{2}\text{'s}$$

$$\sin(\tfrac{3}{16}\pi) = \frac{\sqrt{2-\sqrt{2-\sqrt{2}}}}{2} = \textbf{depends on } \sqrt{2}\text{'s}$$

<u>Proof</u> **Based on** $\pi\cot(\pi z)\dfrac{1}{z+\frac{3}{16}}$,

$$\ldots+\frac{1}{-3+\frac{3}{16}}+\frac{1}{-2+\frac{3}{16}}+\frac{1}{-1+\frac{3}{16}}+\frac{1}{0+\frac{3}{16}}+$$

$$+\frac{1}{1+\frac{3}{16}}+\frac{1}{2+\frac{3}{16}}+\frac{1}{3+\frac{3}{16}}+\ldots-\pi\frac{\sqrt{2+\sqrt{2-\sqrt{2}}}}{\sqrt{2-\sqrt{2-\sqrt{2}}}}=0$$

$$16\left(\frac{1}{3}-\frac{1}{13}+\frac{1}{19}-\frac{1}{29}+\frac{1}{35}-\frac{1}{45}+\frac{1}{51}-\ldots\right)=\pi\frac{\sqrt{2+\sqrt{2-\sqrt{2}}}}{\sqrt{2-\sqrt{2-\sqrt{2}}}}.\square$$

$$\cot(\tfrac{3}{16}\pi) = \frac{\cos(\tfrac{3}{16}\pi)}{\sin(\tfrac{3}{16}\pi)}$$

$$= \frac{\sqrt{\tfrac{1}{2}\Big[1+\cos(\tfrac{3}{8}\pi)\Big]}}{\sqrt{\tfrac{1}{2}\Big[1-\cos(\tfrac{3}{8}\pi)\Big]}}$$

$$= \frac{\sqrt{1+\sqrt{\tfrac{1}{2}[1+\cos\tfrac{3}{4}\pi]}}}{\sqrt{1-\sqrt{\tfrac{1}{2}[1+\cos\tfrac{3}{4}\pi]}}}$$

$$= \frac{\sqrt{1+\sqrt{\tfrac{1}{2}[1-\tfrac{\sqrt{2}}{2}]}}}{\sqrt{1-\sqrt{\tfrac{1}{2}[1-\tfrac{\sqrt{2}}{2}]}}}$$

$$= \frac{\sqrt{2+\sqrt{2-\sqrt{2}}}}{\sqrt{2-\sqrt{2-\sqrt{2}}}}.\square$$

Based on $\pi\dfrac{1}{\sin(\pi z)}\dfrac{1}{z+\frac{3}{16}}$,

$$...+\frac{1}{-4+\frac{3}{16}}-\frac{1}{-3+\frac{3}{16}}+\frac{1}{-2+\frac{3}{16}}-\frac{1}{-1+\frac{3}{16}}+\frac{1}{0+\frac{3}{16}}+$$

$$-\frac{1}{1+\frac{3}{16}}+\frac{1}{2+\frac{3}{16}}-\frac{1}{3+\frac{3}{16}}+...-\pi\frac{1}{\sin(\frac{3}{16}\pi)}=0$$

$$\pi = 16\sin(\tfrac{3}{16}\pi)\left(\frac{1}{3}+\frac{1}{13}-\frac{1}{19}-\frac{1}{29}+\frac{1}{35}+\frac{1}{45}-\frac{1}{51}-\frac{1}{61}...\right).\square$$

$$\sin(\tfrac{3}{16}\pi) = \sqrt{\tfrac{1}{2}\Big[1-\cos(\tfrac{3}{8}\pi)\Big]}$$

$$= \sqrt{\tfrac{1}{2}\Big[1-\sqrt{\tfrac{1}{2}\Big[1+\cos(\tfrac{3}{4}\pi)\Big]}\Big]}$$

$$= \sqrt{\tfrac{1}{2}\Big[1-\sqrt{\tfrac{1}{2}\Big[1-\tfrac{\sqrt{2}}{2}\Big]}\Big]} = \frac{\sqrt{2-\sqrt{2-\sqrt{2}}}}{2}.\square$$

22.

$$\pi = 16\underbrace{\tan(\tfrac{5}{16}\pi)}_{\cot(\frac{3}{16}\pi)}\left(\frac{1}{5}-\frac{1}{11}+\frac{1}{21}-\frac{1}{27}+\frac{1}{37}-\frac{1}{43}+\frac{1}{53}-\frac{1}{59}+\frac{1}{69}-...\right)$$

$$\pi = 16\underbrace{\sin(\tfrac{5}{16}\pi)}_{\cos(\frac{3}{16}\pi)}\left(\frac{1}{5}+\frac{1}{11}-\frac{1}{21}-\frac{1}{27}+\frac{1}{37}+\frac{1}{43}-\frac{1}{53}-\frac{1}{59}+\frac{1}{69}+...\right)$$

$$R_{5/16} = \frac{1}{5}-\frac{1}{11}+\frac{1}{21}-\frac{1}{27}+\frac{1}{37}-\frac{1}{43}+\frac{1}{53}-\frac{1}{59}+...=\frac{\pi}{16}\underbrace{\cot(\tfrac{5}{16})}_{\tan(\frac{3}{16})}$$

$$S_{5/16} = \frac{1}{5}+\frac{1}{11}-\frac{1}{21}-\frac{1}{27}+\frac{1}{37}+\frac{1}{43}-\frac{1}{53}-\frac{1}{59}+.. = \frac{\pi}{16\sin(\frac{5}{16}\pi)}$$

$$\tan(\tfrac{5}{16}\pi) = \frac{\sqrt{2+\sqrt{2-\sqrt{2}}}}{\sqrt{2-\sqrt{2-\sqrt{2}}}} = \text{depends on } \sqrt{2}\text{'s}$$

$$\sin(\tfrac{5}{16}\pi) = \sqrt{2-\sqrt{2+\sqrt{2}}} = \text{depends on } \sqrt{2}\text{'s}$$

Proof: Based on $\pi\cot(\pi z)\dfrac{1}{z+\frac{5}{16}}$,

$$... + \frac{1}{-3+\frac{5}{16}} + \frac{1}{-2+\frac{5}{16}} + \frac{1}{-1+\frac{5}{16}} + \frac{1}{0+\frac{5}{16}} +$$

$$+\frac{1}{1+\frac{5}{16}} + \frac{1}{2+\frac{5}{16}} + \frac{1}{3+\frac{5}{16}} + ... - \pi\frac{\sqrt{2-\sqrt{2-\sqrt{2}}}}{\sqrt{2+\sqrt{2-\sqrt{2}}}} = 0$$

$$16\left(\frac{1}{5}-\frac{1}{11}+\frac{1}{21}-\frac{1}{27}+\frac{1}{37}-\frac{1}{43}+\frac{1}{53}-...\right) = \pi\frac{\sqrt{2-\sqrt{2-\sqrt{2}}}}{\sqrt{2+\sqrt{2-\sqrt{2}}}}.\square$$

$$\cot(\tfrac{5}{16}\pi) = \frac{\cos(\tfrac{5}{16}\pi)}{\sin(\tfrac{5}{16}\pi)}$$

$$= \frac{\sqrt{\tfrac{1}{2}\Big[1+\cos(\tfrac{5}{8}\pi)\Big]}}{\sqrt{\tfrac{1}{2}\Big[1-\cos(\tfrac{5}{8}\pi)\Big]}}$$

$$= \frac{\sqrt{1+\sqrt{\tfrac{1}{2}[1+\cos\tfrac{5}{4}\pi]}}}{\sqrt{1-\sqrt{\tfrac{1}{2}[1+\cos\tfrac{5}{4}\pi]}}}$$

$$= \frac{\sqrt{1+\sqrt{\tfrac{1}{2}[1-\tfrac{\sqrt{2}}{2}]}}}{\sqrt{1-\sqrt{\tfrac{1}{2}[1-\tfrac{\sqrt{2}}{2}]}}}$$

$$= \frac{\sqrt{2-\sqrt{2-\sqrt{2}}}}{\sqrt{2+\sqrt{2-\sqrt{2}}}}.\square$$

Based on $\pi\dfrac{1}{\sin(\pi z)}\dfrac{1}{z+\frac{5}{16}}$,

$$...+\frac{1}{-4+\frac{5}{16}}-\frac{1}{-3+\frac{5}{16}}+\frac{1}{-2+\frac{5}{16}}-\frac{1}{-1+\frac{5}{16}}+\frac{1}{0+\frac{5}{16}}+$$

$$-\frac{1}{1+\frac{5}{16}}+\frac{1}{2+\frac{5}{16}}-\frac{1}{3+\frac{5}{16}}+...-\pi\frac{1}{\sin(\frac{5}{16}\pi)}=0$$

$$\pi = 16\sin(\tfrac{5}{16}\pi)\left(\frac{1}{5}+\frac{1}{11}-\frac{1}{21}-\frac{1}{27}+\frac{1}{37}+\frac{1}{43}-\frac{1}{53}-\frac{1}{59}+\frac{1}{69}+...\right).\square$$

$$\sin(\tfrac{5}{16}\pi) = \sqrt{\tfrac{1}{2}\Big[1-\cos(\tfrac{5}{8}\pi)\Big]}$$

$$= \sqrt{\tfrac{1}{2}\Big[1-\sqrt{\tfrac{1}{2}\Big[1+\cos(\tfrac{5}{4}\pi)\Big]}\Big]}$$

$$= \sqrt{\tfrac{1}{2}\Big[1-\sqrt{\tfrac{1}{2}\Big[1+\tfrac{\sqrt{2}}{2}\Big]}\Big]} = \sqrt{2-\sqrt{2+\sqrt{2}}}.\square$$

23.

$$\pi = 16\underbrace{\tan(\tfrac{7}{16}\pi)}_{\cot(\frac{1}{16}\pi)}\left(\frac{1}{7}-\frac{1}{9}+\frac{1}{23}-\frac{1}{25}+\frac{1}{39}-\frac{1}{41}+\frac{1}{55}-\frac{1}{57}+\frac{1}{71}-...\right)$$

$$\pi = 16\underbrace{\sin(\tfrac{7}{16}\pi)}_{\cos(\frac{1}{16}\pi)}\left(\frac{1}{7}+\frac{1}{9}-\frac{1}{23}-\frac{1}{25}+\frac{1}{39}+\frac{1}{41}-\frac{1}{55}-\frac{1}{57}+\frac{1}{71}+...\right)$$

$$R_{7/16} = \frac{1}{7}-\frac{1}{9}+\frac{1}{23}-\frac{1}{25}+\frac{1}{39}-\frac{1}{41}+\frac{1}{55}-... = \frac{\pi}{16}\underbrace{\cot(\tfrac{7}{16}\pi)}_{\tan(\frac{1}{16}\pi)}$$

$$S_{7/16} = \frac{1}{7}+\frac{1}{9}-\frac{1}{23}-\frac{1}{25}+\frac{1}{39}+\frac{1}{41}-\frac{1}{55}-\frac{1}{57}+\frac{1}{71}+.. = \frac{\pi}{16\sin(\frac{7}{16}\pi)}$$

$$\tan(\tfrac{7}{16}\pi) = \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{\sqrt{2-\sqrt{2+\sqrt{2}}}} = \textbf{depends on } \sqrt{2}\text{'s}$$

$$\sin(\tfrac{7}{16}\pi) = \sqrt{2-\sqrt{2+\sqrt{2}}} \ \textbf{ depends on } \sqrt{2}$$

<u>Proof</u>: **Based on** $\pi\cot(\pi z)\dfrac{1}{z+\frac{7}{16}}$,

$$... + \frac{1}{-3+\frac{7}{16}} + \frac{1}{-2+\frac{7}{16}} + \frac{1}{-1+\frac{7}{16}} + \frac{1}{0+\frac{7}{16}} +$$

$$+\frac{1}{1+\frac{7}{16}} + \frac{1}{2+\frac{7}{16}} + \frac{1}{3+\frac{7}{16}} + ... - \pi\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}} = 0$$

$$16\left(\frac{1}{7}-\frac{1}{9}+\frac{1}{23}-\frac{1}{25}+\frac{1}{39}-\frac{1}{41}+\frac{1}{55}-...\right) = \pi\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

$$\cot(\tfrac{7}{16}\pi) = \frac{\cos(\frac{7}{16}\pi)}{\sin(\frac{7}{16}\pi)}$$

$$= \frac{\sin(\frac{1}{16}\pi)}{\cos(\frac{1}{16}\pi)}$$

$$= \frac{\sqrt{\frac{1}{2}\left[1 - \cos(\frac{1}{8}\pi)\right]}}{\sqrt{\frac{1}{2}\left[1 + \cos(\frac{1}{8}\pi)\right]}}$$

$$= \frac{\sqrt{1 - \sqrt{\frac{1}{2}[1 + \cos(\frac{1}{4}\pi)]}}}{\sqrt{1 + \sqrt{\frac{1}{2}[1 + \cos(\frac{1}{4}\pi)]}}}$$

$$= \frac{\sqrt{1 - \sqrt{\frac{1}{2}[1 + \frac{\sqrt{2}}{2}]}}}{\sqrt{1 + \sqrt{\frac{1}{2}[1 + \frac{\sqrt{2}}{2}]}}}$$

$$= \frac{\sqrt{2 - \sqrt{2 + \sqrt{2}}}}{\sqrt{2 + \sqrt{2 + \sqrt{2}}}}.\square$$

Based on $\pi \frac{1}{\sin(\pi z)} \frac{1}{z + \frac{7}{16}}$,

$$... + \frac{1}{-4 + \frac{7}{16}} - \frac{1}{-3 + \frac{7}{16}} + \frac{1}{-2 + \frac{7}{16}} - \frac{1}{-1 + \frac{7}{16}} + \frac{1}{0 + \frac{7}{16}} +$$

$$- \frac{1}{1 + \frac{7}{16}} + \frac{1}{2 + \frac{7}{16}} - \frac{1}{3 + \frac{7}{16}} + ... - \pi \frac{1}{\sin(\frac{7}{16}\pi)} = 0$$

$$\pi = 16 \sin(\tfrac{7}{16}\pi)\left(\frac{1}{7} + \frac{1}{9} - \frac{1}{23} - \frac{1}{25} + \frac{1}{39} + \frac{1}{41} - \frac{1}{55} - \frac{1}{57} + \frac{1}{71} + ...\right).\square$$

$$\sin(\tfrac{7}{16}\pi) = \sqrt{\tfrac{1}{2}\left[1 - \cos(\tfrac{7}{8}\pi)\right]}$$

$$= \sqrt{\tfrac{1}{2}\left[1 - \sqrt{\tfrac{1}{2}\left[1 + \cos(\tfrac{7}{4}\pi)\right]}\right]}$$

$$= \sqrt{\tfrac{1}{2}\left[1 - \sqrt{\tfrac{1}{2}\left[1 + \tfrac{\sqrt{2}}{2}\right]}\right]} = \sqrt{2 - \sqrt{2 + \sqrt{2}}}$$

24.

$R_{1/16} + R_{3/16} + R_{5/16} + R_{7/16}$

$$R_{1/16} = \boxed{\frac{1}{1} - \frac{1}{15} + \frac{1}{17} - \frac{1}{31} + \frac{1}{33} - \frac{1}{47} + \frac{1}{49} - \ldots = \frac{\pi}{16}\cot(\tfrac{1}{16}\pi) = \frac{\pi}{16}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{\sqrt{2-\sqrt{2+\sqrt{2}}}}}$$

$$R_{3/16} = \boxed{\frac{1}{3} - \frac{1}{13} + \frac{1}{19} - \frac{1}{29} + \frac{1}{35} - \frac{1}{45} + \frac{1}{51} - \ldots = \frac{\pi}{16}\cot(\tfrac{3}{16}\pi) = \frac{\pi}{16}\frac{\sqrt{2+\sqrt{2-\sqrt{2}}}}{\sqrt{2-\sqrt{2-\sqrt{2}}}}}$$

$$R_{5/16} = \boxed{\frac{1}{5} - \frac{1}{11} + \frac{1}{21} - \frac{1}{27} + \frac{1}{37} - \frac{1}{43} + \frac{1}{53} - \ldots = \frac{\pi}{16}\cot(\tfrac{5}{16}) = \frac{\pi}{16}\frac{\sqrt{2-\sqrt{2-\sqrt{2}}}}{\sqrt{2+\sqrt{2-\sqrt{2}}}}}$$

$$R_{7/16} = \boxed{\frac{1}{7} - \frac{1}{9} + \frac{1}{23} - \frac{1}{25} + \frac{1}{39} - \frac{1}{41} + \frac{1}{55} - \ldots = \frac{\pi}{16}\cot(\tfrac{7}{16}\pi) = \frac{\pi}{16}\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}}}$$

$R_{1/16} + R_{3/16} + R_{5/16} + R_{7/16} =$

$$= \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{9} - \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \frac{1}{21} + \frac{1}{23} - \ldots$$

$$= \frac{\pi}{16}\underbrace{\left(\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{\sqrt{2-\sqrt{2+\sqrt{2}}}} + \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}}\right)}_{\frac{4}{\sqrt{2-\sqrt{2}}}} + \frac{\pi}{16}\underbrace{\left(\frac{\sqrt{2+\sqrt{2-\sqrt{2}}}}{\sqrt{2-\sqrt{2-\sqrt{2}}}} + \frac{\sqrt{2-\sqrt{2-\sqrt{2}}}}{\sqrt{2+\sqrt{2-\sqrt{2}}}}\right)}_{\frac{4}{\sqrt{2+\sqrt{2}}}}$$

$$= \frac{\pi}{4}\left(\frac{1}{\sqrt{2-\sqrt{2}}} + \frac{1}{\sqrt{2+\sqrt{2}}}\right)$$

$$= \frac{\pi}{4}\frac{\sqrt{2+\sqrt{2}} + \sqrt{2-\sqrt{2}}}{\sqrt{(2-\sqrt{2})(2+\sqrt{2})}}$$

$$= \frac{\pi}{4}\frac{\sqrt{\left(\sqrt{2+\sqrt{2}} + \sqrt{2-\sqrt{2}}\right)^2}}{\sqrt{2}}$$

$$= \frac{\pi}{4}\frac{\sqrt{2+\sqrt{2}+2-\sqrt{2}+2\sqrt{2^2-2}}}{\sqrt{2}}$$

$$= \frac{\pi}{4}\sqrt{2+\sqrt{2}}$$

25.

$R_{1/16} - R_{3/16}$

$$R_{1/16} = \boxed{\begin{aligned} \frac{1}{1} - \frac{1}{15} + \frac{1}{17} - \frac{1}{31} + \frac{1}{33} - \frac{1}{47} + \frac{1}{49} - ... &= \frac{\pi}{16}\cot(\tfrac{1}{16}\pi) \\ &= \frac{\pi}{16}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{\sqrt{2-\sqrt{2+\sqrt{2}}}} \end{aligned}}$$

$$R_{3/16} = \boxed{\begin{aligned} \frac{1}{3} - \frac{1}{13} + \frac{1}{19} - \frac{1}{29} + \frac{1}{35} - \frac{1}{45} + \frac{1}{51} - ... &= \frac{\pi}{16}\cot(\tfrac{3}{16}\pi) \\ &= \frac{\pi}{16}\frac{\sqrt{2+\sqrt{2-\sqrt{2}}}}{\sqrt{2-\sqrt{2-\sqrt{2}}}} \end{aligned}}$$

$R_{1/16} - R_{3/16} =$

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \frac{1}{29} - \frac{1}{31} + \frac{1}{33} +$$

$$-\frac{1}{35} + \frac{1}{45} - \frac{1}{47} + \frac{1}{49} - \frac{1}{51} + \frac{1}{61} + ... =$$

$$= \frac{\pi}{16}\left\{\cot(\tfrac{1}{16}\pi) - \cot(\tfrac{3}{16}\pi)\right\}$$

26.

$R_{5/16} - R_{7/16}$

$$R_{5/16} = \boxed{\begin{aligned}&\frac{1}{5} - \frac{1}{11} + \frac{1}{21} - \frac{1}{27} + \frac{1}{37} - \frac{1}{43} + \frac{1}{53} - \ldots = \frac{\pi}{16}\cot(\tfrac{5}{16}) \\ &= \frac{\pi}{16}\frac{\sqrt{2-\sqrt{2-\sqrt{2}}}}{\sqrt{2+\sqrt{2-\sqrt{2}}}}\end{aligned}}$$

$$R_{7/16} = \boxed{\begin{aligned}&\frac{1}{7} - \frac{1}{9} + \frac{1}{23} - \frac{1}{25} + \frac{1}{39} - \frac{1}{41} + \frac{1}{55} - \ldots = \frac{\pi}{16}\cot(\tfrac{7}{16}\pi) \\ &= \frac{\pi}{16}\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}}\end{aligned}}$$

$$R_{5/16} - R_{7/16} =$$

$$\frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{21} - \frac{1}{23} + \frac{1}{25} - \frac{1}{27} + \frac{1}{37} +$$

$$-\frac{1}{39} + \frac{1}{41} - \frac{1}{43} + \frac{1}{53} - \frac{1}{55} + \frac{1}{57} + \ldots$$

$$= \frac{\pi}{16}\left\{\cot(\tfrac{5}{16}) - \cot(\tfrac{7}{16})\right\}$$

27.

$R_{3/16} - R_{5/16}$

$$R_{3/16} = \boxed{\begin{aligned} \frac{1}{3} - \frac{1}{13} + \frac{1}{19} - \frac{1}{29} + \frac{1}{35} - \frac{1}{45} + \frac{1}{51} - ... &= \frac{\pi}{16}\cot(\tfrac{3}{16}\pi) \\ &= \frac{\pi}{16}\frac{\sqrt{2+\sqrt{2-\sqrt{2}}}}{\sqrt{2-\sqrt{2-\sqrt{2}}}} \end{aligned}}$$

$$R_{5/16} = \boxed{\begin{aligned} \frac{1}{5} - \frac{1}{11} + \frac{1}{21} - \frac{1}{27} + \frac{1}{37} - \frac{1}{43} + \frac{1}{53} - ... &= \frac{\pi}{16}\cot(\tfrac{5}{16}) \\ &= \frac{\pi}{16}\frac{\sqrt{2-\sqrt{2-\sqrt{2}}}}{\sqrt{2+\sqrt{2-\sqrt{2}}}} \end{aligned}}$$

$R_{3/16} - R_{5/16}$

$$\frac{1}{3} - \frac{1}{5} + \frac{1}{11} - \frac{1}{13} + \frac{1}{19} - \frac{1}{21} + \frac{1}{27} - \frac{1}{29} + \frac{1}{35}$$

$$-\frac{1}{37} + \frac{1}{43} - \frac{1}{45} + \frac{1}{51} - \frac{1}{53} + \frac{1}{59} - ... =$$

$$= \frac{\pi}{16}\left\{\cot(\tfrac{3}{16}\pi) - \cot(\tfrac{5}{16}\pi)\right\}$$

28.

$R_{1/16} - R_{5/16}$

$$R_{1/16} = \boxed{\frac{1}{1} - \frac{1}{15} + \frac{1}{17} - \frac{1}{31} + \frac{1}{33} - \frac{1}{47} + \frac{1}{49} - \ldots = \frac{\pi}{16}\cot(\tfrac{1}{16}\pi) = \frac{\pi}{16}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{\sqrt{2-\sqrt{2+\sqrt{2}}}}}$$

$$R_{5/16} = \boxed{\frac{1}{5} - \frac{1}{11} + \frac{1}{21} - \frac{1}{27} + \frac{1}{37} - \frac{1}{43} + \frac{1}{53} - \ldots = \frac{\pi}{16}\cot(\tfrac{5}{16}) = \frac{\pi}{16}\frac{\sqrt{2-\sqrt{2-\sqrt{2}}}}{\sqrt{2+\sqrt{2-\sqrt{2}}}}}$$

$R_{1/16} - R_{5/16} =$

$$\frac{1}{1} - \frac{1}{5} - \frac{1}{11} + \frac{1}{15} + \frac{1}{17} - \frac{1}{21} - \frac{1}{27} + \frac{1}{31} + \frac{1}{33}$$

$$-\frac{1}{37} - \frac{1}{43} + \frac{1}{47} + \frac{1}{49} - \frac{1}{53} - \frac{1}{59} + \ldots =$$

$$= \frac{\pi}{16}\left\{\cot(\tfrac{1}{16}\pi) - \cot(\tfrac{5}{16}\pi)\right\}$$

29.

$R_{1/16} - R_{7/16}$

$$R_{1/16} = \boxed{\begin{aligned}\frac{1}{1} - \frac{1}{15} + \frac{1}{17} - \frac{1}{31} + \frac{1}{33} - \frac{1}{47} + \frac{1}{49} - ... &= \frac{\pi}{16}\cot(\tfrac{1}{16}\pi) \\ &= \frac{\pi}{16}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{\sqrt{2-\sqrt{2+\sqrt{2}}}}\end{aligned}}$$

$$R_{7/16} = \boxed{\begin{aligned}\frac{1}{7} - \frac{1}{9} + \frac{1}{23} - \frac{1}{25} + \frac{1}{39} - \frac{1}{41} + \frac{1}{55} - ... &= \frac{\pi}{16}\cot(\tfrac{7}{16}\pi) \\ &= \frac{\pi}{16}\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}}\end{aligned}}$$

$R_{1/16} - R_{7/16} =$

$$\frac{1}{1} - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \frac{1}{23} + \frac{1}{25} - \frac{1}{31} + \frac{1}{33}$$

$$-\frac{1}{39} + \frac{1}{41} - \frac{1}{47} + \frac{1}{49} - \frac{1}{55} + \frac{1}{57} - ... =$$

$$= \frac{\pi}{16}\left\{\cot(\tfrac{1}{16}\pi) - \cot(\tfrac{7}{16}\pi)\right\}$$

30.

$(R_{3/16} - R_{5/16})(R_{1/16} - R_{7/16})$

$$R_{3/16} - R_{5/16} = \frac{1}{3} - \frac{1}{5} + \frac{1}{11} - \frac{1}{13} + \frac{1}{19} - \frac{1}{21} + \frac{1}{27} - \frac{1}{29} + \frac{1}{35}$$

$$-\frac{1}{37} + \frac{1}{43} - \frac{1}{45} + \frac{1}{51} - \frac{1}{53} + \frac{1}{59} - ... = \frac{\pi}{16}\left\{\cot(\tfrac{3}{16}\pi) - \cot(\tfrac{5}{16}\pi)\right\}$$

$$R_{1/16} - R_{7/16} = \frac{1}{1} - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \frac{1}{23} + \frac{1}{25} - \frac{1}{31} + \frac{1}{33}$$

$$-\frac{1}{39} + \frac{1}{41} - \frac{1}{47} + \frac{1}{49} - \frac{1}{55} + \frac{1}{57} - ... = \frac{\pi}{16}\left\{\cot(\tfrac{1}{16}\pi) - \cot(\tfrac{7}{16}\pi)\right\}$$

$$(R_{3/16} - R_{5/16})(R_{1/16} - R_{7/16}) =$$

$$= \frac{\pi^2}{16^2}\left\{\cot(\tfrac{3}{16}\pi) - \cot(\tfrac{5}{16}\pi)\right\}\left\{\cot(\tfrac{1}{16}\pi) - \cot(\tfrac{7}{16}\pi)\right\}$$

31.

$R_{3/16} - R_{7/16}$

$$R_{3/16} = \boxed{\begin{aligned} &\frac{1}{3} - \frac{1}{13} + \frac{1}{19} - \frac{1}{29} + \frac{1}{35} - \frac{1}{45} + \frac{1}{51} - ... = \frac{\pi}{16}\cot(\tfrac{3}{16}\pi) \\ &= \frac{\pi}{16}\frac{\sqrt{2+\sqrt{2-\sqrt{2}}}}{\sqrt{2-\sqrt{2-\sqrt{2}}}} \end{aligned}}$$

$$R_{7/16} = \boxed{\begin{aligned} &\frac{1}{7} - \frac{1}{9} + \frac{1}{23} - \frac{1}{25} + \frac{1}{39} - \frac{1}{41} + \frac{1}{55} - ... = \frac{\pi}{16}\cot(\tfrac{7}{16}\pi) \\ &= \frac{\pi}{16}\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}} \end{aligned}}$$

$R_{3/16} - R_{7/16} =$

$$\frac{1}{3} - \frac{1}{7} + \frac{1}{9} - \frac{1}{13} + \frac{1}{19} - \frac{1}{23} + \frac{1}{25} - \frac{1}{29} + \frac{1}{35} -$$

$$-\frac{1}{39} + \frac{1}{41} - \frac{1}{45} + \frac{1}{51} - \frac{1}{55} + \frac{1}{61} - ... = \frac{\pi}{16}\{\cot(\tfrac{1}{16}\pi) - \cot(\tfrac{7}{16}\pi)\}$$

32.

$$\pi = p\tan(\tfrac{1}{p}\pi)\left\{\frac{1}{1} - \frac{1}{p-1} + \frac{1}{p+1} - \frac{1}{2p-1} + \frac{1}{2p+1} - \frac{1}{3p-1} + \frac{1}{3p+1} + \ldots\right\}$$

$$\pi = p\sin(\tfrac{1}{p}\pi)\left\{\frac{1}{1} + \frac{1}{p-1} - \frac{1}{p+1} - \frac{1}{2p-1} + \frac{1}{2p+1} + \frac{1}{3p-1} - \frac{1}{3p+1} + \ldots\right\}$$

$p =$ prime.

$$R_{1/p} = 1 - \frac{1}{p-1} + \frac{1}{p+1} - \frac{1}{2p-1} + \frac{1}{2p+1} - \frac{1}{3p-1} + \frac{1}{3p+1} + \ldots = \frac{\pi}{p}\cot(\tfrac{1}{p}\pi)$$

$$S_{1/p} = 1 + \frac{1}{p-1} - \frac{1}{p+1} - \frac{1}{2p-1} + \frac{1}{2p+1} + \frac{1}{3p-1} - \frac{1}{3p+1} - \ldots = \frac{\pi}{p\sin(\tfrac{1}{p}\pi)}$$

Proof: Based on $\pi\cot(\pi z)\dfrac{1}{z+\frac{1}{p}}$,

$$\ldots + \frac{1}{-3+\frac{1}{p}} + \frac{1}{-2+\frac{1}{p}} + \frac{1}{-1+\frac{1}{p}} + \frac{1}{0+\frac{1}{p}} +$$

$$+\frac{1}{1+\frac{1}{p}} \quad + \quad \frac{1}{2+\frac{1}{p}} \quad + \quad \frac{1}{3+\frac{1}{p}} + ... - \pi\cot(\tfrac{1}{p}\pi) = 0$$

$$\pi = p\tan(\tfrac{1}{p}\pi)\left\{1 - \frac{1}{p-1} + \frac{1}{p+1} - \frac{1}{2p-1} + \frac{1}{2p+1} - \frac{1}{3p-1} + \frac{1}{3p+1} + ...\right\}. \square$$

Based on $\pi\frac{1}{\sin(\pi z)}\frac{1}{z+\frac{1}{p}}$,

$$... - \frac{1}{-3+\frac{1}{p}} + \frac{1}{-2+\frac{1}{p}} - \frac{1}{-1+\frac{1}{p}} + \frac{1}{0+\frac{1}{p}} -$$

$$-\frac{1}{1+\frac{1}{p}} \quad + \quad \frac{1}{2+\frac{1}{p}} - \frac{1}{3+\frac{1}{p}} + ... - \frac{\pi}{\sin(\frac{1}{p}\pi)} = 0$$

$$\pi = p\sin(\tfrac{1}{p}\pi)\left\{1 + \frac{1}{p-1} - \frac{1}{p+1} - \frac{1}{2p-1} + \frac{1}{2p+1} + \frac{1}{3p-1} - \frac{1}{3p+1} - ...\right\}. \square$$

33.

$$\pi = p^n \tan(\tfrac{m}{p^n}\pi)\left\{\frac{1}{m} - \frac{1}{p^n - m} + \frac{1}{p^n + m} - \frac{1}{2p^n - m} + \frac{1}{2p^n + m} - \frac{1}{3p^n - m} + \frac{1}{3p^n + m} + \ldots\right\}$$

$$\pi = p^n \tan(\tfrac{m}{p^n}\pi)\left\{\frac{1}{m} + \frac{1}{p^n - m} - \frac{1}{p^n + m} - \frac{1}{2p^n - m} + \frac{1}{2p^n + m} + \frac{1}{3p^n - m} - \frac{1}{3p^n + m} - \ldots\right\}$$

$m < \frac{1}{2}p^n$, m and p^n have no common factor, $n = 1, 2, 3, \ldots$

$$R_{m/p^n} = \frac{1}{m} - \frac{1}{p^n - m} + \frac{1}{p^n + m} - \frac{1}{2p^n - m} + \frac{1}{2p^n + m} - \frac{1}{3p^n - m} + \frac{1}{3p^n + m} + \ldots = \frac{\pi}{p^n}\cot(\tfrac{m}{p^n}\pi)$$

$$S_{m/p^n} = \frac{1}{m} + \frac{1}{p^n - m} - \frac{1}{p^n + m} - \frac{1}{2p^n - m} + \frac{1}{2p^n + m} + \frac{1}{3p^n - m} - \frac{1}{3p^n + m} - \ldots = \frac{\pi}{p^n}\frac{1}{\sin(\frac{m}{p^n}\pi)}$$

Proof: Based on $\pi\cot(\pi z)\dfrac{1}{z + \frac{m}{p^n}}$,

$$... - \frac{1}{-3 + \frac{m}{p^n}} + \frac{1}{-2 + \frac{m}{p^n}} - \frac{1}{-1 + \frac{m}{p^n}} + \frac{1}{0 + \frac{m}{p^n}}$$

$$- \frac{1}{1 + \frac{m}{p^n}} \quad + \quad \frac{1}{2 + \frac{m}{p^n}} - \frac{1}{3 + \frac{m}{p^n}} + ... - \pi \cot(\tfrac{m}{p^n}\pi) = 0$$

$$p^n \left\{ \frac{1}{m} + \frac{1}{p^n - m} - \frac{1}{p^n + m} \right.$$
$$\left. + \frac{1}{2p^n - m} - \frac{1}{2p^n + m} + \frac{1}{3p^n - m} - .... \right\} = \pi \cot(\tfrac{m}{p^n}\pi)$$

Based on $\pi \dfrac{1}{\sin(\pi z)} \dfrac{1}{z + \frac{m}{p^n}}$,

$$... - \frac{1}{-3 + \frac{m}{p^n}} + \frac{1}{-2 + \frac{m}{p^n}} - \frac{1}{-1 + \frac{m}{p^n}} + \frac{1}{0 + \frac{m}{p^n}}$$

$$- \frac{1}{1 + \frac{m}{p^n}} \quad + \quad \frac{1}{2 + \frac{m}{p^n}} - \frac{1}{3 + \frac{m}{p^n}} + ... - \pi \frac{1}{\sin(\frac{m}{p^n}\pi)} = 0$$

$$p^n \left\{ \frac{1}{m} + \frac{1}{p^n - m} - \frac{1}{p^n + m} \right.$$
$$- \frac{1}{2p^n - m} + \frac{1}{2p^n + m}$$
$$\left. + \frac{1}{3p^n - m} - \frac{1}{3p^n + m} - .... \right\} = \pi \cot(\tfrac{m}{p^n}\pi)$$

34.

$$\boxed{\pi = 3\underbrace{\tan(\tfrac{1}{3}\pi)}_{\cot(\frac{1}{6}\pi)}\left(\frac{1}{1}-\frac{1}{2}+\frac{1}{4}-\frac{1}{5}+\frac{1}{7}-\frac{1}{8}+\frac{1}{10}-\frac{1}{11}+...\right)}$$

$$\boxed{\pi = 3\underbrace{\sin(\tfrac{1}{3}\pi)}_{\cos(\frac{1}{6}\pi)}\left(\frac{1}{1}+\frac{1}{2}-\frac{1}{4}-\frac{1}{5}+\frac{1}{7}+\frac{1}{8}-\frac{1}{10}-\frac{1}{11}+...\right)}$$

$$R_{1/3} = \frac{1}{1}-\frac{1}{2}+\frac{1}{4}-\frac{1}{5}+\frac{1}{7}-\frac{1}{8}+\frac{1}{10}-\frac{1}{11}+... = \frac{\pi}{3\sqrt{3}}$$

$$S_{1/3} = \frac{1}{1}+\frac{1}{2}-\frac{1}{4}-\frac{1}{5}+\frac{1}{7}+\frac{1}{8}-\frac{1}{10}-\frac{1}{11}+... = 2\frac{\pi}{3\sqrt{3}}$$

Proof: **Based on** $\pi\cot(\pi z)\dfrac{1}{z+\frac{1}{3}}$,

$$... + \frac{1}{-3+\frac{1}{3}} + \frac{1}{-2+\frac{1}{3}} + \frac{1}{-1+\frac{1}{3}} +$$

$$+\frac{1}{0+\frac{1}{3}} + \frac{1}{1+\frac{1}{3}} + \frac{1}{2+\frac{1}{3}} + \frac{1}{3+\frac{1}{3}} + ... - \pi\cot(\tfrac{1}{3}\pi) = 0$$

$$3\underbrace{\tan(\tfrac{1}{3}\pi)}_{\sqrt{3}}\left(\frac{1}{1}-\frac{1}{2}+\frac{1}{4}-\frac{1}{5}+\frac{1}{7}-\frac{1}{8}+\frac{1}{10}-...\right) = \pi\,.\square$$

Based on $\dfrac{\pi}{\sin(\pi z)}\dfrac{1}{z+\frac{1}{3}}$,

$$... + \frac{(-1)^3}{-3+\frac{1}{3}} + \frac{(-1)^2}{-2+\frac{1}{3}} + \frac{(-1)^{-1}}{-1+\frac{1}{3}} +$$

$$+\frac{(-1)^0}{0+\frac{1}{3}} + \frac{(-1)^1}{1+\frac{1}{3}} + \frac{(-1)^2}{2+\frac{1}{3}} + \frac{(-1)^3}{3+\frac{1}{3}} + ... - \pi\frac{1}{\sin(\frac{1}{3}\pi)} = 0$$

$$3\left(\frac{1}{1}+\frac{1}{2}-\frac{1}{4}-\frac{1}{5}+\frac{1}{7}+\frac{1}{8}-\frac{1}{10}-\frac{1}{11}+...\right)=\pi\frac{1}{\sin(\frac{1}{3}\pi)}$$

$$\pi=3\underbrace{\sin(\tfrac{1}{3}\pi)}_{\frac{\sqrt{3}}{2}}\left(\frac{1}{1}+\frac{1}{2}-\frac{1}{4}-\frac{1}{5}+\frac{1}{7}+\frac{1}{8}-\frac{1}{10}-\frac{1}{11}+...\right).\square$$

35.

$$\boxed{\pi = 9\underbrace{\tan(\tfrac{1}{9}\pi)}_{\cot(\frac{7}{18}\pi)}\left(\frac{1}{1} - \frac{1}{8} + \frac{1}{10} - \frac{1}{17} + \frac{1}{19} - \frac{1}{26} + \frac{1}{28} - ...\right)}$$

$$\boxed{\pi = 9\underbrace{\sin(\tfrac{1}{9}\pi)}_{\cos(\frac{7}{18}\pi)}\left(\frac{1}{1} + \frac{1}{8} - \frac{1}{10} - \frac{1}{17} + \frac{1}{19} + \frac{1}{26} - \frac{1}{28} - \frac{1}{35} + ...\right)}$$

$$R_{1/9} = \frac{1}{1} - \frac{1}{8} + \frac{1}{10} - \frac{1}{17} + \frac{1}{19} - \frac{1}{26} + \frac{1}{28} - ... = \frac{\pi}{3}\underbrace{\cot(\tfrac{1}{9}\pi)}_{\tan(\frac{7}{18}\pi)}$$

$$S_{1/8} = \frac{1}{1} + \frac{1}{8} - \frac{1}{10} - \frac{1}{17} + \frac{1}{19} + \frac{1}{26} - \frac{1}{28} - \frac{1}{35} + ... = \frac{\pi}{9}\frac{1}{\sin(\frac{1}{9}\pi)}$$

$\sin(\tfrac{1}{9}\pi)$ **solves the cubic** $4x^3 - 3x + \tfrac{1}{2}\sqrt{3} = 0$

<u>Proof</u> **Based on** $\pi\cot(\pi z)\dfrac{1}{z + \frac{1}{9}}$,

$$... + \frac{1}{-3 + \frac{1}{9}} + \frac{1}{-2 + \frac{1}{9}} + \frac{1}{-1 + \frac{1}{9}} +$$

$$+ \frac{1}{0 + \frac{1}{9}} + \frac{1}{1 + \frac{1}{9}} + \frac{1}{2 + \frac{1}{9}} + \frac{1}{3 + \frac{1}{9}} + ... - \pi\cot(\tfrac{1}{9}\pi) = 0$$

$$9\tan(\tfrac{1}{9}\pi)\left(\frac{1}{1} - \frac{1}{8} + \frac{1}{10} - \frac{1}{17} + \frac{1}{19} - \frac{1}{26} + \frac{1}{28} - ...\right) = \pi . \square$$

$$\cot(\tfrac{1}{9}\pi) = \frac{\cos(\frac{1}{9}\pi)}{\sin(\frac{1}{9}\pi)}$$

$$= \frac{\sqrt{\frac{1}{2}[1 + \cos(\frac{2}{9}\pi)]}}{\sqrt{\frac{1}{2}[1 - \cos(\frac{2}{9}\pi)]}}$$

$$= \frac{\sqrt{1+\cos(\frac{2}{9}\pi)}}{\sqrt{1-\cos(\frac{2}{9}\pi)}}.\square$$

Base on $\pi \dfrac{1}{\sin(\pi z)} \dfrac{1}{z+\frac{1}{9}}$

$$... + \frac{1}{-4+\frac{1}{9}} - \frac{1}{-3+\frac{1}{9}} + \frac{1}{-2+\frac{1}{9}} - \frac{1}{-1+\frac{1}{9}} + \frac{1}{0+\frac{1}{9}} +$$

$$- \frac{1}{1+\frac{1}{9}} + \frac{1}{2+\frac{1}{9}} - \frac{1}{3+\frac{1}{9}} + ... - \frac{\pi}{\sin(\frac{1}{9}\pi)} = 0$$

$$\pi = 9\sin(\tfrac{1}{9}\pi)\left(\frac{1}{1} + \frac{1}{8} - \frac{1}{10} - \frac{1}{17} + \frac{1}{19} + \frac{1}{26} - \frac{1}{28} - \frac{1}{35} + ...\right).\square$$

$$\sin 3A = -4\sin^3 A + 3\sin A \Rightarrow$$

$$\underbrace{\sin(\tfrac{1}{3}\pi)}_{\frac{1}{2}\sqrt{3}} = -4\sin^3(\tfrac{1}{9}\pi) + 3\sin(\tfrac{1}{9}\pi) \Rightarrow$$

$\sin(\frac{1}{9}\pi)$ **solves the cubic** $4x^3 - 3x + \frac{1}{2}\sqrt{3} = 0.\square$

36.

$$\boxed{\pi = 9\underbrace{\tan(\tfrac{2}{9}\pi)}_{\cot(\frac{5}{18}\pi)}\left(\frac{1}{2} - \frac{1}{7} + \frac{1}{11} - \frac{1}{16} + \frac{1}{20} - \frac{1}{25} + \frac{1}{29} - ...\right)}$$

$$\boxed{\pi = 9\underbrace{\sin(\tfrac{2}{9}\pi)}_{\cos(\frac{5}{18}\pi)}\left(\frac{1}{2} + \frac{1}{7} - \frac{1}{11} - \frac{1}{16} + \frac{1}{20} + \frac{1}{25} - \frac{1}{29} - \frac{1}{34} + ...\right)}$$

$$R_{2/9} = \frac{1}{2} - \frac{1}{7} + \frac{1}{11} - \frac{1}{16} + \frac{1}{20} - \frac{1}{25} + \frac{1}{29} - ... = \frac{\pi}{9}\underbrace{\cot(\tfrac{2}{9}\pi)}_{\tan(\frac{5}{18}\pi)}$$

$$S_{2/9} = \frac{1}{2} + \frac{1}{7} - \frac{1}{11} - \frac{1}{16} + \frac{1}{20} + \frac{1}{25} - \frac{1}{29} - \frac{1}{34} + ... = \frac{\pi}{9\sin(\frac{2}{9}\pi)}$$

$\sin(\tfrac{2}{9}\pi)$ **solves the cubic** $4x^3 - 3x + \tfrac{1}{2}\sqrt{3} = 0$

Proof **Based on** $\pi\cot(\pi z)\dfrac{1}{z + \frac{2}{9}}$,

$$... + \frac{1}{-3 + \frac{2}{9}} + \frac{1}{-2 + \frac{2}{9}} + \frac{1}{-1 + \frac{2}{9}} +$$

$$+ \frac{1}{0 + \frac{2}{9}} + \frac{1}{1 + \frac{2}{9}} + \frac{1}{2 + \frac{2}{9}} + \frac{1}{3 + \frac{2}{9}} + ... - \pi\cot(\tfrac{2}{9}\pi) = 0$$

$$9\tan(\tfrac{2}{9}\pi)\left(\frac{1}{2} - \frac{1}{7} + \frac{1}{11} - \frac{1}{16} + \frac{1}{20} - \frac{1}{25} + \frac{1}{29} - ...\right) = \pi\,.\square$$

$$\cot(\tfrac{2}{9}\pi) = \frac{\cos(\frac{2}{9}\pi)}{\sin(\frac{2}{9}\pi)}$$

$$= \frac{\sqrt{\frac{1}{2}[1 + \cos(\frac{4}{9}\pi)]}}{\sqrt{\frac{1}{2}[1 - \cos(\frac{4}{9}\pi)]}}$$

$$= \frac{\sqrt{1+\cos(\frac{4}{9}\pi)}}{\sqrt{1-\cos(\frac{4}{9}\pi)}}.\square$$

Base on $\pi\frac{1}{\sin(\pi z)}\frac{1}{z+\frac{2}{9}}$,

$$... + \frac{1}{-4+\frac{2}{9}} - \frac{1}{-3+\frac{2}{9}} + \frac{1}{-2+\frac{2}{9}} - \frac{1}{-1+\frac{2}{9}} + \frac{1}{0+\frac{2}{9}} +$$

$$-\frac{1}{1+\frac{2}{9}} + \frac{1}{2+\frac{2}{9}} - \frac{1}{3+\frac{2}{9}} + ... - \pi\frac{1}{\sin(\frac{2}{9}\pi)} = 0$$

$$\pi = 9\sin(\tfrac{2}{9}\pi)\left(\frac{1}{2}+\frac{1}{7}-\frac{1}{11}-\frac{1}{16}+\frac{1}{20}+\frac{1}{25}-\frac{1}{29}-\frac{1}{34}+...\right).\square$$

$$\sin 3A = -4\sin^3 A + 3\sin A \Rightarrow$$

$$\underbrace{\sin(\tfrac{2}{3}\pi)}_{\frac{1}{2}\sqrt{3}} = -4\sin^3(\tfrac{2}{9}\pi) + 3\sin(\tfrac{2}{9}\pi) \Rightarrow$$

$\sin(\frac{2}{9}\pi)$ **solves the cubic** $4x^3 - 3x + \frac{1}{2}\sqrt{3} = 0.\square$

37.

$$\boxed{\pi = 9\underbrace{\tan(\tfrac{4}{9}\pi)}_{\cot(\frac{1}{18}\pi)}\left(\frac{1}{4} - \frac{1}{5} + \frac{1}{13} - \frac{1}{14} + \frac{1}{22} - \frac{1}{23} + \frac{1}{31} - ...\right)}$$

$$\boxed{\pi = 9\underbrace{\sin(\tfrac{4}{9}\pi)}_{\cos(\frac{1}{18}\pi)}\left(\frac{1}{4} + \frac{1}{5} - \frac{1}{13} - \frac{1}{14} + \frac{1}{22} + \frac{1}{23} - \frac{1}{31} - \frac{1}{32} + ...\right)}$$

$$R_{4/9} = \frac{1}{4} - \frac{1}{5} + \frac{1}{13} - \frac{1}{14} + \frac{1}{22} - \frac{1}{23} + \frac{1}{31} - ... = \frac{\pi}{9}\underbrace{\cot(\tfrac{4}{9}\pi)}_{\tan(\frac{1}{18}\pi)}$$

$$S_{4/9} = \frac{1}{4} + \frac{1}{5} - \frac{1}{13} - \frac{1}{14} + \frac{1}{22} + \frac{1}{23} - \frac{1}{31} - \frac{1}{32} + ... = \frac{\pi}{9\sin(\frac{4}{9}\pi)}$$

$\sin(\tfrac{4}{9}\pi)$ **solves the cubic** $4x^3 - 3x - \tfrac{1}{2}\sqrt{3} = 0$

Proof **Based on** $\pi\cot(\pi z)\dfrac{1}{z + \frac{4}{9}}$

$$... + \frac{1}{-3 + \frac{4}{9}} + \frac{1}{-2 + \frac{4}{9}} + \frac{1}{-1 + \frac{4}{9}} +$$

$$+ \frac{1}{0 + \frac{4}{9}} + \frac{1}{1 + \frac{4}{9}} + \frac{1}{2 + \frac{4}{9}} + \frac{1}{3 + \frac{4}{9}} + ... - \pi\cot(\tfrac{4}{9}\pi) = 0$$

$$9\tan(\tfrac{4}{9}\pi)\left(\frac{1}{4} - \frac{1}{5} + \frac{1}{13} - \frac{1}{14} + \frac{1}{22} - \frac{1}{23} + \frac{1}{31} - ...\right) = \pi\,.\square$$

$$\cot(\tfrac{4}{9}\pi) = \frac{\cos(\frac{4}{9}\pi)}{\sin(\frac{4}{9}\pi)}$$

$$= \frac{\sqrt{\frac{1}{2}[1 + \cos(\frac{8}{9}\pi)]}}{\sqrt{\frac{1}{2}[1 - \cos(\frac{8}{9}\pi)]}}$$

$$= \frac{\sqrt{1-\cos(\frac{1}{9}\pi)}}{\sqrt{1+\cos(\frac{1}{9}\pi)}}.\square$$

Based on $\pi\frac{1}{\sin(\pi z)}\frac{1}{z+\frac{4}{9}}$,

$$... + \frac{1}{-4+\frac{4}{9}} - \frac{1}{-3+\frac{4}{9}} + \frac{1}{-2+\frac{4}{9}} - \frac{1}{-1+\frac{4}{9}} + \frac{1}{0+\frac{4}{9}} +$$

$$-\frac{1}{1+\frac{4}{9}} + \frac{1}{2+\frac{4}{9}} - \frac{1}{3+\frac{4}{9}} + ... - \pi\frac{1}{\sin(\frac{4}{9}\pi)} = 0$$

$$\pi = 9\sin(\tfrac{4}{9}\pi)\left(\frac{1}{4}+\frac{1}{5}-\frac{1}{13}-\frac{1}{14}+\frac{1}{22}+\frac{1}{23}-\frac{1}{31}-\frac{1}{32}+...\right).\square$$

$$\sin 3A = -4\sin^3 A + 3\sin A \Rightarrow$$

$$\underbrace{\sin(3\tfrac{4}{9}\pi)}_{-\frac{1}{2}\sqrt{3}} = -4\sin^3(\tfrac{4}{9}\pi) + 3\sin(\tfrac{4}{9}\pi) \Rightarrow$$

$\sin(\frac{4}{9}\pi)$ **solves the cubic** $4x^3 - 3x - \frac{1}{2}\sqrt{3} = 0.\square$

38.

$$\boxed{\pi = 2\sqrt{3}\left(\frac{1}{1} - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \frac{1}{29} - \frac{1}{23} + ...\right)}$$

$$\boxed{\pi = 3\left(\frac{1}{1} + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \frac{1}{17} - \frac{1}{29} - \frac{1}{23} + ...\right)}$$

$$R_{1/6} = \frac{1}{1} - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \frac{1}{29} - \frac{1}{23} + ... = \frac{\pi}{2\sqrt{3}}$$

$$S_{1/6} = \frac{1}{1} + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \frac{1}{17} - \frac{1}{29} - \frac{1}{23} + ... = \frac{\pi}{3}$$

Proof: Based on $\pi\cot(\pi z)\dfrac{1}{z+\frac{1}{6}}$,

$$... + \frac{1}{-4+\frac{1}{6}} + \frac{1}{-3+\frac{1}{6}} + \frac{1}{-2+\frac{1}{6}} + \frac{1}{-1+\frac{1}{6}} + \frac{1}{0+\frac{1}{6}} +$$

$$+\frac{1}{1+\frac{1}{6}} + \frac{1}{2+\frac{1}{6}} + \frac{1}{3+\frac{1}{6}} + ... - \pi\underbrace{\cot(\tfrac{1}{6}\pi)}_{\sqrt{3}} = 0$$

$$\pi = 2\sqrt{3}\left(\frac{1}{1} - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \frac{1}{29} - \frac{1}{23} + ...\right).\square$$

Based on $\pi\dfrac{1}{\sin(\pi z)}\dfrac{1}{z+\frac{1}{6}}$,

$$... + \frac{1}{-4+\frac{1}{6}} - \frac{1}{-3+\frac{1}{6}} + \frac{1}{-2+\frac{1}{6}} - \frac{1}{-1+\frac{1}{6}} + \frac{1}{0+\frac{1}{6}} +$$

$$-\frac{1}{1+\frac{1}{6}} + \frac{1}{2+\frac{1}{6}} - \frac{1}{3+\frac{1}{6}} + ... - \frac{\pi}{\underbrace{\sin(\tfrac{1}{6}\pi)}_{2\pi}} = 0$$

$$\pi = 3\left(\frac{1}{1} + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \frac{1}{17} - \frac{1}{29} - \frac{1}{23} + ...\right).\square$$

39.

$$\pi = 18\underbrace{\tan(\tfrac{1}{18}\pi)}_{\cot(\frac{8}{18}\pi)}\left(\frac{1}{1}+\frac{1}{17}-\frac{1}{19}-\frac{1}{35}+\frac{1}{37}+\frac{1}{53}-\frac{1}{55}-\frac{1}{71}...\right)$$

$$\pi = 18\underbrace{\sin(\tfrac{1}{18}\pi)}_{\cos(\frac{8}{18}\pi)}\left(\frac{1}{1}+\frac{1}{17}-\frac{1}{19}-\frac{1}{35}+\frac{1}{37}+\frac{1}{53}-\frac{1}{55}-\frac{1}{71}...\right)$$

$$R_{1/18} = \frac{1}{1}-\frac{1}{17}+\frac{1}{19}-\frac{1}{35}+\frac{1}{37}-\frac{1}{53}+\frac{1}{55}-\frac{1}{71}... = \frac{\pi}{18}\cot(\tfrac{1}{18}\pi)$$

$$S_{1/18} = \frac{1}{1}+\frac{1}{17}-\frac{1}{19}-\frac{1}{35}+\frac{1}{37}+\frac{1}{53}-\frac{1}{55}-\frac{1}{71}... = \frac{\pi}{18\sin(\frac{1}{18}\pi)}$$

$$\sin(\tfrac{1}{18}\pi) = \sqrt{\tfrac{1}{2}[1-\cos(\tfrac{1}{9}\pi)]} = \sqrt{\tfrac{1}{2}[1-\sin(\tfrac{4}{9}\pi)]}$$

And $\sin(\frac{4}{9}\pi)$ **solves the cubic** $4x^3 - 3x - \frac{1}{2}\sqrt{3} = 0$

Proof: Based on $\pi\cot(\pi z)\dfrac{1}{z+\frac{1}{18}}$

$$... + \frac{1}{-4+\frac{1}{18}} - \frac{1}{-3+\frac{1}{18}} + \frac{1}{-2+\frac{1}{18}} - \frac{1}{-1+\frac{1}{18}} + \frac{1}{0+\frac{1}{18}} +$$

$$-\frac{1}{1+\frac{1}{18}} + \frac{1}{2+\frac{1}{18}} - \frac{1}{3+\frac{1}{18}} + ... - \pi\cot(\tfrac{1}{18}\pi) = 0$$

$$18\left(\frac{1}{1}+\frac{1}{17}-\frac{1}{19}-\frac{1}{35}+\frac{1}{37}+\frac{1}{53}-\frac{1}{55}-\frac{1}{71}...\right) = \pi\cot(\tfrac{1}{18}\pi).\square$$

Based on $\pi\dfrac{1}{\sin(\pi z)}\dfrac{1}{z+\frac{1}{18}}$,

$$... + \frac{1}{-4+\frac{1}{18}} - \frac{1}{-3+\frac{1}{18}} + \frac{1}{-2+\frac{1}{18}} - \frac{1}{-1+\frac{1}{18}} + \frac{1}{0+\frac{1}{18}} +$$

$$-\frac{1}{1+\frac{1}{18}}+\frac{1}{2+\frac{1}{18}}-\frac{1}{3+\frac{1}{18}}+\ldots-\frac{\pi}{\sin(\frac{1}{18}\pi)}=0$$

$$18\left(\frac{1}{1}+\frac{1}{17}-\frac{1}{19}-\frac{1}{35}+\frac{1}{37}+\frac{1}{53}-\frac{1}{55}-\frac{1}{71}\ldots\right)=\frac{\pi}{\sin(\frac{1}{18}\pi)}.\square$$

40.

$$\pi = 18\underbrace{\tan(\tfrac{5}{18}\pi)}_{\cot(\frac{4}{18}\pi)}\left(\frac{1}{5}-\frac{1}{13}+\frac{1}{23}-\frac{1}{31}+\frac{1}{41}-\frac{1}{49}+\frac{1}{59}-\frac{1}{67}...\right)$$

$$\pi = 18\underbrace{\sin(\tfrac{5}{18}\pi)}_{\cos(\frac{4}{18}\pi)}\left(\frac{1}{5}+\frac{1}{13}-\frac{1}{23}-\frac{1}{31}+\frac{1}{41}+\frac{1}{49}-\frac{1}{59}-\frac{1}{67}...\right)$$

$$R_{5/18} = \frac{1}{5}-\frac{1}{13}+\frac{1}{23}-\frac{1}{31}+\frac{1}{41}-\frac{1}{49}+\frac{1}{59}-\frac{1}{67}... = \frac{\pi}{18}\cot(\tfrac{5}{18}\pi)$$

$$S_{5/18} = \frac{1}{5}+\frac{1}{13}-\frac{1}{23}-\frac{1}{31}+\frac{1}{41}+\frac{1}{49}-\frac{1}{59}-\frac{1}{67}... = \frac{\pi}{18\sin(\frac{5}{18}\pi)}$$

Proof: Base on $\pi\cot(\pi z)\dfrac{1}{z+\frac{5}{18}}$,

$$...+\frac{1}{-4+\frac{5}{18}}+\frac{1}{-3+\frac{5}{18}}+\frac{1}{-2+\frac{5}{18}}+\frac{1}{-1+\frac{5}{18}}+\frac{1}{0+\frac{5}{18}}+$$

$$+\frac{1}{1+\frac{5}{18}}+\frac{1}{2+\frac{5}{18}}+\frac{1}{3+\frac{5}{18}}+...-\pi\frac{1}{\sin(\frac{5}{18}\pi)}=0$$

$$18\left(\frac{1}{5}-\frac{1}{13}+\frac{1}{23}-\frac{1}{31}+\frac{1}{41}-\frac{1}{49}+\frac{1}{59}-\frac{1}{67}...\right)=\pi\frac{1}{\sin(\frac{5}{18}\pi)}.\square$$

Base on $\pi\dfrac{1}{\sin(\pi z)}\dfrac{1}{z+\frac{5}{18}}$,

$$...+\frac{1}{-4+\frac{5}{18}}-\frac{1}{-3+\frac{5}{18}}+\frac{1}{-2+\frac{5}{18}}-\frac{1}{-1+\frac{5}{18}}+\frac{1}{0+\frac{5}{18}}+$$

$$-\frac{1}{1+\frac{5}{18}}+\frac{1}{2+\frac{5}{18}}-\frac{1}{3+\frac{5}{18}}+...-\pi\frac{1}{\sin(\frac{5}{18}\pi)}=0$$

$$18\left(\frac{1}{5}+\frac{1}{13}-\frac{1}{23}-\frac{1}{31}+\frac{1}{41}+\frac{1}{49}-\frac{1}{59}-\frac{1}{67}...\right)=\pi\frac{1}{\sin(\frac{5}{18}\pi)}.\square$$

41.

$$\pi = 18\underbrace{\tan(\tfrac{7}{18}\pi)}_{\cot(\frac{2}{18}\pi)}\left(\frac{1}{7}-\frac{1}{11}+\frac{1}{25}-\frac{1}{29}+\frac{1}{43}-\frac{1}{47}+\frac{1}{61}-\frac{1}{65}+\frac{1}{79}-...\right)$$

$$\pi = 18\underbrace{\sin(\tfrac{7}{18}\pi)}_{\cos(\frac{2}{18}\pi)}\left(\frac{1}{7}+\frac{1}{11}-\frac{1}{25}-\frac{1}{29}+\frac{1}{43}+\frac{1}{47}-\frac{1}{61}-\frac{1}{65}+\frac{1}{79}+...\right)$$

$$R_{7/18} = \frac{1}{7}-\frac{1}{11}+\frac{1}{25}-\frac{1}{29}+\frac{1}{43}-\frac{1}{47}+\frac{1}{61}-\frac{1}{65}+... = \frac{\pi}{18}\cot(\tfrac{7}{18}\pi)$$

$$S_{7/18} = \frac{1}{7}+\frac{1}{11}-\frac{1}{25}-\frac{1}{29}+\frac{1}{43}+\frac{1}{47}-\frac{1}{61}-\frac{1}{65}+... = \frac{\pi}{18\sin(\frac{7}{18}\pi)}$$

Proof: Based on $\pi\cot(\pi z)\dfrac{1}{z+\frac{7}{18}}$,

$$...+\frac{1}{-4+\frac{7}{18}}+\frac{1}{-3+\frac{7}{18}}+\frac{1}{-2+\frac{7}{18}}+\frac{1}{-1+\frac{7}{18}}+\frac{1}{0+\frac{7}{18}}+$$

$$+\frac{1}{1+\frac{7}{18}}+\frac{1}{2+\frac{7}{18}}+\frac{1}{3+\frac{7}{18}}+...-\pi\cot(\tfrac{7}{18}\pi) = 0$$

$$18\left(\frac{1}{7}-\frac{1}{11}+\frac{1}{25}-\frac{1}{29}+\frac{1}{43}-\frac{1}{47}+\frac{1}{61}-\frac{1}{65}+\frac{1}{79}+...\right) = \pi\cot(\tfrac{7}{18}\pi).\square$$

Based on $\pi\dfrac{1}{\sin(\pi z)}\dfrac{1}{z+\frac{7}{18}}$,

$$...+\frac{1}{-4+\frac{7}{18}}-\frac{1}{-3+\frac{7}{18}}+\frac{1}{-2+\frac{7}{18}}-\frac{1}{-1+\frac{7}{18}}+\frac{1}{0+\frac{7}{18}}+$$

$$-\frac{1}{1+\frac{7}{18}}+\frac{1}{2+\frac{7}{18}}-\frac{1}{3+\frac{7}{18}}+...-\pi\frac{1}{\sin(\frac{7}{18}\pi)} = 0$$

$$18\left(\frac{1}{7}+\frac{1}{11}-\frac{1}{25}-\frac{1}{29}+\frac{1}{43}+\frac{1}{47}-\frac{1}{61}-\frac{1}{65}+\frac{1}{79}+...\right) = \pi\frac{1}{\sin(\frac{7}{18}\pi)}.\square$$