All the π Series

H. Vic Dannon
vic0@comcast.net
April, 2022

Abstract By π Series we mean a series like Leibniz's

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+...=\frac{1}{4}\pi,$$

of reciprocals of integers, that sum up to an algebraic number times π

Euler obtained infinitely many such series[1]. But his formulas seem to assign integer values to the tangent function. Thus, it is not clear that he considered all the possible series.

To see the crucial dependence of π series on the trigonometric functions, and to ensure obtaining systematically all the possible series we need to apply Cauchy Residue Theorem, which was discovered a century later.

The Residue Theorem yields all the four families of π series.

Euler's work confirms three of the families that we obtain here by the Residue Theorem. Thus, confirming the Residue Theorem Method as the most effective, and powerful in Mathematics

If $l<\frac{1}{2}m$, l and m are natural numbers with no common factor.

Then, applying the Residue Theorem to $\pi\cot(\pi z)\dfrac{1}{z+\frac{l}{m}}$, we obtain the series

[1] Leonardi Euleri, "Introductio in Analysin Infinitorum", #166, #172, #174, #175, #176, #177, #178, #179

$$R_{l/m} = \frac{1}{l} - \frac{1}{m-l} + \frac{1}{m+l} - \frac{1}{2m-l} + \frac{1}{2m+l} - \frac{1}{3m-l} + ... = \frac{\pi}{m}\cot(\tfrac{l}{m}\pi)$$

Applying the Residue Theorem to $\pi\tan(\pi z)\dfrac{1}{z-\frac{l}{m}}$ yields the series

$$T_{l/m} = \frac{1}{m-2l} - \frac{1}{m+2l} + \frac{1}{3m-2l} - \frac{1}{3m+2l} + \frac{1}{5m-2l} - \frac{1}{5m+2l} + ... = \frac{\pi}{2m}\tan(\tfrac{l}{m}\pi).$$

Applying the Residue theorem to $\pi\dfrac{1}{\sin(\pi)}\dfrac{1}{z+\frac{l}{m}}$ yields the series

$$S_{l/m} = \frac{1}{l} + \frac{1}{m-l} - \frac{1}{m+l} - \frac{1}{2m-l} + \frac{1}{2m+l} + \frac{1}{3m-l} - ... = \frac{\pi}{m\sin(\frac{l}{m}\pi)}$$

Applying the Residue Theorem to $\pi\dfrac{1}{\cos(\pi z)}\dfrac{1}{z-\frac{l}{m}}$, yields the series

$$C_{l/m} = \frac{1}{m-2l} + \frac{1}{m+2l} - \frac{1}{3m-2l} - \frac{1}{3m+2l}$$

$$+\frac{1}{5m-2l}+\frac{1}{5m+2l}-...=\frac{\pi}{2m\cos(\frac{l}{m}\pi)}$$

In a recent paper[2], we derived only $R_{l/m}$, and $S_{l/m}$ because

$$\frac{\pi}{2m}\tan(\tfrac{l}{m}\pi)=\frac{\pi}{2m}\cot(\tfrac{1}{2}\pi-\tfrac{l}{m}\pi)$$

guarantees that every π-series from 7.$_{\tan}$ matches uniquely a π-series from 7.$_{\cot}$, and vice versa.

Thus, the family of π-series from 7.$_{\tan}$ is identical to the family of π-series from 7.$_{\cot}$

And

$$\frac{\pi}{2m\cos(\frac{l}{m}\pi)}=\frac{\pi}{2m\sin(\frac{1}{2}\pi-\frac{l}{m}\pi)}$$

guarantees that every π-series from $\mathbf{7_{cos}}$ matches uniquely a π-series from $\mathbf{7_{sin}}$, and vice versa.

Thus, the family of π-series from $\mathbf{7_{cos}}$ is identical with the family of π-series from $\mathbf{7_{sin}}$

But the importance of π deems that presentation incomplete.

Here, we derive $T_{l/m}$, and $C_{l/m}$ as well.

[2] H. Vic Dannon: "π Series" Gauge Institute Journal of math and Physics, Volume 18. No 1. Feb. 2022

Contents

$7_{\sin}$)

$$\pi = m\sin(\tfrac{l}{m}\pi)\left\{\frac{1}{l} + \frac{1}{m-l} - \frac{1}{m+l} - \frac{1}{2m-l} + \frac{1}{2m+l} + \frac{1}{3m-l} - \ldots\right\}$$

$7_{\cos}$)

$$\pi = 2m\cos(\tfrac{l}{m}\pi)\left\{\frac{1}{m-2l} + \frac{1}{m+2l} - \frac{1}{3m-2l} - \frac{1}{3m+2l} + \frac{1}{5m-2l} + \frac{1}{5m+2l} - \ldots\right\}$$

8) $15 \,/\, 38$

9) $(2k+1) \,/\, 2^n$

10) 1/4

11) 1/8

12) 3/8

13) $R_{1/8} + R_{3/8} = S_{1/4}$

14) $S_{1/8} - S_{3/8}$

15) $R_{1/8} - R_{3/8} = R_{1/4}$

16) 1/16

17) 3/16

18) $R_{1/16} + R_{3/16} + R_{5/16} + R_{7/16}$

19) $R_{1/16} - R_{3/16}$

20) $R_{5/16} - R_{7/16}$

21) $R_{3/16} - R_{5/16}$

22) $R_{1/16} - R_{5/16}$

23) $R_{1/16} - R_{7/16}$

24) $(R_{3/16} - R_{5/16})(R_{1/16} - R_{7/16})$

25) $R_{3/16} - R_{7/16}$

26) $1 / p$

27) 1/3

28) l / p^n

29) 1/9

30) 2/9

31) 4/9

32) 1/6

33) 1/18

1.

Residue of $f(z)$ Singular at z_0

$$\boxed{\mathrm{Res}\left\{f(z)\right\}_{z=z_0} \equiv a_{-1} = \frac{1}{2\pi i}\oint_{\zeta=z_0+\varepsilon e^{i\phi}} f(\zeta)d\zeta}$$

Proof: $f(z) = .. + \dfrac{a_{-k}}{(z-z_0)^k} + .. + \dfrac{a_{-2}}{(z-z_0)^2} +$

$$+\frac{a_{-1}}{z-z_0}+$$

$$+a_0 + ... + a_n(z-z_0)^n + ..$$

$$\Rightarrow \oint_{\zeta=z_0+\rho e^{i\phi}} f(\zeta)d\zeta =$$

$$= ... + a_{-k}\underbrace{\oint_{\zeta=z_0+\varepsilon e^{i\phi}} \frac{1}{(\zeta-z_0)^k}d\zeta}_{\frac{1}{\varepsilon^k}\varepsilon i\underbrace{\oint \frac{e^{i\phi}}{e^{ki\phi}}d\phi}_{0}} + ... + a_{-2}\underbrace{\oint_{\zeta=z_0+\varepsilon e^{i\phi}} \frac{1}{(\zeta-z_0)^2}d\zeta}_{\frac{1}{\varepsilon^2}\varepsilon i\underbrace{\oint \frac{e^{i\phi}}{e^{2i\phi}}d\phi}_{0}}$$

$$+a_{-1}\underbrace{\oint_{\zeta=z_0+\varepsilon e^{i\phi}} \frac{1}{\zeta-z_0}d\zeta}_{\frac{1}{\varepsilon}\varepsilon i\oint \frac{e^{i\phi}}{e^{i\phi}}d\phi = i\underbrace{\oint d\phi}_{2\pi}}$$

$$+a_0 \underbrace{\oint_{\zeta = z_0 + \varepsilon e^{i\phi}} d\zeta}_{\varepsilon i \underbrace{\oint e^{i\phi} d\phi}_{0}} + ... + a_n \underbrace{\oint_{\zeta = z_0 + \varepsilon e^{i\phi}} (\zeta - z_0)^n d\zeta}_{\varepsilon^{n+1} i \underbrace{\oint e^{in\phi} e^{i\phi} d\phi}_{0}} + ...$$

$$\Rightarrow \mathrm{Res}\left\{f(z)\right\}_{z = z_0} \equiv a_{-1} = \frac{1}{2\pi i} \oint_{\zeta = z_0 + \varepsilon e^{i\phi}} f(\zeta) d\zeta . \square$$

2.

Residue at Pole of Order k

2.1 $f(z) = \dfrac{a_{-k}}{(z-z_0)^k} + \ldots + \dfrac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \ldots$

$$\Rightarrow \boxed{\operatorname{Res}_{-1,k}\left\{f(z)\right\}_{z=z_0} = a_{-1} = \left[\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left\{(z-z_0)^k f(z)\right\}\right]_{z=z_0}}$$

2.2 $f(z) = \dfrac{a_{-2}}{(z-z_0)^2} + \dfrac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$

$$\Rightarrow \boxed{\operatorname{Res}_{-1,2}\left\{f(z)\right\}_{z=z_0} = a_{-1} = \left[\frac{d}{dz}\left\{(z-z_0)^2 f(z)\right\}\right]_{z=z_0}}$$

2.3 $f(z) = \dfrac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$

$$\Rightarrow \boxed{\operatorname{Res}_{-1,1}\left\{f(z)\right\}_{z=z_0} = a_{-1} = \left[(z-z_0)f(z)\right]_{z=z_0}}$$

3.

Residue at Pole of Infinite order

3.1 $e^{-\frac{1}{z}} = 1 - \frac{1}{z} + \frac{1}{2!}\left(-\frac{1}{z}\right)^2 + \frac{1}{3!}\left(-\frac{1}{z}\right)^3 + \frac{1}{4!}\left(-\frac{1}{z}\right)^4 + ... \Rightarrow$

$$\Rightarrow \text{Res}_{-2}\{e^{-\frac{1}{z}}\}_{z=0} = \frac{1}{2}$$

$$\Rightarrow \text{Res}_{-1}\{e^{-\frac{1}{z}}\}_{z=0} = -1$$

$$\Rightarrow \text{Res}_{0}\{e^{-\frac{1}{z}}\}_{z=0} = 1$$

4.

$$\text{Res}\left\{\frac{\cot z \coth z}{z^3}\right\}_{z=0} = -\frac{7}{45}$$

Proof: divide the series,

$$\frac{1}{z^3}\frac{\cos z}{\sin z}\frac{\cosh z}{\sinh z} = \frac{1}{z^3}\frac{1-\frac{1}{2!}z^2+\frac{1}{4!}z^4+..}{z-\frac{1}{3!}z^3+\frac{1}{5!}z^5+...}\times\frac{1+\frac{1}{2!}z^2+\frac{1}{4!}z^4+..}{z+\frac{1}{3!}z^3+\frac{1}{5!}z^5+...}$$

$$= \frac{1}{z^3}\frac{1+\frac{1}{4!}z^4-\frac{1}{2!}z^2+..}{z+\frac{1}{5!}z^5-\frac{1}{3!}z^3+...}\times\frac{1+\frac{1}{4!}z^4+\frac{1}{2!}z^2+..}{z+\frac{1}{5!}z^5+\frac{1}{3!}z^3+...}$$

$$\approx \frac{1}{z^5}\frac{\left(1+\frac{1}{4!}z^4\right)^2-\left(\frac{1}{2!}z^2\right)^2}{\left(1+\frac{1}{5!}z^4\right)^2-\left(\frac{1}{3!}z^2\right)^2}$$

$$\approx \frac{1}{z^5}\frac{1-\left(\frac{1}{4}-\frac{1}{12}\right)z^4+\frac{1}{(4!)^2}z^8}{1-\left(\frac{1}{36}-\frac{1}{60}\right)z^4+\frac{1}{(5!)^2}z^8}$$

$$\approx \frac{1}{z^5}\frac{1-\frac{1}{6}z^4}{1-\frac{14}{90}z^4}$$

$$\approx \frac{1}{z^5}\left(1-\frac{1}{6}z^4\right)\left(1+\frac{1}{90}z^4\right)$$

$$= \left(1-\left[\frac{1}{6}-\frac{1}{90}\right]z^4-\frac{1}{540}z^8\right)$$

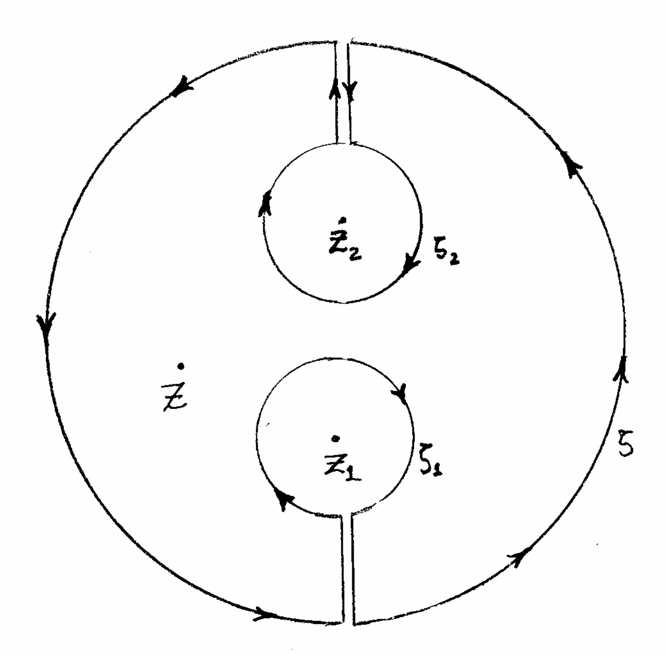
$$= \frac{1}{z^5}\left(1-\frac{14}{90}z^4-\frac{1}{540}z^8\right)$$

$$= \underbrace{1}_{a_{-5}}\frac{1}{z^5}-\underbrace{\frac{14}{90}}_{a_{-1}}\frac{1}{z}+\underbrace{\frac{1}{540}}_{a_3}z^3...$$

5.

Residue Theorem for a_{-1}

5.1



$$\frac{1}{2\pi i}\oint_C f(\zeta)d\zeta = \text{Res}_{-1}\{f(z)\}_{z=z_1} + \text{Res}_{-1}\{f(z)\}_{z=z_2}$$

Proof: $$\oint_{\zeta\in C} f(\zeta)d\zeta + \oint_{\zeta_1\in c_1} f(\zeta_1)d\zeta_1 + \oint_{\zeta_2\in c_2} f(\zeta_2)d\zeta_2 = 0$$

$$\frac{1}{2\pi i}\oint_{\zeta\in C} f(\zeta)d\zeta = \frac{1}{2\pi i}\oint_{\zeta_1=z_1+\rho e^{i\phi}} f(\zeta_1)d\zeta_1 + \frac{1}{2\pi i}\oint_{\zeta_2=z_2+\rho e^{i\phi}} f(\zeta_2)d\zeta_2,$$

For $\zeta_1 \in c_1,\ \ f(\zeta_1) = \dfrac{a_{-k_1,1}}{(\zeta_1 - z_1)^{k_1}} + \ldots + \dfrac{a_{-1,1}}{\zeta_1 - z_1} + a_{0,1} + a_{1,1}(\zeta_1 - z_1) + \ldots$

$$\Rightarrow \frac{1}{2\pi i} \oint_{\zeta_1 = z_1 + \rho e^{i\phi}} f(\zeta_1) d\zeta = a_{-1,1} = \text{Res}_{-1}\left\{ f(z) \right\}_{z=z_1}$$

For $\zeta_2 \in c_2$,

$$f(\zeta_2) = \frac{a_{-k_2,2}}{(\zeta_2 - z_2)^{k_2}} + ... + \frac{a_{-1,2}}{\zeta_2 - z_2} + a_{0,1} + a_{1,2}(\zeta_2 - z_2) + ...$$

$$\Rightarrow \frac{1}{2\pi i} \oint_{\zeta_2 = z_2 + \rho e^{i\phi}} f(\zeta_2) d\zeta_2 = a_{-1,2} = \text{Res}_{-1}\left\{ f(z) \right\}_{z=z_2}$$

$$\frac{1}{2\pi i} \oint_C f(\zeta) d\zeta = \text{Res}_{-1}\left\{ f(z) \right\}_{z=z_1} + \text{Res}_{-1}\left\{ f(z) \right\}_{z=z_2} . \square$$

5.2 $f(z)$ has poles at $z_1, z_2, ..., z_N \Rightarrow$

$$\frac{1}{2\pi i} \oint_C f(\zeta) d\zeta = \text{Res}_{-1}\left\{ f(z) \right\}_{z=z_1} + ... + \text{Res}_{-1}\left\{ f(z) \right\}_{z=z_N}$$

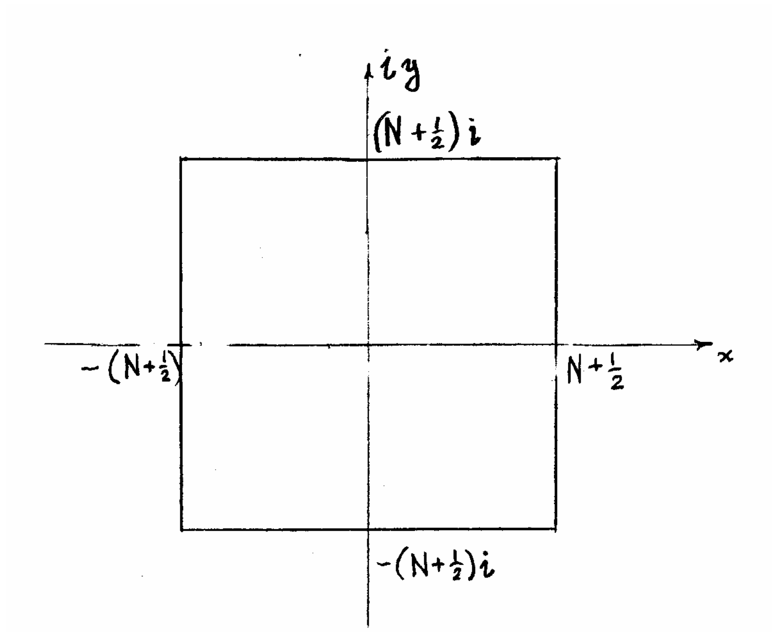
6_{cot}.

$$\boxed{\pi\cot(\pi z)f(z)}$$

$$\boxed{\text{Res}\{\pi\cot(\pi z)f(z)\}_{z=n} = f(n)}$$

$$\boxed{\begin{aligned}&...f(-3)+f(-2)+f(-1)+f(0)+f(1)+f(2)+f(3)+...\\ &\qquad + \sum \text{Res}_{-1}\left\{\pi\cot(\pi\sigma)f(\sigma)\right\}_{\sigma=\text{pole of } f(\sigma)} = 0\end{aligned}}$$

6_{cot}.1 $\left|\cot\pi z\right| \le A$ **on** $\square_{N+\frac{1}{2}}$ **for any** N



$$\underline{y = -N - \frac{1}{2},\ \text{and}\ -N - \frac{1}{2} \le x \le N + \frac{1}{2}} \Rightarrow$$

$$\left|\cot\pi z\right| = \frac{\left|e^{i\pi z} + e^{-i\pi z}\right|}{\left|e^{i\pi z} - e^{-i\pi z}\right|}$$

$$\leq \frac{\left|e^{i\pi z}\right| + \left|e^{-i\pi z}\right|}{\left|\left|e^{i\pi z}\right| - \left|e^{-i\pi z}\right|\right|}$$

$$= \frac{\left|e^{i\pi(x+i[-N-\frac{1}{2}])}\right| + \left|e^{-i\pi(x+i[-N-\frac{1}{2}])}\right|}{\left|\left|e^{i\pi(x+i[-N-\frac{1}{2}])}\right| - \left|e^{-i\pi(x+i[-N-\frac{1}{2}])}\right|\right|}$$

$$= \frac{\left|e^{i\pi x+\pi N+\pi/2}\right| + \left|e^{-i\pi x-\pi N-\pi/2}\right|}{\left|\left|e^{i\pi x+\pi N+\pi/2}\right| - \left|e^{-i\pi x-\pi N-\pi/2}\right|\right|}$$

$$= \frac{\left(e^{\pi N+\pi/2} + e^{-\pi N-\pi/2}\right)}{\left(e^{\pi N+\pi/2} - e^{-\pi N-\pi/2}\right)} \frac{e^{-\pi N+\pi/2}}{e^{-\pi N+\pi/2}}$$

$$= \frac{e^{\pi} + e^{-2\pi N}}{e^{\pi} - e^{-2\pi N}}$$

$$= \frac{e^{\pi} + \frac{1}{e^{2\pi N}}}{e^{\pi} - \frac{1}{e^{2\pi N}}}$$

$$< \frac{e^{\pi} + 1}{e^{\pi} - 1}$$

$$= 1 + \frac{2}{e^{\pi} - 1} < 1.1. \square$$

$$\underline{\left. y = N + \frac{1}{2}, \text{ and } -N - \frac{1}{2} \leq x \leq N + \frac{1}{2} \right|} \Rightarrow$$

$$\left|\cot \pi z\right| = \left|\frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}}\right|$$

$$\leq \frac{\left|e^{i\pi z}\right| + \left|e^{-i\pi z}\right|}{\left|\left|e^{i\pi z}\right| - \left|e^{-i\pi z}\right|\right|}$$

$$= \frac{\left|e^{i\pi(x+i[N+\frac{1}{2}])}\right| + \left|e^{-i\pi(x+i[N+\frac{1}{2}])}\right|}{\left|\left|e^{i\pi(x+i[N+\frac{1}{2}])}\right| - \left|e^{-i\pi(x+i[N+\frac{1}{2}])}\right|\right|}$$

$$= \frac{\left|e^{i\pi x-\pi N-\pi/2}\right| + \left|e^{-i\pi x+\pi N+\pi/2}\right|}{\left|\left|e^{i\pi x-\pi N-\pi/2}\right| - \left|e^{-i\pi x+\pi N+\pi/2}\right|\right|}$$

$$= \frac{(e^{-\pi N-\pi/2} + e^{\pi N+\pi/2})}{(e^{\pi N+\pi/2} - e^{-\pi N-\pi/2})} \frac{e^{-\pi N+\pi/2}}{e^{-\pi N+\pi/2}}$$

$$= \frac{e^{\pi} + e^{-2\pi N}}{e^{\pi} - e^{-2\pi N}}$$

$$= \frac{e^{\pi} + \frac{1}{e^{2\pi N}}}{e^{\pi} - \frac{1}{e^{2\pi N}}}$$

$$< \frac{e^{\pi} + 1}{e^{\pi} - 1}$$

$$= 1 + \frac{2}{e^{\pi} - 1} < 1.1.\square$$

$\underline{x = N + \frac{1}{2},\ \text{and}\ -N - \frac{1}{2} \leq y \leq N + \frac{1}{2}\Big|} \Rightarrow$

For $y = 0$, $\left|\cot \pi z\right| = \left|\cot \pi(N + \tfrac{1}{2})\right|$

$$= \left|\cot(\pi N + \tfrac{\pi}{2})\right|$$

$$= \left|\cot \frac{\pi}{2}\right|$$

$$= \tan 0 = 0.$$

For $y \neq 0$, $\left|\cot \pi z\right| = \left|\cot\left(\pi N + \tfrac{\pi}{2} + i\pi y\right)\right|$

$$= \left|\cot\left(\tfrac{\pi}{2} + i\pi y\right)\right|$$

$$= \left|\tan\left(i\pi y\right)\right|$$

$$= \left|\frac{\sin i\pi y}{\cos i\pi y}\right|$$

$$= \left|\frac{\dfrac{e^{i\pi i y} - e^{-i\pi i y}}{2i}}{\dfrac{e^{i\pi i y} + e^{-i\pi i y}}{2}}\right|$$

$$= \left|\frac{e^{-\pi y} - e^{\pi y}}{e^{-\pi y} + e^{\pi y}}\right|$$

$$= \left|\frac{1 - e^{2\pi y}}{1 + e^{2\pi y}}\right| = \left|\frac{e^{2\pi y} - 1}{e^{2\pi y} + 1}\right|$$

For $y > 0$, $\quad \dfrac{e^{2\pi y} - 1}{e^{2\pi y} + 1} < 1.\square$

For $y < 0$, $\quad \dfrac{1 - e^{2\pi y}}{1 + e^{2\pi y}} = \dfrac{1 - \dfrac{1}{e^{2\pi(-y)}}}{1 + \dfrac{1}{e^{2\pi(-y)}}} < 1.\square$

$\underline{x = -N - \frac{1}{2},\ \text{and}\ -N - \frac{1}{2} \le y \le N + \frac{1}{2}}\Big| \Rightarrow$

For $y = 0$, $\quad \left|\cot \pi z\right| = \left|\cot \pi(-N - \tfrac{1}{2})\right|$

$$= \left|\cot(-\pi N - \tfrac{\pi}{2})\right|$$

$$= \left|\cot(-\tfrac{\pi}{2})\right|$$

$$= \tan 0 = 0.$$

For $y \neq 0$, $\quad \left|\cot \pi z\right| = \left|\cot\left(-\pi N - \tfrac{\pi}{2} + i\pi y\right)\right|$

$$= \left|\cot\left(-\tfrac{\pi}{2} + i\pi y\right)\right|$$

$$= \left|\tan\left(i\pi y\right)\right|$$

$$= \left|\frac{\sin i\pi y}{\cos i\pi y}\right|$$

$$= \left|\frac{\dfrac{e^{i\pi i y} - e^{-i\pi i y}}{2i}}{\dfrac{e^{i\pi i y} + e^{-i\pi i y}}{2}}\right|$$

$$= \left|\frac{e^{-\pi y} - e^{\pi y}}{e^{-\pi y} + e^{\pi y}}\right|$$

$$= \left|\frac{1 - e^{2\pi y}}{1 + e^{2\pi y}}\right| = \left|\frac{e^{2\pi y} - 1}{e^{2\pi y} + 1}\right|$$

For $y > 0$, $\quad \dfrac{e^{2\pi y} - 1}{e^{2\pi y} + 1} < 1.\square$

For $y < 0$, $\quad \dfrac{1 - e^{2\pi y}}{1 + e^{2\pi y}} = \dfrac{1 - \dfrac{1}{e^{2\pi(-y)}}}{1 + \dfrac{1}{e^{2\pi(-y)}}} < 1.\square$

6_{cot}.2 $\quad \cot \pi z = \dfrac{\cos \pi z}{\sin \pi z}$ **has poles at** $z = n = ... - 2, -1, 0, 1, 2, ...$

6_{cot}.3

$$\boxed{\operatorname{Res}\left\{\pi \cot(\pi z)\right\}_{z=n} = 1}$$

Proof: $\operatorname{Res}\left\{\pi \cot(\pi z)\right\}_{z=n} = \left[(z-n)\pi \dfrac{\cos \pi z}{\sin \pi z}\right]_{z=n}$

$$= \left[\frac{D_z(\pi z - \pi n)}{D_z \sin(\pi z)} \cos(\pi z) \right]_{z=n}$$

$$= \left[\frac{\pi}{\pi \cos(\pi z)} \cos(\pi z) \right]_{z=n} = 1. \square$$

6_{cot}.4

$$\boxed{\operatorname{Res}\left\{ \pi \cot(\pi z) f(z) \right\}_{z=n} = f(n)}$$

6_{cot}.5 $\left| f(z) \right|_{\square_{N+\frac{1}{2}}} \leq \dfrac{M}{z^k} \Rightarrow$

$$\boxed{\begin{aligned} &...f(-3) + f(-2) + f(-1) + f(0) + f(1) + f(2) + f(3) + ... \\ &\qquad + \sum \operatorname{Res}_{-1} \left\{ \pi \cot(\pi\sigma) f(\sigma) \right\}_{\sigma = \text{pole of } f(\sigma)} = 0 \end{aligned}}$$

<u>Proof</u>:

$$\oint_{\square_{N+\frac{1}{2}}} \pi \cot(\pi\zeta) f(\zeta) d\zeta = \sum \operatorname{Res}_{-1} \left\{ \pi \cot(\pi z) f(z) \right\}_{z = \text{pole of } \cot \pi z \text{ in } \square_{N+\frac{1}{2}}}$$

$$+ \sum \operatorname{Res}_{-1} \left\{ \pi \cot(\pi\sigma) f(\sigma) \right\}_{\sigma = \text{pole of } f(\sigma) \text{ in } \square_{N+\frac{1}{2}}}$$

$$\left| \oint_{\square_{N+\frac{1}{2}}} \pi \cot(\pi\zeta) f(\zeta) d\zeta \right| \leq \pi \underbrace{\left| \cot \pi\zeta \right|}_{\leq A} \frac{M}{N^k} \underbrace{(\text{length} \square_{N+\frac{1}{2}})}_{8\left(N+\frac{1}{2}\right)} \xrightarrow[N \to \infty]{} 0$$

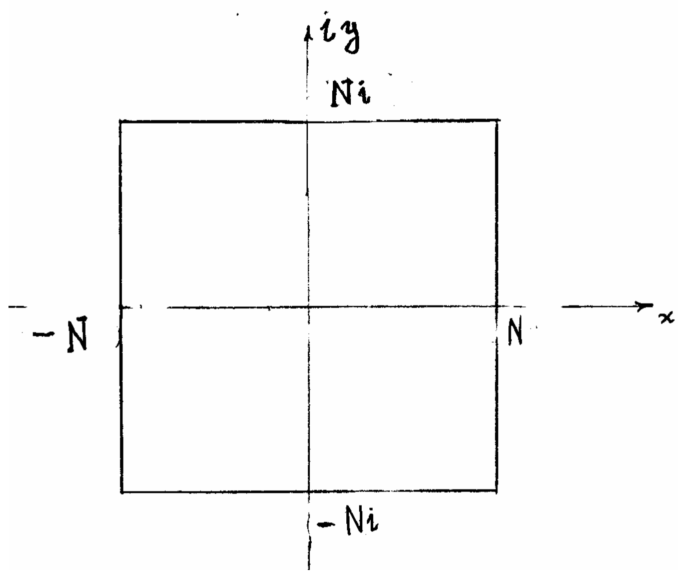
$$\operatorname{Res}\left\{ \pi \cot(\pi z) f(z) \right\}_{z=n} = f(n). \square$$

$\mathbf{6_{tan}}$.

$$\boxed{\pi \tan(\pi z) f(z)}$$

$$\boxed{\begin{aligned} &.. + f(-\tfrac{5}{2}) + f(-\tfrac{3}{2}) + f(-\tfrac{1}{2}) + f(\tfrac{1}{2}) + f(\tfrac{3}{2}) + f(\tfrac{5}{2}) + .. = \\ &\qquad = \sum \underset{-1}{\text{Res}} \left\{ \pi \tan(\pi\sigma) f(\sigma) \right\}_{\sigma = pole\ \ of\ \ f(\sigma)} \end{aligned}}$$

$\mathbf{6_{tan}.1}$ $\left|\tan \pi z\right| \leq A$ on $\square_N$ for any N



$\underline{y = -N, \text{ and } -N \leq x \leq N} \Rightarrow$

$$\left|\tan \pi z\right| = \left|\frac{e^{i\pi z} - e^{-i\pi z}}{e^{i\pi z} + e^{-i\pi z}}\right|$$

$$= \left|\frac{1 - e^{-2i\pi z}}{1 + e^{-2i\pi z}}\right|$$

$$= \left|\frac{1 - e^{-2i\pi x - 2\pi N}}{1 + e^{-2i\pi x - 2\pi N}}\right|$$

$$< \frac{1+e^{-2\pi N}}{1-e^{-2\pi N}}$$

$$< \frac{1+e^{-\pi}}{1-e^{-\pi}}.\square$$

$\underline{y = N, \text{ and } -N \le x \le N}\Big| \Rightarrow$

$$\left|\tan \pi z\right| = \left|\frac{e^{i\pi z} - e^{-i\pi z}}{e^{i\pi z} + e^{-i\pi z}}\right|$$

$$= \left|\frac{e^{2i\pi z} - 1}{e^{2i\pi z} + 1}\right|$$

$$= \left|\frac{e^{-2i\pi x + 2\pi N} - 1}{e^{-2i\pi x + 2\pi N} + 1}\right|$$

$$< \frac{e^{2\pi N} + 1}{e^{2\pi N} - 1}\frac{e^{-2\pi N}}{e^{-2\pi N}}$$

$$< \frac{1+\dfrac{1}{e^{2\pi N}}}{1-\dfrac{1}{e^{2\pi N}}}$$

$$< \frac{1+\dfrac{1}{e^{\pi}}}{1-\dfrac{1}{e^{\pi}}}$$

$$< \frac{e^{\pi}+1}{e^{\pi}-1}$$

$$< 1 + 2\frac{1}{e^{\pi}-1}.\square$$

$\underline{x = N, \text{ and } -N \le y \le N}\Big| \Rightarrow$

For $y = 0$, $\left|\tan \pi z\right| = \left|\tan \pi N\right|$

$$= \tan 0 = 0.$$

For $y \neq 0$, $\left|\tan \pi z\right| = \left|\tan\left(\pi N + i\pi y\right)\right|$

$$= \left|\tan\left(i\pi y\right)\right|$$

$$= \left|\frac{\sin i\pi y}{\cos i\pi y}\right|$$

$$= \left|\frac{\dfrac{e^{i\pi i y} - e^{-i\pi i y}}{2i}}{\dfrac{e^{i\pi i y} + e^{-i\pi i y}}{2}}\right|$$

$$= \left|\frac{e^{-\pi y} - e^{\pi y}}{e^{-\pi y} + e^{\pi y}}\right|$$

$$= \left|\frac{1 - e^{2\pi y}}{1 + e^{2\pi y}}\right| = \left|\frac{e^{2\pi y} - 1}{e^{2\pi y} + 1}\right|$$

For $y > 0$, $\dfrac{e^{2\pi y} - 1}{e^{2\pi y} + 1} < 1.\square$

For $y < 0$, $\dfrac{1 - e^{2\pi y}}{1 + e^{2\pi y}} = \dfrac{1 - \dfrac{1}{e^{2\pi(-y)}}}{1 + \dfrac{1}{e^{2\pi(-y)}}} < 1.\square$

$\underline{x = -N, \text{ and } -N \leq y \leq N}\Rightarrow$

For $y = 0$, $\left|\tan \pi z\right| = \left|\tan \pi(-N)\right| = \tan 0 = 0$

For $y \neq 0$, $\left|\tan \pi z\right| = \left|\tan\left(-\pi N + i\pi y\right)\right|$

$$= \left|\tan\left(i\pi y\right)\right|$$

$$= \left| \frac{\sin i\pi y}{\cos i\pi y} \right|$$

$$= \left| \frac{\dfrac{e^{i\pi iy} - e^{-i\pi iy}}{2i}}{\dfrac{e^{i\pi iy} + e^{-i\pi iy}}{2}} \right|$$

$$= \left| \frac{e^{-\pi y} - e^{\pi y}}{e^{-\pi y} + e^{\pi y}} \right|$$

$$= \left| \frac{1 - e^{2\pi y}}{1 + e^{2\pi y}} \right| = \left| \frac{e^{2\pi y} - 1}{e^{2\pi y} + 1} \right|$$

For $y > 0$, $\quad \dfrac{e^{2\pi y} - 1}{e^{2\pi y} + 1} < 1.\square$

For $y < 0$, $\quad \dfrac{1 - e^{2\pi y}}{1 + e^{2\pi y}} = \dfrac{1 - \dfrac{1}{e^{2\pi(-y)}}}{1 + \dfrac{1}{e^{2\pi(-y)}}} < 1.\square$

6_{tan}.2 $\quad \pi \tan(\pi\zeta)$ **has poles at** $z = n + \frac{1}{2}$

6_{tan}.3

$$\boxed{\operatorname{Res}\left\{\pi \tan(\pi z)\right\}_{z=n+\frac{1}{2}} = -1}$$

Proof: $\operatorname{Res}\left\{\pi \tan(\pi z)\right\}_{z=n+\frac{1}{2}} = \left[(z - [n + \tfrac{1}{2}])\dfrac{\pi \sin(\pi z)}{\cos(\pi z)}\right]_{z=n+\frac{1}{2}}$

$$= \pi \left[\frac{D_z(z-n)}{D_z \cos(\pi z)} \sin(\pi z)\right]_{z=n+\frac{1}{2}}$$

$$= \left[\frac{1}{-\sin(\pi z)} \sin(\pi z)\right]_{z=n+\frac{1}{2}} = -1$$

$6_{\mathbf{tan}}$.4 $$\boxed{\mathrm{Res}\left\{\pi\tan(\pi z)f(z)\right\}_{z=n+\frac{1}{2}}=-f(n+\tfrac{1}{2})}$$

$6_{\mathbf{tan}}$.5 $\left|f(z)\right|_{\square_N}\le\dfrac{M}{z^k}\Rightarrow$

$$\boxed{\begin{aligned}&..+f(-\tfrac{5}{2})+f(-\tfrac{3}{2})+f(-\tfrac{1}{2})+f(\tfrac{1}{2})+f(\tfrac{3}{2})+f(\tfrac{5}{2})+..=\\&=\sum\mathrm{Res}_{-1}\left\{\pi\tan(\pi\sigma)f(\sigma)\right\}_{\sigma=\text{pole of }f(\sigma)}\end{aligned}}$$

Proof:

$$\oint_{\square_N}\pi\tan(\pi\zeta)f(\zeta)d\zeta=\sum\mathrm{Res}_{-1}\left\{\pi\tan(\pi\zeta)f(z)\right\}_{z=\text{pole of }\pi\tan(\pi\zeta)\text{ in }\square_N}$$

$$+\sum\mathrm{Res}_{-1}\left\{\pi\tan(\pi\zeta)f(\sigma)\right\}_{\sigma=\text{pole of }f(\sigma)\text{ in }\square_N}$$

$$\left|\oint_{\square_N}\pi\tan(\pi\zeta)f(\zeta)d\zeta\right|\le\underbrace{\left|\pi\tan(\pi\zeta)\right|}_{\le A}\frac{M}{N^k}\underbrace{(\text{length}\square_N)}_{8N}\xrightarrow[N\to\infty]{}0$$

$\pi\tan(\pi\zeta)$ **has poles at** $z=n+\frac{1}{2}$

$$\mathrm{Res}_{-1}\left\{\pi\tan(\pi z)f(z)\right\}_{z=n+\frac{1}{2}}=-f(n+\tfrac{1}{2})$$

$$\Rightarrow\ ..+f(-\tfrac{5}{2})+f(-\tfrac{3}{2})+f(-\tfrac{1}{2})+f(\tfrac{1}{2})+f(\tfrac{3}{2})+f(\tfrac{5}{2})+...=$$

$$=\sum\mathrm{Res}_{-1}\left\{\pi\tan(\pi\sigma)f(\sigma)\right\}_{\sigma=pole\ of\ f(\sigma)}$$

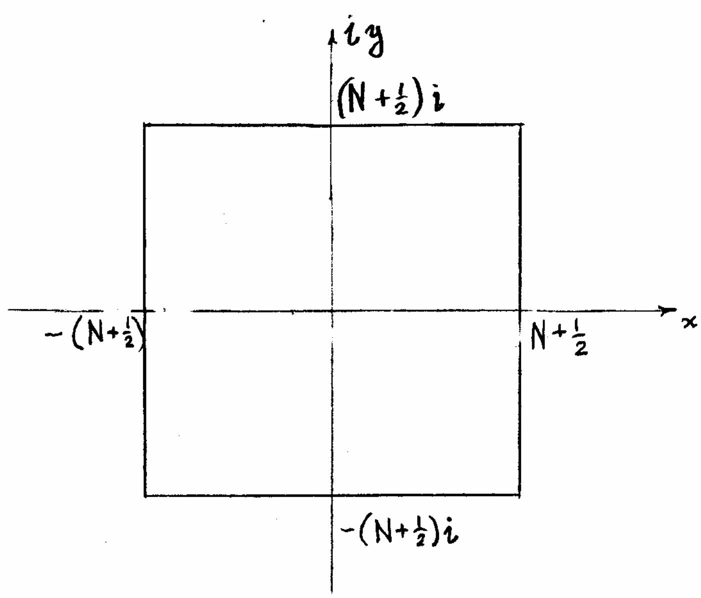
$\mathbf{6_{sin}}$.

$$\boxed{\frac{\pi}{\sin \pi z} f(z)}$$

$$\boxed{\operatorname{Res}\left\{\pi \frac{1}{\sin \pi z} f(z)\right\}_{z=n} = (-1)^n f(n)}$$

$$\boxed{\begin{aligned} &... - f(-3) + f(-2) - f(-1) + f(0) - f(1) + f(2) - f(3) + ... \\ &\qquad + \sum \operatorname{Res}_{-1}\left\{\frac{\pi}{\sin(\pi\sigma)} f(\sigma)\right\}_{\sigma=\text{pole of } f(\sigma)} = 0 \end{aligned}}$$

$\mathbf{6_{sin}.1}$ $\left|\dfrac{1}{\sin \pi z}\right| \le A$ on $\square_{N+\frac{1}{2}}$ for any N



$$\underline{y = -N - \frac{1}{2}, \text{ and } -N - \frac{1}{2} \le x \le N + \frac{1}{2}} \Rightarrow$$

$$\left|\frac{1}{\sin \pi z}\right| = \frac{2}{\left|e^{i\pi z} - e^{-i\pi z}\right|}$$

$$= \frac{2}{\left|e^{i\pi(x+iy)} - e^{-i\pi(x+iy)}\right|}$$

$$\leq \frac{2}{\left|\left|e^{i\pi(x+iy)}\right| - \left|e^{-i\pi(x+iy)}\right|\right|}$$

$$= \frac{2}{\left|e^{-\pi y} - e^{\pi y}\right|}$$

$$= \frac{2}{e^{\pi N + \frac{\pi}{2}} - e^{-\pi N - \frac{\pi}{2}}} \frac{e^{-\pi N + \frac{\pi}{2}}}{e^{-\pi N + \frac{\pi}{2}}}$$

$$= \frac{2e^{-\pi N + \frac{\pi}{2}}}{e^{\pi} - e^{-2\pi N}}$$

$$= \frac{\frac{2e^{\frac{\pi}{2}}}{e^{\pi N}}}{e^{\pi} - \frac{1}{e^{2\pi N}}}$$

$$< \frac{2e^{\frac{\pi}{2}}}{e^{\pi} - 1} \equiv A. \square$$

$$\underline{y = N + \frac{1}{2}, \text{ and } -N - \frac{1}{2} \leq x \leq N + \frac{1}{2}\Big|} \Rightarrow$$

$$\left|\frac{1}{\sin \pi z}\right| = \frac{2}{\left|e^{i\pi z} - e^{-i\pi z}\right|}$$

$$= \frac{2}{\left|e^{i\pi(x+iy)} - e^{-i\pi(x+iy)}\right|}$$

$$\leq \frac{2}{\left|\left|e^{i\pi(x+iy)}\right| - \left|e^{-i\pi(x+iy)}\right|\right|}$$

$$= \frac{2}{\left|e^{-\pi y} - e^{\pi y}\right|}$$

$$= \frac{2}{e^{\pi N + \frac{\pi}{2}} - e^{-\pi N - \frac{\pi}{2}}} \frac{e^{-\pi N + \frac{\pi}{2}}}{e^{-\pi N + \frac{\pi}{2}}}$$

$$= \frac{2e^{-\pi N + \frac{\pi}{2}}}{e^{\pi} - e^{-2\pi N}}$$

$$= \frac{\dfrac{2e^{\frac{\pi}{2}}}{e^{\pi N}}}{e^{\pi} - \dfrac{1}{e^{2\pi N}}}$$

$$< \frac{2e^{\frac{\pi}{2}}}{e^{\pi} - 1} \equiv A. \square$$

$$\underline{x = N + \frac{1}{2}, \text{ and } -N - \frac{1}{2} \leq y \leq N + \frac{1}{2}} \Rightarrow$$

$$\left|\frac{1}{\sin \pi z}\right| = \frac{1}{\left|\sin(\pi N + \frac{\pi}{2} + i\pi y)\right|}$$

$$= \left|\csc(\pi N + \tfrac{\pi}{2} + i\pi y)\right|$$

$$= \left|\csc(\tfrac{\pi}{2} + i\pi y)\right|$$

$$= \frac{1}{\left|\sin(\frac{\pi}{2} + i\pi y)\right|}$$

$$= \frac{1}{\left|\cos(i\pi y)\right|}$$

$$= \frac{1}{\left|\dfrac{e^{i\pi(iy)} + e^{-i\pi(iy)}}{2}\right|}$$

$$= \frac{2}{e^{-\pi y} + e^{\pi y}}$$

For $y = 0$, $\quad \dfrac{2}{e^{-\pi y} + e^{\pi y}} = 1$

For $y > 0$, $\quad \dfrac{2}{e^{-\pi y} + e^{\pi y}} \dfrac{e^{\pi y}}{e^{\pi y}} = 2\dfrac{e^{\pi y}}{e^{2\pi y} + 1} < 2$

For $y < 0$, $\quad \dfrac{2}{e^{-\pi y} + e^{\pi y}} \dfrac{e^{-\pi y}}{e^{-\pi y}} = 2\dfrac{e^{\pi(-y)}}{e^{2\pi(-y)} + 1} < 2\,.\square$

$\underline{x = -N - \frac{1}{2}, \text{ and } -N - \frac{1}{2} \le y \le N + \frac{1}{2}}\Big| \Rightarrow$

$$\left|\frac{1}{\sin \pi z}\right| = \frac{1}{\left|\sin(-\pi N - \frac{\pi}{2} + i\pi y)\right|}$$

$$= \left|\csc(-\pi N - \tfrac{\pi}{2} + i\pi y)\right|$$

$$= \left|\csc(-\tfrac{\pi}{2} + i\pi y)\right|$$

$$= \frac{1}{\left|\sin(\frac{\pi}{2} + i\pi y)\right|}$$

$$= \frac{1}{\left|\cos(i\pi y)\right|}$$

$$= \frac{1}{\left|\dfrac{e^{i\pi(iy)} + e^{-i\pi(iy)}}{2}\right|}$$

$$= \frac{2}{e^{-\pi y} + e^{\pi y}}$$

For $y = 0$**,** $\quad \frac{2}{e^{-\pi y} + e^{\pi y}} = 1$

For $y > 0$**,** $\quad \frac{2}{e^{-\pi y} + e^{\pi y}} \frac{e^{\pi y}}{e^{\pi y}} = 2\frac{e^{\pi y}}{e^{2\pi y} + 1} < 2$

For $y < 0$**,** $\quad \frac{2}{e^{-\pi y} + e^{\pi y}} \frac{e^{-\pi y}}{e^{-\pi y}} = 2\frac{e^{\pi(-y)}}{e^{2\pi(-y)} + 1} < 2\,.\square$

$6_{\sin}$.2 $\quad \frac{1}{\sin \pi z}$ **has poles of order 1 at** $z = n = ... - 2, -1, 0, 1, 2, ...$

$6_{\sin}$.3

$$\boxed{\operatorname{Res}\left\{\pi \frac{1}{\sin \pi z}\right\}_{z=n} = (-1)^n}$$

Proof: $\operatorname{Res}\left\{\pi \frac{1}{\sin \pi z}\right\}_{z=n} = \left[(z-n)\pi \frac{1}{\sin \pi z}\right]_{z=n}$

$$= \left[\frac{D_z(\pi z - \pi n)}{D_z \sin(\pi z)}\right]_{z=n}$$

$$= \left[\frac{\pi}{\pi \cos(\pi z)}\right]_{z=n} = (-1)^n.\square$$

$6_{\sin}$.4

$$\boxed{\operatorname{Res}\left\{\pi \frac{1}{\sin \pi z} f(z)\right\}_{z=n} = (-1)^n f(n)}$$

$6_{\sin}$.5 $\left|f(z)\right|_{\square_{N+\frac{1}{2}}} \le \dfrac{M}{z^k} \Rightarrow$

$$\boxed{\begin{aligned}&...-f(-3)+f(-2)-f(-1)+f(0)-f(1)+f(2)-f(3)+...\\&\qquad\qquad+\sum \text{Res}_{-1}\left\{\frac{\pi}{\sin(\pi\sigma)}f(\sigma)\right\}_{\sigma=\text{pole of } f(\sigma)}=0\end{aligned}}$$

Proof:

$$\oint_{\square_{N+\frac{1}{2}}} \frac{\pi}{\sin(\pi\zeta)} f(\zeta)d\zeta = \sum \text{Res}_{-1}\left\{\frac{\pi}{\sin(\pi z)}f(z)\right\}_{z=\text{pole of } \frac{\pi}{\sin(\pi\zeta)} \text{ in } \square_{N+\frac{1}{2}}}$$

$$+\sum \text{Res}_{-1}\left\{\frac{\pi}{\sin(\pi\sigma)}f(\sigma)\right\}_{\sigma=\text{pole of } f(\sigma) \text{ in } \square_{N+\frac{1}{2}}}$$

$$\left|\oint_{\square_{N+\frac{1}{2}}} \frac{\pi}{\sin(\pi\zeta)} f(\zeta)d\zeta\right| \le \pi \underbrace{\left|\frac{1}{\sin(\pi\zeta)}\right|}_{\le A} \frac{M}{N^k} \underbrace{(\text{length}\square_{N+\frac{1}{2}})}_{8\left(N+\frac{1}{2}\right)} \xrightarrow[N\to\infty]{} 0$$

$\dfrac{1}{\sin \pi z}$ **has poles at** $z = n$

$$\text{Res}_{-1}\left\{\frac{\pi}{\sin(\pi z)}f(z)\right\}_{z=\text{pole of } \frac{\pi}{\sin(\pi z)} \text{ in } \square_{N+\frac{1}{2}}} = (-1)^n f(n).\square$$

$\Rightarrow$

$$...-f(-3)+f(-2)-f(-1)+f(0)-f(1)+f(2)-f(3)+...$$

$$+\sum \text{Res}_{-1}\left\{\frac{\pi}{\sin(\pi\sigma)}f(\sigma)\right\}_{\sigma=pole\ \ of\ \ f(\sigma)} = 0$$

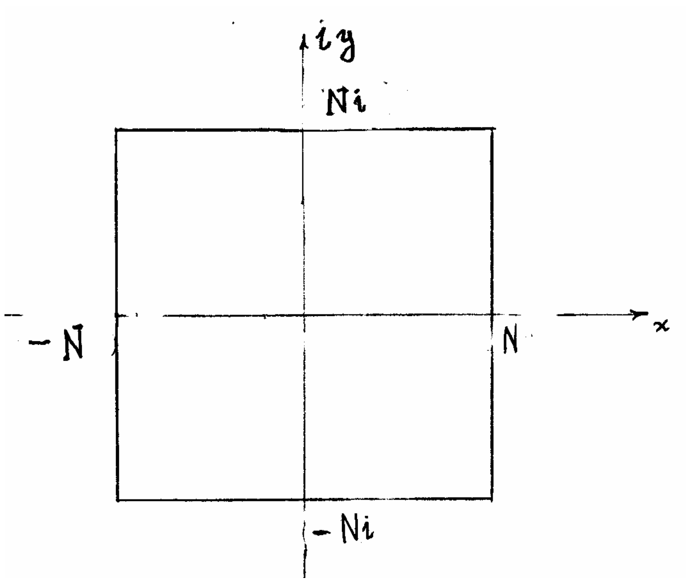
$\mathbf{6_{cos}.}$

$$\boxed{\frac{\pi}{\cos \pi z} f(z)}$$

$$\boxed{\text{Res}\left\{\frac{\pi}{\cos(\pi z)} f(z)\right\}_{z=n+\frac{1}{2}} = -(-1)^n f(n+\tfrac{1}{2})}$$

$$\boxed{\begin{aligned}&... - f(-\tfrac{5}{2}) + f(-\tfrac{3}{2}) - f(-\tfrac{1}{2}) + f(\tfrac{1}{2}) - f(\tfrac{3}{2}) + f(\tfrac{5}{2}) - ... = \\ &= \sum \text{Res}_{-1}\left\{\frac{\pi}{\cos(\pi\sigma)} f(\sigma)\right\}_{\sigma=\text{pole of } f(\sigma)}\end{aligned}}$$

$6_{cos}.1$ $\left|\frac{1}{\cos \pi z}\right| \le A$ **on** $\square_N$ **for any** N



$\underline{y = -N, \text{ and } -N \le x \le N|} \Rightarrow$

$$\frac{1}{|\cos \pi z|} = \frac{2}{|e^{i\pi z} + e^{-i\pi z}|}$$

$$= \frac{2}{\left|e^{i\pi(x+iy)} + e^{-i\pi(x+iy)}\right|}$$

$$\leq \frac{2}{\left|\left|e^{i\pi(x+iy)}\right| - \left|e^{-i\pi(x+iy)}\right|\right|}$$

$$= \frac{2}{\left|e^{-\pi y} - e^{\pi y}\right|}$$

$$= \frac{2}{\left|e^{-\pi(-N)} - e^{\pi(-N)}\right|}$$

$$= \frac{2}{e^{\pi N} - e^{-\pi N}}$$

$$\leq 2\frac{1}{e^{\pi} - 1}.\square$$

$\underline{y = N, \text{ and } -N \leq x \leq N}\Big| \Rightarrow$

$$\left|\frac{1}{\cos \pi z}\right| = \frac{2}{\left|e^{-i\pi z} + e^{i\pi z}\right|}$$

$$= \frac{2}{\left|e^{i\pi(x+iy)} + e^{-i\pi(x+iy)}\right|}$$

$$\leq \frac{2}{\left|\left|e^{i\pi(x+iy)}\right| - \left|e^{-i\pi(x+iy)}\right|\right|}$$

$$= \frac{2}{\left|e^{-\pi y} - e^{\pi y}\right|}$$

$$= \frac{2}{\left|e^{-\pi N} - e^{\pi N}\right|}$$

$$\leq 2\frac{1}{e^{\pi}-1}.\square$$

$\underline{x = N, \text{ and } -N \leq y \leq N|} \Rightarrow$

$$\left|\frac{1}{\cos \pi z}\right| = \frac{1}{\left|\cos(\pi N + i\pi y)\right|}$$

$$= \left|\sec(\pi N + i\pi y)\right|$$

$$= \left|\sec(i\pi y)\right|$$

$$= \frac{1}{\left|\cos(i\pi y)\right|}$$

$$= \frac{2}{\left|e^{i(i\pi y)} + e^{-i(i\pi y)}\right|}$$

$$= \frac{2}{\left|e^{-\pi y} + e^{\pi y}\right|}$$

$$= \frac{2}{e^{-\pi y} + e^{\pi y}}$$

For $y = 0$, $\quad \dfrac{2}{e^{-\pi y} + e^{\pi y}} = 1$

For $y > 0$, $\quad \dfrac{2}{e^{-\pi y} + e^{\pi y}}\dfrac{e^{\pi y}}{e^{\pi y}} = 2\dfrac{e^{\pi y}}{e^{2\pi y}+1} < 2$

For $y < 0$, $\quad \dfrac{2}{e^{-\pi y} + e^{\pi y}}\dfrac{e^{-\pi y}}{e^{-\pi y}} = 2\dfrac{e^{\pi(-y)}}{e^{2\pi(-y)}+1} < 2.\square$

$\underline{x = -N, \text{ and } -N \leq y \leq N|} \Rightarrow$

$$\left|\frac{1}{\cos \pi z}\right| = \frac{1}{\left|\cos(-\pi N + i\pi y)\right|}$$

$$= \left|\sec(-\pi N + i\pi y)\right|$$

$$= \left|\sec(i\pi y)\right|$$

$$= \frac{1}{\left|\cos(i\pi y)\right|}$$

$$= \frac{2}{\left|e^{i(i\pi y)} + e^{-i(i\pi y)}\right|}$$

$$= \frac{2}{\left|e^{-\pi y} + e^{\pi y}\right|}$$

$$= \frac{2}{e^{-\pi y} + e^{\pi y}}$$

For $y = 0$, $\quad \dfrac{2}{e^{-\pi y} + e^{\pi y}} = 1$

For $y > 0$, $\quad \dfrac{2}{e^{-\pi y} + e^{\pi y}} \dfrac{e^{\pi y}}{e^{\pi y}} = 2 \dfrac{e^{\pi y}}{e^{2\pi y} + 1} < 2$

For $y < 0$, $\quad \dfrac{2}{e^{-\pi y} + e^{\pi y}} \dfrac{e^{-\pi y}}{e^{-\pi y}} = 2 \dfrac{e^{\pi(-y)}}{e^{2\pi(-y)} + 1} < 2\,.\square$

$6_{\cos}$.2 $\quad \dfrac{1}{\cos \pi z}$ **has poles at** $z = n + \frac{1}{2} = ..., -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ...$

$6_{\cos}$.3

$$\boxed{\operatorname{Res}\left\{\frac{\pi}{\cos(\pi z)}\right\}_{z=n+\frac{1}{2}} = -(-1)^n}$$

<u>Proof</u>: $\operatorname{Res}\left\{\dfrac{\pi}{\cos(\pi z)}\right\}_{z=n+\frac{1}{2}} = \left[(z - [n + \tfrac{1}{2}])\dfrac{\pi}{\cos(\pi z)}\right]_{z=n+\frac{1}{2}}$

$$= \pi \left[\frac{D_z(z - n - \frac{1}{2})}{D_z \cos(\pi z)} \right]_{z=n+\frac{1}{2}}$$

$$= \left[\frac{1}{-\sin(\pi z)} \right]_{z=n+\frac{1}{2}}$$

$$= \frac{1}{-\sin(\pi n + \frac{\pi}{2})}$$

$$= -(-1)^n . \square$$

$6_{\cos}$.4

$$\boxed{\operatorname{Res}\left\{ \frac{\pi}{\cos(\pi z)} f(z) \right\}_{z=n+\frac{1}{2}} = -(-1)^n f(n + \tfrac{1}{2})}$$

$6_{\cos}$.5 $\left| f(z) \right|_{\square_N} \le \frac{M}{z^k} \Rightarrow$

$$\boxed{\begin{array}{r} ... - f(-\frac{5}{2}) + f(-\frac{3}{2}) - f(-\frac{1}{2}) + f(\frac{1}{2}) - f(\frac{3}{2}) + f(\frac{5}{2}) - ... = \\ = \sum \operatorname{Res}_{-1} \left\{ \frac{\pi}{\cos(\pi\sigma)} f(\sigma) \right\}_{\sigma=\text{pole of } f(\sigma)} \end{array}}$$

Proof:

$$\oint_{\square_N} \frac{\pi}{\cos(\pi\zeta)} f(\zeta) d\zeta = \sum \operatorname{Res}_{-1} \left\{ \frac{\pi}{\cos(\pi z)} f(z) \right\}_{z=\text{pole of } \frac{\pi}{\cos(\pi z)} \text{ in } \square_N}$$

$$+ \sum \operatorname{Res}_{-1} \left\{ \frac{\pi}{\cos(\pi\sigma)} f(\sigma) \right\}_{\sigma=\text{pole of } f(\sigma) \text{ in } \square_N}$$

$$\left|\oint_{\square_N} \frac{\pi}{\cos(\pi\zeta)} f(\zeta)d\zeta\right| \leq \underbrace{\left|\frac{\pi}{\cos(\pi\zeta)}\right|}_{\leq A} \frac{M}{N^k} \underbrace{(\text{length}\square_N)}_{8(N)} \xrightarrow[N\to\infty]{} 0$$

$\dfrac{1}{\cos \pi z}$ **has poles at** $z = n + \frac{1}{2}$

$$\text{Res}_{-1}\left\{\frac{\pi}{\cos(\pi z)} f(z)\right\}_{z=\text{pole of } \frac{\pi}{\cos(\pi z)} \text{ in } \square_N} = \text{Res}_{-1}\left\{\frac{\pi}{\cos(\pi z)} f(z)\right\}_{z=n+\frac{1}{2}}$$

$$= -(-1)^n f(n + \tfrac{1}{2})$$

$$\Rightarrow \quad \sum(-1)^n f(n + \tfrac{1}{2}) = \sum \text{Res}_{-1}\left\{\frac{\pi}{\cos(\pi\sigma)} f(\sigma)\right\}_{\sigma=\text{pole of } f(z)}$$

$$\Rightarrow \quad ... + f(-\tfrac{7}{2}) - f(-\tfrac{5}{2}) + f(-\tfrac{3}{2}) - f(-\tfrac{1}{2}) + f(\tfrac{1}{2}) - f(\tfrac{3}{2}) + f(\tfrac{5}{2}) + ... =$$

$$= \sum \text{Re}\, s_{-1}\left\{\frac{\pi}{\cos(\pi\sigma)} f(\sigma)\right\}_{\sigma=pole\ \ of\ \ f(z)} . \square$$

7_{cot}.

$$\pi = m\tan(\tfrac{l}{m}\pi)\left\{\frac{1}{l} - \frac{1}{m-l} + \frac{1}{m+l} - \frac{1}{2m-l} + \frac{1}{2m+l} - \frac{1}{3m-l} + \ldots\right\}$$

$l < \frac{1}{2}m$, l and m have no common factor

The associated π series is

$$R_{l/m} = \frac{1}{l} - \frac{1}{m-l} + \frac{1}{m+l} - \frac{1}{2m-l} + \frac{1}{2m+l} - \frac{1}{3m-l} + \ldots. = \frac{\pi}{m}\cot(\tfrac{l}{m}\pi)$$

Proof: $\pi\cot(\pi z)\dfrac{1}{z+\frac{l}{m}}$ has poles of order 1 at $z = n$,

and a pole at $z = -\frac{l}{m}$

$$\oint_{\square_{N+\frac{1}{2}}} \pi\cot(\pi\zeta)\frac{1}{\zeta+\frac{l}{m}}d\zeta = \sum \text{Res}_{-1}\left\{\pi\cot(\pi z)\frac{1}{z+\frac{l}{m}}\right\}_{z=n} + \text{Res}_{-1}\left\{\pi\cot(\pi z)\frac{1}{z+\frac{l}{m}}\right\}_{z=-\frac{l}{m}}$$

$$\left|\oint_{\square_{N+\frac{1}{2}}} \pi\cot(\pi\zeta)\frac{1}{\zeta+\frac{l}{m}}d\zeta\right| \le \underbrace{\left|\pi\cot\pi\zeta\right|}_{\le A}\underbrace{\oint_{\square_{N+\frac{1}{2}}}\frac{1}{\zeta+\frac{l}{m}}d\zeta}_{\left[\log(\zeta+\frac{l}{m})\right]_{\square_{N+\frac{1}{2}}}=0} = 0.\square$$

$$\mathrm{Res}_{-1}\left\{\pi\cot(\pi z)\frac{1}{z+\frac{l}{m}}\right\}_{z=n} = \left[(z-n)\pi\frac{\cos\pi z}{\sin\pi z}\frac{1}{z+\frac{l}{m}}\right]_{z=n}$$

$$= \left[\frac{\pi D_z(z-n)}{D_z\sin(\pi z)}\right]_{z=n}\left(\cos(\pi n)\frac{1}{n+\frac{l}{m}}\right)$$

$$= \left[\frac{\pi}{\pi\cos(\pi z)}\right]_{z=n}\left(\cos(\pi n)\frac{1}{n+\frac{l}{m}}\right)$$

$$= \frac{1}{n+\frac{l}{m}}.\square$$

To find $\mathrm{Res}_{-1}\left\{\pi\cot(\pi z)\frac{1}{z+\frac{l}{m}}\right\}_{z=-\frac{l}{m}}$**, divide the series**

$$\pi\frac{\cos(\pi z)}{\sin(\pi z)}\frac{1}{z+\frac{l}{m}} = \pi\frac{\sin(\pi z+\frac{\pi}{2})}{-\cos(\pi z+\frac{\pi}{2})}\frac{1}{z+\frac{l}{m}}$$

$$= -\pi\frac{\sin\left[\pi(z+\frac{l}{m})+(\frac{1}{2}-\frac{l}{m})\pi\right]}{\cos\left[\pi(z+\frac{l}{m})+(\frac{1}{2}-\frac{l}{m})\pi\right]}\frac{1}{z+\frac{l}{m}}$$

$$= -\pi\frac{\sin\left[\pi u+(\frac{1}{2}-\frac{l}{m})\pi\right]}{\cos\left[\pi u+(\frac{1}{2}-\frac{l}{m})\pi\right]}\frac{1}{u}$$

$$= -\pi\frac{\sin(\pi u)\cos(\frac{1}{2}-\frac{l}{m})\pi+\cos(\pi u)\sin(\frac{1}{2}-\frac{l}{m})\pi}{\cos(\pi u)\cos(\frac{1}{2}-\frac{l}{m})\pi-\sin(\pi u)\sin(\frac{1}{2}-\frac{l}{m})\pi}\frac{1}{u}$$

$$= -\pi\frac{[\pi u-\frac{1}{3!}\pi^3u^3+...]\sin(\frac{l}{m}\pi)+[1-\frac{1}{2}\pi^2u^2+...]\cos(\frac{l}{m}\pi)}{[1-\frac{1}{2}\pi^2u^2+...]\sin(\frac{l}{m}\pi)-[\pi u-\frac{1}{3!}\pi^3u^3+...]\cos(\frac{l}{m}\pi)}\frac{1}{u}$$

$$= -\pi \frac{\cos(\frac{l}{m}\pi) + u\pi \sin(\frac{l}{m}\pi) + ...}{\sin(\frac{l}{m}\pi) - \pi u \cos(\frac{l}{m}\pi) + ...} \frac{1}{u}$$

$$= -\pi \cot(\tfrac{l}{m}\pi) \frac{1 + u\pi \tan(\frac{l}{m}\pi) + ...}{1 - \pi u \cot(\frac{l}{m}\pi) + ...} \frac{1}{u}$$

$$\approx -\pi \cot(\tfrac{l}{m}\pi)[1 + u\pi \tan(\tfrac{l}{m}\pi)][1 + \pi u \cot(\tfrac{l}{m}\pi)] \frac{1}{u}$$

$$= -\pi \cot(\tfrac{l}{m}\pi) \frac{1}{u} - \pi^2 \cot(\tfrac{l}{m}\pi)[\cot(\tfrac{l}{m}\pi) + \tan(\tfrac{l}{m}\pi)] - \pi^3 u \cot(\tfrac{l}{m}\pi) + ...$$

$$= -\pi \cot(\tfrac{l}{m}\pi) \frac{1}{u} - \pi^2[\cot^2(\tfrac{l}{m}\pi) + 1] - \pi^3 u \cot(\tfrac{l}{m}\pi) + ...$$

$$\text{Res}_{-1} \left\{ \pi \cot(\pi z) \frac{1}{z + \frac{l}{m}} \right\}_{z = -\frac{l}{m}} = \boxed{-\pi \cot(\tfrac{l}{m}\pi)}$$

Therefore,

$$... + \frac{1}{-3 + \frac{l}{m}} + \frac{1}{-2 + \frac{l}{m}} + \frac{1}{-1 + \frac{l}{m}} + \frac{1}{0 + \frac{l}{m}} +$$

$$+ \frac{1}{1 + \frac{l}{m}} + \frac{1}{2 + \frac{l}{m}} + \frac{1}{3 + \frac{l}{m}} + ... - \pi \cot(\tfrac{l}{m}\pi) = 0$$

$$\pi = m \tan(\tfrac{l}{m}\pi) \left\{ \frac{1}{l} - \frac{1}{m - l} + \frac{1}{m + l} - \frac{1}{2m - l} + \frac{1}{2m + l} - \frac{1}{3m - l} + ... \right\}$$

$$R_{l/m} = \frac{1}{l} - \frac{1}{m-l} + \frac{1}{m+l} - \frac{1}{2m-l} + \frac{1}{2m+l} - \frac{1}{3m-l} + ... = \frac{\pi}{m}\cot(\tfrac{l}{m}\pi)$$

$\mathbf{7_{cot}.2}$ Euler Almost Obtained $\mathbf{7_{cot}}$

In #174, Euler has in his notations

$$\frac{\pi}{2n_1 k} = \frac{1}{m_1} - \frac{1}{2n_1 - m_1} + \frac{1}{2n_1 + m_1} - \frac{1}{4n_1 - m_1} + \frac{1}{4n_1 + m_1} - \frac{1}{6n_1 - m_1} + ...$$

Euler's Notation	Our Notation
$2n_1$	m
m_1	l
k	$\tan(\tfrac{l}{m}\pi)$

Replacing Euler's Notation with ours we obtain $\mathbf{7_{tan}}$.

But Euler's notation

$$\tan(\tfrac{m_1}{2n_1}\pi) = k$$

insinuates that $\tan(\tfrac{m_1}{2n_1}\pi)$ is always an integer.

This amounts to Euler's hypothesizing that π series are parameterized by three integers

$$n_1, \quad m_1, \text{ and } k,$$

In fact, π series are parameterized by the first two, and the crucial role of the tangent function is hidden by a notation that is reserved to integers.

$7_{\tan}$.

$7_{\tan}$.1

$$\boxed{\pi = 2m\cot(\tfrac{l}{m}\pi)\left\{\frac{1}{m-2l}-\frac{1}{m+2l} + \frac{1}{3m-2l}-\frac{1}{3m+2l} + \frac{1}{5m-2l}-\frac{1}{5m+2l}+...\right\}}$$

$l < \frac{1}{2}m$, l and m have no common factor.

The associated π series is

$$T_{l/m} = \frac{1}{m-2l}-\frac{1}{m+2l}$$
$$+\frac{1}{3m-2l}-\frac{1}{3m+2l}$$
$$+\frac{1}{5m-2l}-\frac{1}{5m+2l}+... = \frac{\pi}{2m}\tan(\tfrac{l}{m}\pi).$$

Proof: $\pi\tan(\pi z)\dfrac{1}{z-\frac{l}{m}}$ has poles of order 1 at $z = n+\frac{1}{2}$,

and a pole of order 1 at $z = \frac{l}{m}$

$$\oint_{\square_N}\pi\tan(\pi\zeta)\frac{1}{\zeta-\frac{l}{m}}d\zeta = \sum \mathrm{Res}_{-1}\left\{\pi\tan(\pi z)\frac{1}{z-\frac{l}{m}}\right\}_{z=n+\frac{1}{2}}$$
$$+\mathrm{Res}_{-1}\left\{\pi\tan(\pi z)\frac{1}{z-\frac{l}{m}}\right\}_{z=\frac{l}{m}}$$

$$\left|\oint_{\square_N} \pi\tan(\pi\zeta)\frac{1}{\zeta-\frac{l}{m}}d\zeta\right| \le \underbrace{\left|\pi\tan(\pi\zeta)\right|}_{\le A}\underbrace{\oint_{\square_N}\frac{1}{\zeta-\frac{l}{m}}d\zeta}_{\left[\log(\zeta-\frac{l}{m})\right]_{\square_N}=0} = 0\,.\square$$

$$\mathrm{Res}_{-1}\left\{\pi\tan(\pi z)\frac{1}{z-\frac{l}{m}}\right\}_{z=n+\frac{1}{2}} = \left[(z-n-\tfrac{1}{2})\pi\frac{\sin(\pi z)}{\cos(\pi z)}\frac{1}{z-\frac{l}{m}}\right]_{z=n+\frac{1}{2}}$$

$$= \left[\frac{\pi D_z(z-n-\frac{1}{2})}{D_z\cos(\pi z)}\right]_{z=n+\frac{1}{2}}\sin\pi(n+\tfrac{1}{2})\frac{1}{n+\frac{1}{2}-\frac{l}{m}}$$

$$= \left[\frac{\pi}{-\pi\sin(\pi z)}\right]_{z=n+\frac{1}{2}}\sin\pi(n+\tfrac{1}{2})\frac{1}{n+\frac{1}{2}-\frac{l}{m}}$$

$$= \frac{-1}{n+\frac{1}{2}-\frac{l}{m}}.\square$$

To find $\mathrm{Res}_{-1}\left\{\pi\tan(\pi z)\frac{1}{z-\frac{l}{m}}\right\}_{z=\frac{l}{m}}$**, divide the series**

$$\pi\frac{\sin(\pi z)}{\cos(\pi z)}\frac{1}{z-\frac{l}{m}} = \pi\frac{\cos(\frac{\pi}{2}-\pi z)}{\sin(\frac{\pi}{2}-\pi z)}\frac{1}{z-\frac{l}{m}}$$

$$= \pi\frac{\cos\left[\pi(\frac{1}{2}-\frac{l}{m})-\pi(z-\frac{l}{m})\right]}{\sin\left[\pi(\frac{1}{2}-\frac{l}{m})-\pi(z-\frac{l}{m})\right]}\frac{1}{z-\frac{l}{m}}$$

$$= \pi\frac{\cos\left[\pi(\frac{1}{2}-\frac{l}{m})-\pi u\right]}{\sin\left[\pi(\frac{1}{2}-\frac{l}{m})-\pi u\right]}\frac{1}{u}$$

$$= \pi\frac{\cos\pi(\frac{1}{2}-\frac{l}{m})\cos(\pi u)+\sin\pi(\frac{1}{2}-\frac{l}{m})\sin(\pi u)}{\sin\pi(\frac{1}{2}-\frac{l}{m})\cos(\pi u)-\cos\pi(\frac{1}{2}-\frac{l}{m})\sin(\pi u)}\frac{1}{u}$$

$$= \pi \frac{\sin(\frac{l}{m}\pi)[1 - \frac{1}{2!}\pi^2 u^2 + ...] + \sin(\frac{l}{m}\pi)[\pi u - \frac{1}{3!}\pi^3 u^3 + ...]}{\cos(\frac{l}{m}\pi)[1 - \frac{1}{2!}\pi^2 u^2 + ...] - \sin(\frac{l}{m}\pi)[\pi u - \frac{1}{3!}\pi^3 u^3 + ...]} \frac{1}{u}$$

$$= \pi \frac{\sin(\frac{l}{m}\pi)[1 + \pi u] + ...}{\cos(\frac{l}{m}\pi) - \pi u \sin(\frac{l}{m}\pi) + ...} \frac{1}{u}$$

$$= \pi \frac{\sin(\frac{l}{m}\pi)}{\cos(\frac{l}{m}\pi)} \frac{1 + \pi u}{1 - \pi u \tan(\frac{l}{m}\pi) + ...} \frac{1}{u}$$

$$\approx \pi \tan(\tfrac{l}{m}\pi)[1 + \pi u][1 + \pi u \tan(\tfrac{l}{m}\pi) + ...] \frac{1}{u}$$

$$= \pi \tan(\tfrac{l}{m}\pi)\frac{1}{u} + \pi^2 \tan(\tfrac{l}{m}\pi)\left\{1 + \tan(\tfrac{l}{m}\pi)\right\} + ...$$

$$\operatorname{Res}_{-1}\left\{\pi \tan(\tfrac{l}{m}\pi)\frac{1}{z - \frac{l}{m}}\right\}_{z=\frac{l}{m}} = \pi \tan(\tfrac{l}{m}\pi)$$

Therefore,

$$... - \frac{1}{-3 + \frac{1}{2} - \frac{l}{m}} - \frac{1}{-2 + \frac{1}{2} - \frac{l}{m}} - \frac{1}{-1 + \frac{1}{2} - \frac{l}{m}} - \frac{1}{0 + \frac{1}{2} - \frac{l}{m}} -$$

$$- \frac{1}{1 + \frac{1}{2} - \frac{l}{m}} - \frac{1}{2 + \frac{1}{2} - \frac{l}{m}} - \frac{1}{3 + \frac{1}{2} - \frac{l}{m}} - ... + \pi \tan(\tfrac{l}{m}\pi) = 0$$

$$\pi = 2m \cot(\tfrac{l}{m}\pi)\left\{\frac{1}{m - 2l} - \frac{1}{m + 2l} + \frac{1}{3m - 2l} - \frac{1}{3m + 2l} + \frac{1}{5m - 2l} - \frac{1}{5m + 2l} + ...\right\}$$

The Associated Series is

$$T_{l/m} = \frac{1}{m-2l} - \frac{1}{m+2l}$$

$$+\frac{1}{3m-2l} - \frac{1}{3m+2l}$$

$$+\frac{1}{5m-2l} + \frac{1}{5m+2l} + ... = \frac{\pi}{2m}\tan(\tfrac{l}{m}\pi)$$

$7_{\mathbf{tan}}$.2 $$\frac{\pi}{2m}\tan(\tfrac{l}{m}\pi) = \frac{\pi}{2m}\cot(\tfrac{1}{2}\pi - \tfrac{l}{m}\pi)$$

guarantees that every π-series from 7.$_{\mathbf{tan}}$ matches uniquely a π-series from 7.$_{\mathbf{cot}}$, and vice versa.

The family of π-series from 7.$_{\mathbf{tan}}$ is identical to the family of π-series from 7.$_{\mathbf{cot}}$

Proof: We use **7.$_{\mathbf{cot}}$** to find the associated series from **7.$_{\mathbf{tan}}$**:

$$\frac{\pi}{2m}\tan(\tfrac{l}{m}\pi) = \frac{\pi}{2m}\cot(\tfrac{1}{2}\pi - \tfrac{l}{m}\pi)$$

$$= \frac{\pi}{2m}\cot(\tfrac{m-2l}{2m}\pi)$$

By **$7_{\mathbf{cot}}$**,

$$\frac{\pi}{m_1}\cot(\tfrac{l_1}{m_1}\pi) = \frac{1}{l_1} - \frac{1}{m_1 - l_1}$$

$$+\frac{1}{m_1 + l_1} - \frac{1}{2m_1 - l_1}$$

$$+\frac{1}{2m_1 + l_1} - \frac{1}{3m_1 - l_1} + ...$$

Substituting $m_1 = 2m$, $l_1 = m - 2l$

$$\frac{\pi}{2m}\cot(\tfrac{m-2l}{2m}\pi) = \frac{1}{m-2l} - \frac{1}{2m-(m-2l)} + \frac{1}{2m+m-2l} - \frac{1}{2(2m)-(m-2l)} + \frac{1}{2(2m)+(m-2l)} - \frac{1}{3(2m)-(m-2l)} + \dots$$

Therefore,

$$\frac{\pi}{2m}\tan(\tfrac{l}{m}\pi) = \frac{1}{m-2l} - \frac{1}{m+2l} + \frac{1}{3m-2l} - \frac{1}{3m+2l} + \frac{1}{5m-2l} - \frac{1}{5m+2l} + \dots \quad .\square$$

$7_{\tan}$.3 Euler Almost Obtained $7_{\tan}$

In #172, Euler has in his notations

$$\frac{\pi}{2n_1}k = \frac{1}{n_1-m_1} - \frac{1}{n_1+m_1} + \frac{1}{3n_1-m_1} - \frac{1}{3n_1+m_1} + \frac{1}{5n_1-m_1} - \frac{1}{5n_1+m_1} + \dots$$

Euler's Notation	Our Notation
n_1	m
m_1	$2l$
k	$\tan(\frac{l}{m}\pi)$

Replacing Euler's Notation with ours we obtain $\mathbf{7_{tan}}$.

But Euler's notation

$$\tan(\frac{m_1}{2n_1}\pi) = k$$

insinuates that $\tan(\frac{m_1}{2n_1}\pi)$ is always an integer.

This amounts to Euler's hypothesizing that π series are parameterized by three integers

$$n_1, \quad m_1, \text{ and } k,$$

In fact, π series are parameterized by the first two, and the crucial role of the tangent function is hidden by a notation that is reserved to integers.

$\mathbf{7_{sin}}$.

$$\boxed{\pi = m\sin(\tfrac{l}{m}\pi)\left\{\frac{1}{l}+\frac{1}{m-l} - \frac{1}{m+l}-\frac{1}{2m-l} + \frac{1}{2m+l}+\frac{1}{3m-l}-...\right\}}$$

$l < \frac{1}{2}m$, l and m have no common factor.

The associated π series is

$$S_{l/m} = \frac{1}{l}+\frac{1}{m-l} - \frac{1}{m+l}-\frac{1}{2m-l} + \frac{1}{2m+l}+\frac{1}{3m-l}-... = \frac{\pi}{m\sin(\frac{l}{m}\pi)}$$

<u>*Proof*</u>: $\pi\frac{1}{\sin(\pi z)}\frac{1}{z+\frac{l}{m}}$ has poles of order 1 at $z = n$,

and a pole at $z = -\frac{l}{m}$

$$\oint_{\square_{N+\frac{1}{2}}} \pi\frac{1}{\sin(\pi\zeta)}\frac{1}{\zeta+\frac{l}{m}}d\zeta = \sum \mathrm{Res}_{-1}\left\{\pi\frac{1}{\sin(\pi z)}\frac{1}{z+\frac{l}{m}}\right\}_{z=n}$$

$$+\mathrm{Res}_{-1}\left\{\pi\frac{1}{\sin(\pi z)}\frac{1}{z+\frac{l}{m}}\right\}_{z=-\frac{4}{9}}$$

$$\left|\oint_{\square_{N+\frac{1}{2}}} \frac{\pi}{\sin(\pi\zeta)}\frac{1}{\zeta+\frac{l}{m}}d\zeta\right| \le \underbrace{\left|\frac{\pi}{\sin(\pi\zeta)}\right|}_{\le A}\underbrace{\oint_{\square_{N+\frac{1}{2}}}\frac{1}{\zeta+\frac{l}{m}}d\zeta}_{\left[\log(\zeta+\frac{l}{m})\right]_{\square_{N+\frac{1}{2}}}=0} = 0.\square$$

$$\text{Res}_{-1}\left\{\pi\frac{1}{\sin(\pi z)}\frac{1}{z+\frac{l}{m}}\right\}_{z=n} = \left[(z-n)\pi\frac{1}{\sin\pi z}\frac{1}{z+\frac{l}{m}}\right]_{z=n}$$

$$= \left[\frac{\pi D_z(z-n)}{D_z\sin(\pi z)}\right]_{z=n}\left(\frac{1}{n+\frac{l}{m}}\right)$$

$$= \left[\frac{\pi}{\pi\cos(\pi z)}\right]_{z=n}\left(\frac{1}{n+\frac{l}{m}}\right)$$

$$= \frac{(-1)^n}{n+\frac{l}{m}}.\square$$

To find $\text{Res}_{-1}\left\{\pi\frac{1}{\sin(\pi z)}\frac{1}{z+\frac{l}{m}}\right\}_{z=-\frac{l}{m}}$**, divide the series**

$$\pi\frac{1}{\sin(\pi z)}\frac{1}{z+\frac{l}{m}} = \pi\frac{1}{-\cos(\pi z+\frac{\pi}{2})}\frac{1}{z+\frac{l}{m}}$$

$$= -\pi\frac{1}{\cos\left[\pi(z+\frac{l}{m})+(\frac{1}{2}-\frac{l}{m})\pi\right]}\frac{1}{z+\frac{l}{m}}$$

$$= -\pi\frac{1}{\cos\left[\pi u+(\frac{1}{2}-\frac{l}{m})\pi\right]}\frac{1}{u}$$

$$= -\pi\frac{1}{\cos(\pi u)\cos(\frac{1}{2}-\frac{l}{m})\pi-\sin(\pi u)\sin(\frac{1}{2}-\frac{l}{m})\pi}\frac{1}{u}$$

$$= -\pi\frac{1}{[1-\frac{1}{2}\pi^2u^2+\ldots]\sin(\frac{l}{m}\pi)-[\pi u-\frac{1}{3!}\pi^3u^3+\ldots]\cos(\frac{l}{m}\pi)}\frac{1}{u}$$

$$= -\pi\frac{1}{\sin(\frac{l}{m}\pi)-\pi u\cos(\frac{l}{m}\pi)+\ldots}\frac{1}{u}$$

$$= -\pi \frac{1}{\sin(\frac{l}{m}\pi)} \frac{1}{1 - \pi u \cot(\frac{l}{m}\pi) + ...} \frac{1}{u}$$

$$\approx -\pi \frac{1}{\sin(\frac{l}{m}\pi)} [1 + \pi u \cot(\frac{l}{m}\pi) + ...] \frac{1}{u}$$

$$= -\pi \frac{1}{\sin(\frac{l}{m}\pi)} \frac{1}{u} - \pi^2 \frac{1}{\sin(\frac{l}{m}\pi)} \cot(\frac{l}{m}\pi) + ...$$

$$\text{Res}_{-1}\left\{\pi \frac{1}{\sin(\pi z)} \frac{1}{z + \frac{l}{m}}\right\}_{z=-\frac{l}{m}} = -\pi \frac{1}{\sin(\frac{l}{m}\pi)}$$

Therefore,

$$... + \frac{1}{-4 + \frac{l}{m}} - \frac{1}{-3 + \frac{l}{m}} + \frac{1}{-2 + \frac{l}{m}} - \frac{1}{-1 + \frac{l}{m}} + \frac{1}{0 + \frac{l}{m}} +$$

$$- \frac{1}{1 + \frac{l}{m}} + \frac{1}{2 + \frac{l}{m}} - \frac{1}{3 + \frac{l}{m}} + ... - \pi \frac{1}{\sin(\frac{l}{m}\pi)} = 0$$

$$\pi = m \sin(\tfrac{l}{m}\pi) \left\{ \frac{1}{l} + \frac{1}{m-l} - \frac{1}{m+l} - \frac{1}{2m-l} + \frac{1}{2m+l} + \frac{1}{3m-l} - ... \right\} . \square$$

$7_{\sin}.2$ Euler Almost Obtained $7._{\sin}$

In #178, Euler has in his notations

$$\frac{\pi}{n_1}\frac{k^2+1}{2k} = \frac{1}{m_1} + \frac{1}{n_1 - m_1} - \frac{1}{n_1 + m_1} - \frac{1}{2n_1 - m_1} + \frac{1}{2n_1 + m_1} + \frac{1}{3n_1 - m_1} + \dots$$

Euler's Notation	Our Notation
n_1	m
m_1	l
k	$\tan(\frac{l}{2m}\pi)$

Therefore,

$$\frac{k^2+1}{2k} = \frac{\tan^2\left(\frac{l}{2m}\pi\right)+1}{2\tan\left(\frac{l}{2m}\pi\right)}$$

$$= \frac{1}{2}\left\{\tan\left(\frac{l}{2m}\pi\right) + \cot\left(\frac{l}{2m}\pi\right)\right\}$$

$$= \frac{1}{2}\left\{\frac{\sin\left(\frac{l}{2m}\pi\right)}{\cos\left(\frac{l}{2m}\pi\right)} + \frac{\cos\left(\frac{l}{2m}\pi\right)}{\sin\left(\frac{l}{2m}\pi\right)}\right\}$$

$$= \frac{1}{2\sin\left(\frac{l}{2m}\pi\right)\cos\left(\frac{l}{2m}\pi\right)}$$

$$= \frac{1}{\sin\left(\frac{l}{m}\pi\right)}$$

Replacing Euler's Notation with ours we obtain $\mathbf{7_{tan}}$.

But the notation

$$\tan(\tfrac{l}{2m}\pi) = k$$

insinuates that $\tan(\frac{l}{2m}\pi)$ is always an integer.

This amounts to Euler's hypothesizing that π series are parameterized by three integers

$$n_1, \quad m_1, \text{ and } k,$$

In fact, π series are parameterized by the first two, and the crucial role of the tangent function is hidden by a notation that is reserved to integers.

$7_{\cos}$.

$7._{\cos}.1$

$$\boxed{\pi = 2m\cos(\tfrac{l}{m}\pi)\left\{\frac{1}{m-2l}+\frac{1}{m+2l} - \frac{1}{3m-2l}-\frac{1}{3m+2l} + \frac{1}{5m-2l}+\frac{1}{5m+2l}-...\right\}}$$

$l < \frac{1}{2}m$, l and m have no common factor.

The associated π series is

$$C_{l/m} = \frac{1}{m-2l}+\frac{1}{m+2l} - \frac{1}{3m-2l}-\frac{1}{3m+2l} + \frac{1}{5m-2l}+\frac{1}{5m+2l}-... = \frac{\pi}{2m\cos(\frac{l}{m}\pi)}$$

Proof: $\pi\dfrac{1}{\cos(\pi z)}\dfrac{1}{z-\frac{l}{m}}$ has poles of order 1 at $z = n+\frac{1}{2}$,

and a pole of order 1 at $z = \frac{l}{m}$

$$\oint_{\square_N}\pi\frac{1}{\cos(\pi\zeta)}\frac{1}{\zeta-\frac{l}{m}}d\zeta = \sum \mathrm{Res}_{-1}\left\{\pi\frac{1}{\cos(\pi z)}\frac{1}{z-\frac{l}{m}}\right\}_{z=n+\frac{1}{2}} + \mathrm{Res}_{-1}\left\{\pi\frac{1}{\cos(\pi z)}\frac{1}{z-\frac{l}{m}}\right\}_{z=\frac{l}{m}}$$

$$\left|\oint_{\square_N}\frac{\pi}{\cos(\pi\zeta)}\frac{1}{\zeta-\frac{l}{m}}d\zeta\right| \le \underbrace{\left|\frac{\pi}{\cos(\pi\zeta)}\right|}_{\le A}\underbrace{\oint_{\square_N}\frac{1}{\zeta-\frac{l}{m}}d\zeta}_{\left[\log(\zeta-\frac{l}{m})\right]_{\square_N}=0} = 0.\square$$

$$\mathrm{Res}_{-1}\left\{\pi\frac{1}{\cos(\pi z)}\frac{1}{z-\frac{l}{m}}\right\}_{z=n+\frac{1}{2}}=\left[(z-n-\tfrac{1}{2})\pi\frac{1}{\cos(\pi z)}\frac{1}{z-\frac{l}{m}}\right]_{z=n+\frac{1}{2}}$$

$$=\left[\frac{\pi D_z(z-n-\frac{1}{2})}{D_z\cos(\pi z)}\right]_{z=n+\frac{1}{2}}\frac{1}{n+\frac{1}{2}-\frac{l}{m}}$$

$$=\left[\frac{\pi}{-\pi\sin(\pi z)}\right]_{z=n+\frac{1}{2}}\frac{1}{n+\frac{1}{2}-\frac{l}{m}}$$

$$=\frac{1}{-\sin\pi(n+\frac{1}{2})}\frac{1}{n+\frac{1}{2}-\frac{l}{m}}$$

$$=-(-1)^n\frac{1}{n+\frac{1}{2}-\frac{l}{m}}$$

$$=(-1)^{n+1}\frac{1}{n+\frac{1}{2}-\frac{l}{m}}.\square$$

To find $\mathrm{Res}_{-1}\left\{\pi\frac{1}{\cos(\pi z)}\frac{1}{z-\frac{l}{m}}\right\}_{z=\frac{l}{m}}$**, divide the series**

$$\pi\frac{1}{\cos(\pi z)}\frac{1}{z-\frac{l}{m}}=\pi\frac{1}{\sin(\frac{\pi}{2}-\pi z)}\frac{1}{z-\frac{l}{m}}$$

$$=\pi\frac{1}{\sin\left[\pi(\frac{1}{2}-\frac{l}{m})-\pi(z-\frac{l}{m})\right]}\frac{1}{z-\frac{l}{m}}$$

$$=\pi\frac{1}{\sin\left[\pi(\frac{1}{2}-\frac{l}{m})-\pi u\right]}\frac{1}{u}$$

$$=\pi\frac{1}{\sin\pi(\frac{1}{2}-\frac{l}{m})\cos(\pi u)-\cos\pi(\frac{1}{2}-\frac{l}{m})\sin(\pi u)}\frac{1}{u}$$

$$= \pi \frac{1}{\cos(\frac{l}{m}\pi)[1 - \frac{1}{2!}\pi^2 u^2 + ...] - \sin(\frac{l}{m}\pi)[\pi u - \frac{1}{3!}\pi^3 u^3 + ...]} \frac{1}{u}$$

$$= \pi \frac{1}{\cos(\frac{l}{m}\pi) - \pi u \sin(\frac{l}{m}\pi) + ...} \frac{1}{u}$$

$$= \pi \frac{1}{\cos(\frac{l}{m}\pi)} \frac{1}{1 - \pi u \tan(\frac{l}{m}\pi) + ...} \frac{1}{u}$$

$$\approx \pi \frac{1}{\cos(\frac{l}{m}\pi)} [1 + \pi u \tan(\frac{l}{m}\pi) + ...] \frac{1}{u}$$

$$= \pi \frac{1}{\cos(\frac{l}{m}\pi)} \frac{1}{u} + \pi^2 \frac{1}{\cos(\frac{l}{m}\pi)} \tan(\frac{l}{m}\pi) + ...$$

$$\text{Res}_{-1} \left\{ \pi \frac{1}{\cos(\pi z)} \frac{1}{z - \frac{l}{m}} \right\}_{z = \frac{l}{m}} = \pi \frac{1}{\cos(\frac{l}{m}\pi)}$$

Therefore,

$$... + \frac{1}{-3 + \frac{1}{2} - \frac{l}{m}} - \frac{1}{-2 + \frac{1}{2} - \frac{l}{m}} + \frac{1}{-1 + \frac{1}{2} - \frac{l}{m}} - \frac{1}{0 + \frac{1}{2} - \frac{l}{m}} +$$

$$+ \frac{1}{1 + \frac{1}{2} - \frac{l}{m}} - \frac{1}{2 + \frac{1}{2} - \frac{l}{m}} + \frac{1}{3 + \frac{1}{2} - \frac{l}{m}} - ... + \frac{\pi}{\cos(\frac{l}{m}\pi)} = 0$$

$$\pi = 2m \cos(\tfrac{l}{m}\pi) \left\{ \frac{1}{m - 2l} + \frac{1}{m + 2l} - \frac{1}{3m - 2l} - \frac{1}{3m + 2l} + \frac{1}{5m - 2l} + \frac{1}{5m + 2l} - ... \right\}$$

The Associated π Series is

$$C_{l/m} = \frac{1}{m-2l} + \frac{1}{m+2l} - \frac{1}{3m-2l} - \frac{1}{3m+2l} + \frac{1}{5m-2l} + \frac{1}{5m+2l} - ... = \frac{\pi}{2m\cos(\frac{l}{m}\pi)}$$

$7_{\cos}$.2 $$\frac{\pi}{2m\cos(\frac{l}{m}\pi)} = \frac{\pi}{2m\sin(\frac{1}{2}\pi - \frac{l}{m}\pi)}$$

guarantees that every π-series from $7_{\cos}$ matches uniquely a π-series from $7_{\sin}$, and vice versa.

The family of π-series from $7_{\cos}$ is identical to the family of π-series from $7_{\sin}$

Proof: We use **$7_{\sin}$** to find the associated series from **$7_{\cos}$**:

$$\frac{\pi}{2m\cos(\frac{l}{m}\pi)} = \frac{\pi}{2m\sin(\frac{1}{2}\pi - \frac{l}{m}\pi)}$$

$$= \frac{\pi}{2m\sin(\frac{m-2l}{2m}\pi)}$$

By **$7_{\sin}$**,

$$\frac{\pi}{m_1\sin(\frac{l_1}{m_1}\pi)} = \frac{1}{l_1} + \frac{1}{m_1 - l_1} - \frac{1}{m_1 + l_1} - \frac{1}{2m_1 - l_1} + \frac{1}{2m_1 + l_1} + \frac{1}{3m_1 - l_1} - ....$$

Substituting $m_1 = 2m$, $l_1 = m - 2l$

$$\frac{\pi}{2m\sin(\frac{m-2l}{2m}\pi)} = \frac{1}{m-2l} + \frac{1}{2m-(m-2l)} - \frac{1}{2m+m-2l} - \frac{1}{2(2m)-(m-2l)} + \frac{1}{2(2m)+(m-2l)} + \frac{1}{3(2m)-(m-2l)} - ....$$

Therefore,

$$\frac{\pi}{2m\cos(\frac{l}{m}\pi)} = \frac{1}{m-2l} + \frac{1}{m+2l} - \frac{1}{3m-2l} - \frac{1}{3m+2l} + \frac{1}{5m-2l} + \frac{1}{5m-2l} - .... \quad .\square$$

$7_{\cos}$.3 Euler did not obtain $7_{\cos}$.

8.

8.1

$$\pi = 38\tan(\tfrac{15}{38}\pi)\left\{\frac{1}{15} - \frac{1}{38-15} + \frac{1}{38+15} - \frac{1}{2(38)-15} + \frac{1}{2(38)+15} - \frac{1}{3(38)-15} + \dots\right\}$$

$$R_{15/38} = \frac{1}{15} - \frac{1}{38-15} + \frac{1}{38+15} - \frac{1}{2(38)-15} + \frac{1}{2(38)+15} - \frac{1}{3(38)-15} + .. = \frac{\pi}{38}\cot(\tfrac{15}{38}\pi)$$

Proof: By $\mathbf{7_{cot}}$.

8.2

$$\pi = 2(38)\cot(\tfrac{15}{38}\pi)\left\{\frac{1}{38-2(15)} - \frac{1}{38+2(15)} + \frac{1}{3(38)-2(15)} - \frac{1}{3(38)+2(15)} + \frac{1}{5(38)-2(15)} - \frac{1}{5(38)+2(15)} + \dots\right\}$$

$$T_{15/38} = \frac{1}{38-2(15)} - \frac{1}{38+2(15)} + \frac{1}{3(38)-2(15)} - \frac{1}{3(38)+2(15)} + \frac{1}{5(38)-2(15)} - \frac{1}{5(38)+2(15)} + \dots = \frac{\pi}{2(38)}\tan(\tfrac{15}{38}\pi)$$

Proof: By $\mathbf{7_{tan}}$.

8.3

$$\pi = 38\sin(\tfrac{15}{38}\pi)\left\{\frac{1}{15}+\frac{1}{38-15}-\frac{1}{38+15}-\frac{1}{2(38)-15}+\frac{1}{2(38)+15}+\frac{1}{3(38)-15}-\frac{1}{3(38)+15}-...\right\}$$

$$S_{15/38} = \frac{1}{15}+\frac{1}{38-15}-\frac{1}{38+15}-\frac{1}{2(38)-15}+\frac{1}{2(38)+15}+\frac{1}{3(38)-15}-.. = \frac{\pi}{38\sin(\tfrac{15}{38}\pi)}$$

Proof: By $\mathbf{7_{sin}}$.

8.4

$$\pi = 2(38)\cos(\tfrac{15}{38}\pi)\left\{\frac{1}{38-2(15)}+\frac{1}{38+2(15)}-\frac{1}{3(38)-2(15)}-\frac{1}{3(38)+2(15)}+\frac{1}{5(38)-2(15)}+\frac{1}{5(38)+2(15)}+...\right\}$$

$$C_{15/38} = \frac{1}{38-2(15)}+\frac{1}{38+2(15)}-\frac{1}{3(38)-2(15)}-\frac{1}{3(38)+2(15)}+\frac{1}{5(38)-2(15)}+\frac{1}{5(38)+2(15)}-... = \frac{\pi}{2(38)\cos(\tfrac{15}{38}\pi)}$$

Proof: By $\mathbf{7_{cos}}$.

9.

9.1

$$\pi = 2^n \tan(\tfrac{2k+1}{2^n}\pi)\left\{\frac{1}{2k+1} - \frac{1}{2^n - 2k - 1} + \frac{1}{2^n + 2k + 1} - \frac{1}{2^{n+1} - 2k - 1} + \frac{1}{2^{n+1} + 2k + 1} - \frac{1}{2^{n+2} - 2k - 1} + \dots\right\}$$

$$R_{(2k+1)/2^n} = \frac{1}{2k+1} - \frac{1}{2^n - 2k - 1} + \frac{1}{2^n + 2k + 1} - \frac{1}{2^{n+1} - 2k - 1} + \frac{1}{2^{n+1} + 2k + 1} - \frac{1}{2^{n+2} - 2k - 1} + \dots = \frac{\pi}{2^n}\cot(\tfrac{2k+1}{2^n}\pi)$$

Proof: By $\mathbf{7_{cot}}$.

9.2

$$\pi = 2^{n+1} \cot(\tfrac{2k+1}{2^n}\pi)\left\{\frac{1}{2^n - 2(2k+1)} - \frac{1}{2^n + 2(2k+1)} + \frac{1}{3(2^n) - 2(2k+1)} - \frac{1}{3(2^n) + 2(2k+1)} + \frac{1}{5(2^n) - 2(2k+1)} - \frac{1}{5(2^n) + 2(2k+1)} - \dots\right\}$$

$$T_{(2k+1)/2^n} = \frac{1}{2^n - 2(2k+1)} - \frac{1}{2^n + 2(2k+1)} + \frac{1}{3(2^n) - 2(2k+1)} - \frac{1}{3(2^n) + 2(2k+1)} + \frac{1}{5(2^n) - 2(2k+1)} - \frac{1}{5(2^n) + 2(2k+1)} + \dots = \frac{\pi}{2^{n+1}}\tan(\tfrac{2k+1}{2^n}\pi)$$

Proof: By $\mathbf{7_{tan}}$.

9.3
$$\pi = 2^n \sin(\tfrac{2k+1}{2^n}\pi)\left\{\frac{1}{2k+1} + \frac{1}{2^n - 2k - 1} - \frac{1}{2^n + 2k + 1} - \frac{1}{2^{n+1} - 2k - 1} + \frac{1}{2^{n+1} + 2k + 1} + \frac{1}{2^{n+2} - 2k - 1} - \frac{1}{2^{n+2} + 2k + 1} - \ldots\right\}$$

$$S_{(2k+1)/2^n} = \frac{1}{2k+1} + \frac{1}{2^n - 2k - 1} - \frac{1}{2^n + 2k + 1} - \frac{1}{2^{n+1} - 2k - 1} + \frac{1}{2^{n+1} + 2k + 1} + \frac{1}{2^{n+2} - 2k - 1} + \ldots = \frac{\pi}{2^n \sin(\frac{2k+1}{2^n}\pi)}$$

Proof: By $\mathbf{7_{sin}}$.

9.4
$$\pi = 2^{n+1} \cos(\tfrac{2k+1}{2^n}\pi)\left\{\frac{1}{2^n - 2(2k+1)} + \frac{1}{2^n + 2(2k+1)} - \frac{1}{3(2^n) - 2(2k+1)} - \frac{1}{3(2^n) + 2(2k+1)} + \frac{1}{5(2^n) - 2(2k+1)} + \frac{1}{5(2^n) + 2(2k+1)} - \ldots\right\}$$

$$C_{(2k+1)/2^n} = \frac{1}{2^n - 2(2k+1)} + \frac{1}{2^n + 2(2k+1)} - \frac{1}{3(2^n) - 2(2k+1)} - \frac{1}{3(2^n) + 2(2k+1)} + \frac{1}{5(2^n) - 2(2k+1)} + \frac{1}{5(2^n) + 2(2k+1)} + \ldots = \frac{\pi}{2^{n+1} \cos(\frac{2k+1}{2^n}\pi)}$$

$$2k + 1 < 2^{n-1}$$

Proof: By $\mathbf{7_{cos}}$.

$$\tan(\tfrac{2k+1}{2^n}\pi) = \frac{\sqrt{\tfrac{1}{2}[1-\cos(\tfrac{2k+1}{2^{n-1}}\pi)]}}{\sqrt{\tfrac{1}{2}[1+\cos(\tfrac{2k+1}{2^{n-1}}\pi)]}}$$

$$= \frac{\sqrt{1-\tfrac{1}{\sqrt{2}}\sqrt{1+\cos(\tfrac{2k+1}{2^{n-2}}\pi)}}}{\sqrt{1+\tfrac{1}{\sqrt{2}}\sqrt{1+\cos(\tfrac{2k+1}{2^{n-2}}\pi)}}} = ...$$

$$= \text{ a number composed of } \sqrt{2}\text{'s}$$

$$\sin(\tfrac{2k+1}{2^n}\pi) = \sqrt{\tfrac{1}{2}[1-\cos(\tfrac{2k+1}{2^{n-1}}\pi)]}$$

$$= \tfrac{1}{\sqrt{2}}\sqrt{1-\tfrac{1}{\sqrt{2}}\sqrt{1+\cos(\tfrac{2k+1}{2^{n-2}}\pi)}} = ...$$

$$= \text{ a number composed of } \sqrt{2}\text{'s}$$

10.

10.1 $$\boxed{\pi = 4\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots\right)}$$ **(Leibniz)**

$$R_{1/4} = T_{1/4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \ldots = \frac{\pi}{4}$$ **(Leibniz)**

__Proof__: by $\mathbf{7_{cot}}$•,

$$\pi = 4\tan(\tfrac{1}{4}\pi)\left\{\frac{1}{1} - \frac{1}{4-1} + \frac{1}{4+1} - \frac{1}{2(4)-1} + \frac{1}{2(4)+1} - \frac{1}{3(4)-1} + \ldots\right\}$$

or by $\mathbf{7_{tan}}$•,

$$\pi = 2(4)\cot(\tfrac{1}{4}\pi)\left\{\frac{1}{4-2} - \frac{1}{4+2} + \frac{1}{3(4)-2} - \frac{1}{3(4)+2} + \frac{1}{5(4)-2} - \frac{1}{5(4)+2} + \ldots\right\}$$

10.2 $$\boxed{\pi = 2\sqrt{2}\left(\frac{1}{1} + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \ldots\right)}$$

$$S_{1/4}=C_{1/4}=\frac{1}{1}+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}+...=\frac{\pi}{4}\sqrt{2}$$

Proof: By $\mathbf{7_{sin^{\bullet}}}$,

$$\pi=4\sin(\tfrac{1}{4}\pi)\left\{\frac{1}{1}+\frac{1}{4-1}\right.$$
$$-\frac{1}{4+1}-\frac{1}{2(4)-1}$$
$$\left.+\frac{1}{2(4)+1}+\frac{1}{3(4)-1}-...\right\}$$

or by $\mathbf{7_{cos^{\bullet}}}$,

$$\pi=2(4)\cos(\tfrac{1}{4}\pi)\left\{\frac{1}{4-2}+\frac{1}{4+2}\right.$$
$$-\frac{1}{3(4)-2}-\frac{1}{3(4)+2}$$
$$\left.+\frac{1}{5(4)-2}+\frac{1}{5(4)+2}-...\right\}$$

10.3
$$\boxed{\sqrt{2}=\frac{1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}+...}{1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-...}}$$

11.

11.1

$$\boxed{\pi = 8\tan(\tfrac{1}{8}\pi)\left(\frac{1}{1}-\frac{1}{7} + \frac{1}{9}-\frac{1}{15} + \frac{1}{17}-\frac{1}{23}+\dots\right)}$$

$$R_{1/8} = \frac{1}{1}-\frac{1}{7}+\frac{1}{9}-\frac{1}{15}+\frac{1}{17}-\frac{1}{23}+\dots = \frac{\pi}{8}\cot(\tfrac{1}{8}\pi)$$

$$\tan(\tfrac{1}{8}\pi) = \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}} = \text{depends on } \sqrt{2}\text{'s}$$

Proof: By $\mathbf{7_{cot^{\bullet}}}$, based on $\pi\cot(\pi z)\dfrac{1}{z+\frac{1}{8}}$,

$$\pi = 8\tan(\tfrac{1}{8}\pi)\left\{\frac{1}{1}-\frac{1}{8-1} + \frac{1}{8+1}-\frac{1}{2(8)-1} + \frac{1}{2(8)+1}-\frac{1}{3(8)-1}+\dots\right\}$$

$$\cot(\tfrac{1}{8}\pi) = \frac{\cos(\frac{1}{8}\pi)}{\sin(\frac{1}{8}\pi)}$$

$$= \frac{\sqrt{\frac{1}{2}\left[1+\cos(\frac{1}{4}\pi)\right]}}{\sqrt{\frac{1}{2}\left[1-\cos(\frac{1}{4}\pi)\right]}}$$

$$= \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}}.\square$$

11.2
$$\boxed{\pi = 8\cot(\tfrac{1}{8}\pi)\left(\frac{1}{3}-\frac{1}{5} + \frac{1}{11}-\frac{1}{13} + \frac{1}{19}-\frac{1}{21}+...\right)}$$

$$T_{1/8} = \frac{1}{3}-\frac{1}{5}+\frac{1}{11}-\frac{1}{13}+\frac{1}{19}-\frac{1}{21}+... = \frac{\pi}{8}\tan(\tfrac{1}{8}\pi)$$

Proof: By $\mathbf{7_{tan^{\bullet}}}$, based on $\pi\tan(\pi z)\dfrac{1}{z-\frac{1}{8}}$,

$$\pi = 2(8)\cot(\tfrac{1}{8}\pi)\left\{\frac{1}{8-2}-\frac{1}{8+2} + \frac{1}{3(8)-2}-\frac{1}{3(8)+2} + \frac{1}{5(8)-2}-\frac{1}{5(8)+2}+...\right\}$$

11.3
$$\boxed{\pi = 8\sin(\tfrac{1}{8}\pi)\left(\frac{1}{1}+\frac{1}{7} - \frac{1}{9}-\frac{1}{15} + \frac{1}{17}+\frac{1}{23}+...\right)}$$

$$S_{1/8} = \frac{1}{1}+\frac{1}{7}-\frac{1}{9}-\frac{1}{15}+\frac{1}{17}+\frac{1}{23}-... = \frac{\pi}{8}\frac{1}{\sin(\frac{1}{8}\pi)}$$

$\sin(\tfrac{1}{8}\pi) = \dfrac{\sqrt{2-\sqrt{2}}}{2}$ depends on $\sqrt{2}$'s

Proof: By $\mathbf{7_{sin^{\bullet}}}$, based on $\pi\dfrac{1}{\sin(\pi z)}\dfrac{1}{z+\frac{1}{8}}$,

$$\pi = 8\sin(\tfrac{1}{8}\pi)\Big\{\frac{1}{1} + \frac{1}{8-1} - \frac{1}{8+1} - \frac{1}{2(8)-1} + \frac{1}{2(8)+1} + \frac{1}{3(8)-1} - \dots\Big\}$$

$$\sin(\tfrac{1}{8}\pi) = \sqrt{\tfrac{1}{2}\Big[1-\cos(\tfrac{1}{4}\pi)\Big]} = \sqrt{\tfrac{1}{2}\Big[1-\tfrac{\sqrt{2}}{2}\Big]} = \frac{\sqrt{2-\sqrt{2}}}{2}.\square$$

11.4

$$\boxed{\pi = 8\cos(\tfrac{1}{8}\pi)\Big(\frac{1}{3} + \frac{1}{5} - \frac{1}{11} - \frac{1}{13} + \frac{1}{19} + \frac{1}{21} + \dots\Big)}$$

$$C_{1/8} = \frac{1}{3} + \frac{1}{5} - \frac{1}{11} - \frac{1}{13} + \frac{1}{19} + \frac{1}{21} + \dots = \frac{\pi}{8\cos(\tfrac{1}{8}\pi)}$$

Proof: By $\mathbf{7_{cos}}$, based on $\pi\cos(\pi z)\dfrac{1}{z-\frac{1}{8}}$,

$$\pi = 2m\cos(\tfrac{1}{8}\pi)\Big\{\frac{1}{8-2} + \frac{1}{8+2} - \frac{1}{3(8)-2} - \frac{1}{3(8)+2} + \frac{1}{5(8)-2} + \frac{1}{5(8)+2} - \dots\Big\}$$

12.

12.1
$$\boxed{\pi = 8\tan(\tfrac{3}{8}\pi)\left(\frac{1}{3}-\frac{1}{5}+\frac{1}{11}-\frac{1}{13}+\frac{1}{19}-\frac{1}{21}+...\right)}$$

$$R_{3/8} = \frac{1}{3}-\frac{1}{5}+\frac{1}{11}-\frac{1}{13}+\frac{1}{19}-\frac{1}{21}+\frac{1}{27}-\frac{1}{29}+...=\frac{\pi}{8}\cot(\tfrac{3}{8}\pi)$$

$$\tan(\tfrac{3}{8}\pi) = \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}} = \textbf{depends on } \sqrt{2}\text{'s}$$

Proof: By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\dfrac{1}{z+\frac{3}{8}}$,

$$\pi = 8\tan(\tfrac{3}{8}\pi)\left\{\frac{1}{3}-\frac{1}{8-3}+\frac{1}{8+3}-\frac{1}{2(8)-3}+\frac{1}{2(8)+3}-\frac{1}{3(8)-3}+...\right\}$$

$$\cot(\tfrac{3}{8}\pi) = \frac{\cos(\tfrac{3}{8}\pi)}{\sin(\tfrac{3}{8}\pi)}$$

$$= \frac{\sqrt{\tfrac{1}{2}\left[1+\cos(\tfrac{3}{4}\pi)\right]}}{\sqrt{\tfrac{1}{2}\left[1-\cos(\tfrac{3}{4}\pi)\right]}}$$

$$= \frac{\sqrt{1-\tfrac{\sqrt{2}}{2}}}{\sqrt{1+\tfrac{\sqrt{2}}{2}}}$$

$$= \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}.\square$$

12.2

$$\boxed{\begin{aligned}\pi = 8\cot(\tfrac{3}{8}\pi)\Big(&\frac{1}{1}-\frac{1}{7}\\ &+\frac{1}{9}-\frac{1}{15}\\ &+\frac{1}{17}-\frac{1}{23}+...\Big)\end{aligned}}$$

$$T_{3/8} = \frac{1}{1}-\frac{1}{7}+\frac{1}{9}-\frac{1}{15}+\frac{1}{17}-\frac{1}{23}+\frac{1}{25}-\frac{1}{31}+...=\frac{\pi}{8}\cot(\tfrac{3}{8}\pi)$$

$$\tan(\tfrac{3}{8}\pi) = \frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}} = \text{depends on } \sqrt{2}\text{'s}$$

Proof: By $\mathbf{7_{tan}}$. based on $\pi\tan(\pi z)\dfrac{1}{z-\frac{3}{8}}$,

$$\begin{aligned}\pi = 2(8)\cot(\tfrac{3}{8}\pi)\Big\{&\frac{1}{8-2(3)}-\frac{1}{8+2(3)}\\ &+\frac{1}{3(8)-2(3)}-\frac{1}{3(8)+2(3)}\\ &+\frac{1}{5(8)-2(3)}-\frac{1}{5(8)+2(3)}+...\Big\}\end{aligned}$$

$$\pi = 8\cot(\tfrac{3}{8}\pi)\left(\frac{1}{1}-\frac{1}{7}+\frac{1}{9}-\frac{1}{15}+\frac{1}{17}-\frac{1}{23}+\frac{1}{25}-\frac{1}{31}+...\right)$$

12.3

$$\boxed{\begin{aligned}\pi = 8\sin(\tfrac{3}{8}\pi)\Big(&\frac{1}{3}+\frac{1}{5}\\ &-\frac{1}{11}-\frac{1}{13}\\ &+\frac{1}{19}+\frac{1}{21}-...\Big)\end{aligned}}$$

$$S_{3/8} = \frac{1}{3}+\frac{1}{5}-\frac{1}{11}-\frac{1}{13}+\frac{1}{19}+\frac{1}{21}-\frac{1}{27}-\frac{1}{29}+... = \frac{\pi}{8\sin(\frac{3}{8}\pi)}$$

$$\sin(\tfrac{3}{8}\pi) = \frac{\sqrt{2+\sqrt{2}}}{2} \text{ depends on } \sqrt{2}\text{'s}$$

Proof: By $\mathbf{7_{sin}}$. based on $\pi \dfrac{1}{\sin(\pi z)} \dfrac{1}{z+\frac{3}{8}}$,

$$\pi = 8\sin(\tfrac{3}{8}\pi)\left\{\frac{1}{3} + \frac{1}{8-3} - \frac{1}{8+3} - \frac{1}{2(8)-3} + \frac{1}{2(8)+3} + \frac{1}{3(8)-3} - \dots\right\}$$

$$\pi = 8\sin(\tfrac{3}{8}\pi)\left(\frac{1}{3} + \frac{1}{5} - \frac{1}{11} - \frac{1}{13} + \frac{1}{19} + \frac{1}{21} - \frac{1}{27} - \frac{1}{29} + \dots\right)$$

$$\sin(\tfrac{3}{8}\pi) = \sqrt{\tfrac{1}{2}\left[1-\cos(\tfrac{3}{4}\pi)\right]}$$

$$= \sqrt{\tfrac{1}{2}\left[1+\tfrac{\sqrt{2}}{2}\right]}$$

$$= \frac{\sqrt{2+\sqrt{2}}}{2}. \square$$

12.4

$$\boxed{\pi = 8\cos(\tfrac{3}{8}\pi)\left(\frac{1}{1} + \frac{1}{7} - \frac{1}{9} - \frac{1}{15} + \frac{1}{17} + \frac{1}{23} - \dots\right)}$$

$$C_{3/8} = \frac{1}{1} + \frac{1}{7} - \frac{1}{9} - \frac{1}{15} + \frac{1}{17} + \frac{1}{23} - \frac{1}{25} - \frac{1}{31} + \dots = \frac{\pi}{8\cos(\frac{3}{8}\pi)}$$

Proof: By $\mathbf{7_{cos}}$. based on $\pi \dfrac{1}{\cos(\pi z)} \dfrac{1}{z-\frac{3}{8}}$,

$$\pi = 2(8)\cos(\tfrac{3}{8}\pi)\left\{\frac{1}{8-2(3)} + \frac{1}{8+2(3)} - \frac{1}{3(8)-2(3)} - \frac{1}{3(8)+2(3)} + \frac{1}{5(8)-2(3)} + \frac{1}{5(8)+2(3)} - ...\right\}$$

$$\pi = 8\cos(\tfrac{3}{8}\pi)\left(\frac{1}{1} + \frac{1}{7} - \frac{1}{9} - \frac{1}{15} + \frac{1}{17} + \frac{1}{23} - \frac{1}{25} - \frac{1}{31} + ...\right)$$

13.

$R_{1/8} + R_{3/8} = S_{1/4}$

$$\frac{1}{1}+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}-\frac{1}{15}+\frac{1}{17}+\frac{1}{19}-\frac{1}{21}-\frac{1}{23}+\ldots=$$
$$=\frac{\pi}{8}[\cot(\tfrac{1}{8}\pi)+\cot(\tfrac{3}{8}\pi)]$$
$$=\frac{\pi}{4}\sqrt{2}$$

Proof: Based on $\pi\cot(\pi z)\dfrac{1}{z+\frac{1}{8}}$,

$$R_{1/8}=\frac{1}{1}-\frac{1}{7}+\frac{1}{9}-\frac{1}{15}+\frac{1}{17}-\frac{1}{23}+\frac{1}{25}-\ldots=\frac{\pi}{8}\sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}}$$

Based on $\pi\cot(\pi z)\dfrac{1}{z+\frac{3}{8}}$,

$$R_{3/8}=\frac{1}{3}-\frac{1}{5}+\frac{1}{11}-\frac{1}{13}+\frac{1}{19}-\frac{1}{21}+\frac{1}{27}+\ldots=\frac{\pi}{8}\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}$$

$$R_{1/8}+R_{3/8}=$$

$$\frac{1}{1}+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\frac{1}{13}-\frac{1}{15}+\frac{1}{17}+\frac{1}{19}-\frac{1}{21}-\frac{1}{23}+\ldots=$$

$$=\frac{\pi}{8}\left(\sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}}+\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}\right)$$

$$=\frac{\pi}{8}\frac{[2+\sqrt{2}]+[2-\sqrt{2}]}{\sqrt{2^2-2}}$$

$$=\frac{\pi}{8}\frac{4}{\sqrt{2}}=\frac{\pi}{4}\sqrt{2}=S_{1/4}$$

14.

$S_{1/8} - S_{3/8}$

$$1-\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{9}+\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\frac{1}{17}-\frac{1}{19}-\frac{1}{21}-\frac{1}{23}-\frac{1}{25}+\frac{1}{27}+\frac{1}{29}+\ldots = \frac{\pi}{8}\left\{\frac{1}{\sin(\frac{1}{8}\pi)}-\frac{1}{\sin(\frac{3}{8}\pi)}\right\}$$

Proof: Based on $\pi\frac{1}{\sin(\pi z)}\frac{1}{z+\frac{1}{8}}$,

$$S_{1/8} = \frac{1}{1}+\frac{1}{7}-\frac{1}{9}-\frac{1}{15}+\frac{1}{17}-\frac{1}{23}-\frac{1}{25}+\ldots = \frac{\pi}{8}\frac{1}{\sin(\frac{1}{8}\pi)}$$

Based on $\pi\frac{1}{\sin(\pi z)}\frac{1}{z+\frac{3}{8}}$,

$$S_{3/8} = \frac{1}{3}+\frac{1}{5}-\frac{1}{11}-\frac{1}{13}+\frac{1}{19}+\frac{1}{21}-\frac{1}{27}-\frac{1}{29}+\ldots = \frac{\pi}{8\sin(\frac{3}{8}\pi)}$$

$S_{1/8} - S_{3/8} =$

$$1-\frac{1}{3}-\frac{1}{5}+\frac{1}{7}-\frac{1}{9}+\frac{1}{11}+\frac{1}{13}-\frac{1}{15}+\frac{1}{17}-\frac{1}{19}-\frac{1}{21}-\frac{1}{23}-\frac{1}{25}+\frac{1}{27}+\frac{1}{29}\ldots$$

$$= \frac{\pi}{8}\left\{\frac{1}{\sin(\frac{1}{8}\pi)}-\frac{1}{\sin(\frac{3}{8}\pi)}\right\}$$

15.

$R_{1/8} - R_{3/8} = R_{1/4}$

$$\boxed{\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \frac{1}{21} - \frac{1}{23} + \frac{1}{25} - .. = \frac{\pi}{4}}$$

Proof:

based on $\pi\cot(\pi z)\dfrac{1}{z+\frac{1}{8}}$,

$$R_{1/8} = \frac{1}{1} - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \frac{1}{23} + \frac{1}{25} - ... = \frac{\pi}{8}\sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}}$$

Based on $\pi\cot(\pi z)\dfrac{1}{z+\frac{3}{8}}$,

$$R_{3/8} = \frac{1}{3} - \frac{1}{5} + \frac{1}{11} - \frac{1}{13} + \frac{1}{19} - \frac{1}{21} + \frac{1}{27} + ... = \frac{\pi}{8}\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}$$

$$R_{1/8} - R_{3/8} =$$

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \frac{1}{21} - \frac{1}{23} + \frac{1}{25} - ... =$$

$$= \frac{\pi}{8}\left(\sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}} - \sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}\right)$$

$$= \frac{\pi}{8}\frac{[2+\sqrt{2}] - [2-\sqrt{2}]}{\sqrt{2^2-2}}$$

$$= \frac{\pi}{8}\frac{2\sqrt{2}}{\sqrt{2}} = \frac{\pi}{4} = R_{1/4}$$

16.

16.1

$$\boxed{\pi = 16\tan(\tfrac{1}{16}\pi)\left(\frac{1}{1}-\frac{1}{15} + \frac{1}{17}-\frac{1}{31} + \frac{1}{33}-\frac{1}{47}+...\right)}$$

$$R_{1/16} = \frac{1}{1}-\frac{1}{15}+\frac{1}{17}-\frac{1}{31}+\frac{1}{33}-\frac{1}{47}+\frac{1}{49}-...=\frac{\pi}{16}\cot(\tfrac{1}{16}\pi)$$

$$\tan(\tfrac{1}{16}\pi) = \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}} = \textbf{depends on } \sqrt{2}\text{'s}$$

<u>*Proof*</u>: By $\mathbf{7_{cot}}$, based on $\pi\cot(\pi z)\dfrac{1}{z+\frac{1}{16}}$,

$$\pi = 16\tan(\tfrac{1}{16}\pi)\left\{\frac{1}{1}-\frac{1}{16-1} + \frac{1}{16+1}-\frac{1}{2(16)-1} + \frac{1}{2(16)+1}-\frac{1}{3(16)-1}+...\right\}$$

$$\pi = 16\tan(\tfrac{1}{16}\pi)\left(\frac{1}{1}-\frac{1}{15}+\frac{1}{17}-\frac{1}{31}+\frac{1}{33}-\frac{1}{47}+\frac{1}{49}-...\right).\square$$

$$\cot(\tfrac{1}{16}\pi) = \frac{\cos(\tfrac{1}{16}\pi)}{\sin(\tfrac{1}{16}\pi)}$$

$$= \frac{\sqrt{\tfrac{1}{2}\left[1+\cos(\tfrac{1}{8}\pi)\right]}}{\sqrt{\tfrac{1}{2}\left[1-\cos(\tfrac{1}{8}\pi)\right]}}$$

$$= \frac{\sqrt{1+\sqrt{\frac{1}{2}[1+\cos(\frac{1}{4}\pi)]}}}{\sqrt{1-\sqrt{\frac{1}{2}[1+\cos(\frac{1}{4}\pi)]}}}$$

$$= \frac{\sqrt{1+\sqrt{\frac{1}{2}[1+\frac{\sqrt{2}}{2}]}}}{\sqrt{1-\sqrt{\frac{1}{2}[1+\frac{\sqrt{2}}{2}]}}}$$

$$= \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{\sqrt{2-\sqrt{2+\sqrt{2}}}}.\square$$

16.2

$$\boxed{\pi = 16\cot(\tfrac{1}{16}\pi)\left(\frac{1}{7}-\frac{1}{9} + \frac{1}{23}-\frac{1}{25} + \frac{1}{39}-\frac{1}{41}+...\right)}$$

$$T_{1/16} = \frac{1}{7}-\frac{1}{9}+\frac{1}{23}-\frac{1}{25}+\frac{1}{39}-\frac{1}{41}-...=\frac{\pi}{16}\tan(\tfrac{1}{16}\pi)$$

$$\tan(\tfrac{1}{16}\pi) = \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}} = \textbf{depends on } \sqrt{2}\text{'s}$$

Proof: By $\mathbf{7_{tan}}$., based on $\pi\tan(\pi z)\dfrac{1}{z-\frac{1}{16}}$,

$$\pi = 2(16)\cot(\tfrac{1}{16}\pi)\left\{\frac{1}{16-2}-\frac{1}{16+2} + \frac{1}{3(16)-2}-\frac{1}{3(16)+2} + \frac{1}{5(16)-2}-\frac{1}{5(16)+2}+...\right\}.\square$$

16.3

$$\boxed{\pi = 16\sin(\tfrac{1}{16}\pi)\left(\frac{1}{1} + \frac{1}{15} - \frac{1}{17} - \frac{1}{31} + \frac{1}{33} + \frac{1}{47} - ...\right)}$$

$$S_{1/16} = \frac{1}{1} + \frac{1}{15} - \frac{1}{17} - \frac{1}{31} + \frac{1}{33} + \frac{1}{47} - \frac{1}{49} - \frac{1}{51} + ... = \frac{\pi}{16\sin(\frac{1}{16}\pi)}$$

$$\sin(\tfrac{1}{16}\pi) = \frac{\sqrt{2 - \sqrt{2 + \sqrt{2}}}}{2} = \textbf{depends on } \sqrt{2}$$

Proof: By $\mathbf{7_{sin}}$. based on $\pi \dfrac{1}{\sin(\pi z)} \dfrac{1}{z + \frac{1}{16}}$,

$$\pi = 16\sin(\tfrac{1}{16}\pi)\left\{\frac{1}{1} + \frac{1}{16-1} - \frac{1}{16+1} - \frac{1}{2(16)-1} + \frac{1}{2(16)+1} + \frac{1}{3(16)-1} - ...\right\}$$

$$\sin(\tfrac{1}{16}\pi) = \sqrt{\tfrac{1}{2}\left[1 - \cos(\tfrac{1}{8}\pi)\right]}$$

$$= \sqrt{\tfrac{1}{2}\left[1 - \sqrt{\tfrac{1}{2}\left[1 + \cos(\tfrac{1}{4}\pi)\right]}\right]}$$

$$= \sqrt{\tfrac{1}{2}\left[1 - \sqrt{\tfrac{1}{2}\left[1 + \tfrac{\sqrt{2}}{2}\right]}\right]}$$

$$= \frac{\sqrt{2 - \sqrt{2 + \sqrt{2}}}}{2}$$

16.4

$$\boxed{\pi = 16\cos\left(\tfrac{1}{16}\pi\right)\left(\frac{1}{7}+\frac{1}{9} - \frac{1}{23}-\frac{1}{25} + \frac{1}{39}+\frac{1}{41}-...\right)}$$

$$C_{1/16} = \frac{1}{7}+\frac{1}{9}-\frac{1}{23}-\frac{1}{25}+\frac{1}{39}+\frac{1}{41}-... = \frac{\pi}{16\cos(\frac{1}{16}\pi)}$$

Proof: By $\mathbf{7_{cos}}$. based on $\pi\dfrac{1}{\cos(\pi z)}\dfrac{1}{z-\frac{1}{16}}$,

$$\pi = 2(16)\cos(\tfrac{1}{16}\pi)\left\{\frac{1}{16-2}+\frac{1}{16+2} - \frac{1}{3(16)-2}-\frac{1}{3(16)+2} + \frac{1}{5(16)-2}+\frac{1}{5(16)+2}-...\right\}$$

17.

17.1

$$\boxed{\pi = 16\tan(\tfrac{3}{16}\pi)\left(\frac{1}{3}-\frac{1}{13}+\frac{1}{19}-\frac{1}{29}+\frac{1}{35}-\frac{1}{45}+\ldots\right)}$$

$$R_{3/16} = \frac{1}{3}-\frac{1}{13}+\frac{1}{19}-\frac{1}{29}+\frac{1}{35}-\frac{1}{45}+\frac{1}{51}-\ldots=\frac{\pi}{16}\cot(\tfrac{3}{16}\pi)$$

$$\cot(\tfrac{3}{16}\pi) = \frac{\sqrt{2+\sqrt{2-\sqrt{2}}}}{\sqrt{2-\sqrt{2-\sqrt{2}}}} = \textbf{depends on } \sqrt{2}\text{'s}$$

Proof: By $\mathbf{7_{cot}}$. based on $\pi\cot(\pi z)\dfrac{1}{z+\frac{3}{16}}$,

$$\pi = m\tan(\tfrac{l}{m}\pi)\left\{\frac{1}{3}-\frac{1}{16-3}+\frac{1}{16+3}-\frac{1}{2(16)-3}+\frac{1}{2(16)+3}-\frac{1}{3(16)-3}+\ldots\right\}.\square$$

$$\cot(\tfrac{3}{16}\pi) = \frac{\cos(\frac{3}{16}\pi)}{\sin(\frac{3}{16}\pi)}$$

$$= \frac{\sqrt{\frac{1}{2}\left[1+\cos(\frac{3}{8}\pi)\right]}}{\sqrt{\frac{1}{2}\left[1-\cos(\frac{3}{8}\pi)\right]}}$$

$$= \frac{\sqrt{1+\sqrt{\frac{1}{2}[1+\cos\frac{3}{4}\pi]}}}{\sqrt{1-\sqrt{\frac{1}{2}[1+\cos\frac{3}{4}\pi]}}}$$

$$= \frac{\sqrt{1+\sqrt{\frac{1}{2}[1-\frac{\sqrt{2}}{2}]}}}{\sqrt{1-\sqrt{\frac{1}{2}[1-\frac{\sqrt{2}}{2}]}}}$$

$$= \frac{\sqrt{2+\sqrt{2-\sqrt{2}}}}{\sqrt{2-\sqrt{2-\sqrt{2}}}}.\square$$

17.2

$$\boxed{\pi = 16\cot(\tfrac{3}{16}\pi)\left\{\frac{1}{5}-\frac{1}{11} + \frac{1}{21}-\frac{1}{27} + \frac{1}{37}-\frac{1}{43}+...\right\}}$$

$$T_{3/16} = \frac{1}{5}-\frac{1}{11}+\frac{1}{21}-\frac{1}{27}+\frac{1}{37}-\frac{1}{43}+... = \frac{\pi}{16}\tan(\tfrac{3}{16}\pi)$$

Proof: By $\mathbf{7_{tan}}$. based on $\pi\tan(\pi z)\dfrac{1}{z-\frac{3}{16}}$,

$$\pi = 2(16)\cot(\tfrac{3}{16}\pi)\left\{\frac{1}{16-2(3)}-\frac{1}{16+2(3)} + \frac{1}{3(16)-2(3)}-\frac{1}{3(16)+2(3)} + \frac{1}{5(16)-2(3)}-\frac{1}{5(16)+2(3)}+...\right\}$$

17.3
$$\boxed{\pi = 16\sin(\tfrac{3}{16}\pi)\left(\frac{1}{3}+\frac{1}{13} - \frac{1}{19}-\frac{1}{29} + \frac{1}{35}+\frac{1}{45}-...\right)}$$

$$S_{3/16} = \frac{1}{3}+\frac{1}{13}-\frac{1}{19}-\frac{1}{29}+\frac{1}{35}+\frac{1}{45}-... = \frac{\pi}{16\sin(\frac{3}{16}\pi)}$$

$$\sin(\tfrac{3}{16}\pi) = \frac{\sqrt{2-\sqrt{2-\sqrt{2}}}}{2} = \textbf{depends on } \sqrt{2}\textbf{'s}$$

Proof: By $\mathbf{7_{sin}}$. based on $\pi\frac{1}{\sin(\pi z)}\frac{1}{z+\frac{3}{16}}$,

$$\pi = 16\sin(\tfrac{3}{16}\pi)\left\{\frac{1}{3}+\frac{1}{16-3} - \frac{1}{16+3}-\frac{1}{2(16)-3} + \frac{1}{2(16)+3}+\frac{1}{3(16)-3}-...\right\}. \square$$

$$\sin(\tfrac{3}{16}\pi) = \sqrt{\tfrac{1}{2}\left[1-\cos(\tfrac{3}{8}\pi)\right]}$$

$$= \sqrt{\tfrac{1}{2}\left[1-\sqrt{\tfrac{1}{2}\left[1+\cos(\tfrac{3}{4}\pi)\right]}\right]}$$

$$= \sqrt{\tfrac{1}{2}\left[1-\sqrt{\tfrac{1}{2}\left[1-\tfrac{\sqrt{2}}{2}\right]}\right]} = \frac{\sqrt{2-\sqrt{2-\sqrt{2}}}}{2}. \square$$

17.4

$$\boxed{\pi = 16\cos(\tfrac{3}{16}\pi)\left\{\frac{1}{5}+\frac{1}{11} -\frac{1}{21}-\frac{1}{27} +\frac{1}{37}+\frac{1}{43}+...\right\}}$$

$$C_{3/16} = \frac{1}{5}+\frac{1}{11}-\frac{1}{21}-\frac{1}{27}+\frac{1}{37}+\frac{1}{43}-...=\frac{\pi}{16\cos(\frac{3}{16}\pi)}$$

Proof: By $\mathbf{7_{cos}}$. based on $\pi\dfrac{1}{\cos(\pi z)}\dfrac{1}{z-\frac{3}{16}}$,

$$\pi = 2(16)\cos(\tfrac{3}{16}\pi)\left\{\frac{1}{16-2(3)}+\frac{1}{16+2(3)} -\frac{1}{3(16)-2(3)}-\frac{1}{3(16)+2(3)} +\frac{1}{5(16)-2(3)}+\frac{1}{5(16)+2(3)}+...\right\}$$

18.

$R_{1/16} + R_{3/16} + R_{5/16} + R_{7/16}$

$$R_{1/16} = \boxed{\begin{aligned}\frac{1}{1} - \frac{1}{15} + \frac{1}{17} - \frac{1}{31} + \frac{1}{33} - \frac{1}{47} + \frac{1}{49} - ... &= \frac{\pi}{16}\cot(\tfrac{1}{16}\pi) \\ &= \frac{\pi}{16}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{\sqrt{2-\sqrt{2+\sqrt{2}}}}\end{aligned}}$$

$$R_{3/16} = \boxed{\begin{aligned}\frac{1}{3} - \frac{1}{13} + \frac{1}{19} - \frac{1}{29} + \frac{1}{35} - \frac{1}{45} + \frac{1}{51} - ... &= \frac{\pi}{16}\cot(\tfrac{3}{16}\pi) \\ &= \frac{\pi}{16}\frac{\sqrt{2+\sqrt{2-\sqrt{2}}}}{\sqrt{2-\sqrt{2-\sqrt{2}}}}\end{aligned}}$$

$$R_{5/16} = \boxed{\begin{aligned}\frac{1}{5} - \frac{1}{11} + \frac{1}{21} - \frac{1}{27} + \frac{1}{37} - \frac{1}{43} + \frac{1}{53} - ... &= \frac{\pi}{16}\cot(\tfrac{5}{16}) \\ &= \frac{\pi}{16}\frac{\sqrt{2-\sqrt{2-\sqrt{2}}}}{\sqrt{2+\sqrt{2-\sqrt{2}}}}\end{aligned}}$$

$$R_{7/16} = \boxed{\begin{aligned}\frac{1}{7} - \frac{1}{9} + \frac{1}{23} - \frac{1}{25} + \frac{1}{39} - \frac{1}{41} + \frac{1}{55} - ... &= \frac{\pi}{16}\cot(\tfrac{7}{16}\pi) \\ &= \frac{\pi}{16}\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}}\end{aligned}}$$

$R_{1/16} + R_{3/16} + R_{5/16} + R_{7/16} =$

$$= \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{9} - \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \frac{1}{21} + \frac{1}{23} - ...$$

$$= \frac{\pi}{16}\underbrace{\left(\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{\sqrt{2-\sqrt{2+\sqrt{2}}}} + \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}}\right)}_{\frac{4}{\sqrt{2-\sqrt{2}}}} + \frac{\pi}{16}\underbrace{\left(\frac{\sqrt{2+\sqrt{2-\sqrt{2}}}}{\sqrt{2-\sqrt{2-\sqrt{2}}}} + \frac{\sqrt{2-\sqrt{2-\sqrt{2}}}}{\sqrt{2+\sqrt{2-\sqrt{2}}}}\right)}_{\frac{4}{\sqrt{2+\sqrt{2}}}}$$

$$= \frac{\pi}{4}\left(\frac{1}{\sqrt{2-\sqrt{2}}} + \frac{1}{\sqrt{2+\sqrt{2}}}\right)$$

$$= \frac{\pi}{4}\frac{\sqrt{2+\sqrt{2}} + \sqrt{2-\sqrt{2}}}{\sqrt{(2-\sqrt{2})(2+\sqrt{2})}}$$

$$= \frac{\pi}{4}\frac{\sqrt{\left(\sqrt{2+\sqrt{2}} + \sqrt{2-\sqrt{2}}\right)^2}}{\sqrt{2}}$$

$$= \frac{\pi}{4}\frac{\sqrt{2+\sqrt{2}+2-\sqrt{2}+2\sqrt{2^2-2}}}{\sqrt{2}}$$

$$= \frac{\pi}{4}\sqrt{2+\sqrt{2}}$$

19.

$R_{1/16} - R_{3/16}$

$$R_{1/16} = \boxed{\begin{aligned} &\frac{1}{1} - \frac{1}{15} + \frac{1}{17} - \frac{1}{31} + \frac{1}{33} - \frac{1}{47} + \frac{1}{49} - \ldots = \frac{\pi}{16}\cot(\tfrac{1}{16}\pi) \\ &= \frac{\pi}{16}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{\sqrt{2-\sqrt{2+\sqrt{2}}}} \end{aligned}}$$

$$R_{3/16} = \boxed{\begin{aligned} &\frac{1}{3} - \frac{1}{13} + \frac{1}{19} - \frac{1}{29} + \frac{1}{35} - \frac{1}{45} + \frac{1}{51} - \ldots = \frac{\pi}{16}\cot(\tfrac{3}{16}\pi) \\ &= \frac{\pi}{16}\frac{\sqrt{2+\sqrt{2-\sqrt{2}}}}{\sqrt{2-\sqrt{2-\sqrt{2}}}} \end{aligned}}$$

$R_{1/16} - R_{3/16} =$

$$\frac{1}{1} - \frac{1}{3} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \frac{1}{29} - \frac{1}{31} + \frac{1}{33} +$$

$$-\frac{1}{35} + \frac{1}{45} - \frac{1}{47} + \frac{1}{49} - \frac{1}{51} + \frac{1}{61} + \ldots =$$

$$= \frac{\pi}{16}\left\{\cot(\tfrac{1}{16}\pi) - \cot(\tfrac{3}{16}\pi)\right\}$$

20.

$R_{5/16} - R_{7/16}$

$$R_{5/16} = \boxed{\begin{aligned} &\frac{1}{5} - \frac{1}{11} + \frac{1}{21} - \frac{1}{27} + \frac{1}{37} - \frac{1}{43} + \frac{1}{53} - \ldots = \frac{\pi}{16}\cot(\tfrac{5}{16}) \\ &= \frac{\pi}{16}\frac{\sqrt{2-\sqrt{2-\sqrt{2}}}}{\sqrt{2+\sqrt{2-\sqrt{2}}}} \end{aligned}}$$

$$R_{7/16} = \boxed{\begin{aligned} &\frac{1}{7} - \frac{1}{9} + \frac{1}{23} - \frac{1}{25} + \frac{1}{39} - \frac{1}{41} + \frac{1}{55} - \ldots = \frac{\pi}{16}\cot(\tfrac{7}{16}\pi) \\ &= \frac{\pi}{16}\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}} \end{aligned}}$$

$R_{5/16} - R_{7/16} =$

$$\frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{21} - \frac{1}{23} + \frac{1}{25} - \frac{1}{27} + \frac{1}{37} +$$

$$-\frac{1}{39} + \frac{1}{41} - \frac{1}{43} + \frac{1}{53} - \frac{1}{55} + \frac{1}{57} + \ldots$$

$$= \frac{\pi}{16}\Big\{\cot(\tfrac{5}{16}) - \cot(\tfrac{7}{16})\Big\}$$

21.

$R_{3/16} - R_{5/16}$

$$R_{3/16} = \boxed{\begin{aligned}\frac{1}{3}-\frac{1}{13}+\frac{1}{19}-\frac{1}{29}+\frac{1}{35}-\frac{1}{45}+\frac{1}{51}-...&=\frac{\pi}{16}\cot(\tfrac{3}{16}\pi)\\ &=\frac{\pi}{16}\frac{\sqrt{2+\sqrt{2-\sqrt{2}}}}{\sqrt{2-\sqrt{2-\sqrt{2}}}}\end{aligned}}$$

$$R_{5/16} = \boxed{\begin{aligned}\frac{1}{5}-\frac{1}{11}+\frac{1}{21}-\frac{1}{27}+\frac{1}{37}-\frac{1}{43}+\frac{1}{53}-...&=\frac{\pi}{16}\cot(\tfrac{5}{16})\\ &=\frac{\pi}{16}\frac{\sqrt{2-\sqrt{2-\sqrt{2}}}}{\sqrt{2+\sqrt{2-\sqrt{2}}}}\end{aligned}}$$

$R_{3/16} - R_{5/16}$

$$\frac{1}{3}-\frac{1}{5}+\frac{1}{11}-\frac{1}{13}+\frac{1}{19}-\frac{1}{21}+\frac{1}{27}-\frac{1}{29}+\frac{1}{35}$$

$$-\frac{1}{37}+\frac{1}{43}-\frac{1}{45}+\frac{1}{51}-\frac{1}{53}+\frac{1}{59}-... =$$

$$= \frac{\pi}{16}\left\{\cot(\tfrac{3}{16}\pi)-\cot(\tfrac{5}{16}\pi)\right\}$$

22.

$R_{1/16} - R_{5/16}$

$$R_{1/16} = \boxed{\begin{aligned}\frac{1}{1} - \frac{1}{15} + \frac{1}{17} - \frac{1}{31} + \frac{1}{33} - \frac{1}{47} + \frac{1}{49} - \ldots &= \frac{\pi}{16}\cot(\tfrac{1}{16}\pi) \\ &= \frac{\pi}{16}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{\sqrt{2-\sqrt{2+\sqrt{2}}}}\end{aligned}}$$

$$R_{5/16} = \boxed{\begin{aligned}\frac{1}{5} - \frac{1}{11} + \frac{1}{21} - \frac{1}{27} + \frac{1}{37} - \frac{1}{43} + \frac{1}{53} - \ldots &= \frac{\pi}{16}\cot(\tfrac{5}{16}) \\ &= \frac{\pi}{16}\frac{\sqrt{2-\sqrt{2-\sqrt{2}}}}{\sqrt{2+\sqrt{2-\sqrt{2}}}}\end{aligned}}$$

$R_{1/16} - R_{5/16} =$

$$\frac{1}{1} - \frac{1}{5} - \frac{1}{11} + \frac{1}{15} + \frac{1}{17} - \frac{1}{21} - \frac{1}{27} + \frac{1}{31} + \frac{1}{33}$$

$$-\frac{1}{37} - \frac{1}{43} + \frac{1}{47} + \frac{1}{49} - \frac{1}{53} - \frac{1}{59} + \ldots =$$

$$= \frac{\pi}{16}\left\{\cot(\tfrac{1}{16}\pi) - \cot(\tfrac{5}{16}\pi)\right\}$$

23.

$R_{1/16} - R_{7/16}$

$$R_{1/16} = \boxed{\begin{aligned} \frac{1}{1} - \frac{1}{15} + \frac{1}{17} - \frac{1}{31} + \frac{1}{33} - \frac{1}{47} + \frac{1}{49} - ... &= \frac{\pi}{16}\cot(\tfrac{1}{16}\pi) \\ &= \frac{\pi}{16}\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{\sqrt{2-\sqrt{2+\sqrt{2}}}} \end{aligned}}$$

$$R_{7/16} = \boxed{\begin{aligned} \frac{1}{7} - \frac{1}{9} + \frac{1}{23} - \frac{1}{25} + \frac{1}{39} - \frac{1}{41} + \frac{1}{55} - ... &= \frac{\pi}{16}\cot(\tfrac{7}{16}\pi) \\ &= \frac{\pi}{16}\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}} \end{aligned}}$$

$R_{1/16} - R_{7/16} =$

$$\frac{1}{1} - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \frac{1}{23} + \frac{1}{25} - \frac{1}{31} + \frac{1}{33}$$

$$-\frac{1}{39} + \frac{1}{41} - \frac{1}{47} + \frac{1}{49} - \frac{1}{55} + \frac{1}{57} - ... =$$

$$= \frac{\pi}{16}\left\{\cot(\tfrac{1}{16}\pi) - \cot(\tfrac{7}{16}\pi)\right\}$$

24.

$(R_{3/16} - R_{5/16})(R_{1/16} - R_{7/16})$

$$R_{3/16} - R_{5/16} = \frac{1}{3} - \frac{1}{5} + \frac{1}{11} - \frac{1}{13} + \frac{1}{19} - \frac{1}{21} + \frac{1}{27} - \frac{1}{29} + \frac{1}{35}$$

$$-\frac{1}{37} + \frac{1}{43} - \frac{1}{45} + \frac{1}{51} - \frac{1}{53} + \frac{1}{59} - ... = \frac{\pi}{16}\left\{\cot(\tfrac{3}{16}\pi) - \cot(\tfrac{5}{16}\pi)\right\}$$

$$R_{1/16} - R_{7/16} = \frac{1}{1} - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \frac{1}{23} + \frac{1}{25} - \frac{1}{31} + \frac{1}{33}$$

$$-\frac{1}{39} + \frac{1}{41} - \frac{1}{47} + \frac{1}{49} - \frac{1}{55} + \frac{1}{57} - ... = \frac{\pi}{16}\left\{\cot(\tfrac{1}{16}\pi) - \cot(\tfrac{7}{16}\pi)\right\}$$

$$(R_{3/16} - R_{5/16})(R_{1/16} - R_{7/16}) =$$

$$= \frac{\pi^2}{16^2}\left\{\cot(\tfrac{3}{16}\pi) - \cot(\tfrac{5}{16}\pi)\right\}\left\{\cot(\tfrac{1}{16}\pi) - \cot(\tfrac{7}{16}\pi)\right\}$$

25.

$R_{3/16} - R_{7/16}$

$$R_{3/16} = \boxed{\begin{aligned} &\frac{1}{3} - \frac{1}{13} + \frac{1}{19} - \frac{1}{29} + \frac{1}{35} - \frac{1}{45} + \frac{1}{51} - ... = \frac{\pi}{16}\cot(\tfrac{3}{16}\pi) \\ &= \frac{\pi}{16}\frac{\sqrt{2+\sqrt{2-\sqrt{2}}}}{\sqrt{2-\sqrt{2-\sqrt{2}}}} \end{aligned}}$$

$$R_{7/16} = \boxed{\begin{aligned} &\frac{1}{7} - \frac{1}{9} + \frac{1}{23} - \frac{1}{25} + \frac{1}{39} - \frac{1}{41} + \frac{1}{55} - ... = \frac{\pi}{16}\cot(\tfrac{7}{16}\pi) \\ &= \frac{\pi}{16}\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{\sqrt{2+\sqrt{2+\sqrt{2}}}} \end{aligned}}$$

$R_{3/16} - R_{7/16} =$

$$\frac{1}{3} - \frac{1}{7} + \frac{1}{9} - \frac{1}{13} + \frac{1}{19} - \frac{1}{23} + \frac{1}{25} - \frac{1}{29} + \frac{1}{35} -$$

$$-\frac{1}{39} + \frac{1}{41} - \frac{1}{45} + \frac{1}{51} - \frac{1}{55} + \frac{1}{61} - ... = \frac{\pi}{16}\{\cot(\tfrac{1}{16}\pi) - \cot(\tfrac{7}{16}\pi)\}$$

26.

26.1

$$\boxed{\pi = p\tan(\tfrac{1}{p}\pi)\left\{\frac{1}{1}-\frac{1}{p-1} + \frac{1}{p+1}-\frac{1}{2p-1} + \frac{1}{2p+1}-\frac{1}{3p-1}+...\right\}}$$

$p=$ prime.

$$R_{1/p} = 1-\frac{1}{p-1}+\frac{1}{p+1}-\frac{1}{2p-1}+\frac{1}{2p+1}-\frac{1}{3p-1}+... = \frac{\pi}{p}\cot(\tfrac{1}{p}\pi)$$

Proof: By $\mathbf{7._{cot}}$ based on $\pi\cot(\pi z)\frac{1}{z+\frac{1}{p}}$.

26.2

$$\boxed{\pi = 2p\cot(\tfrac{1}{p}\pi)\left\{\frac{1}{p-2}-\frac{1}{p+2} + \frac{1}{3p-2}-\frac{1}{3p+2} + \frac{1}{5p-2}-\frac{1}{5p+2}+...\right\}}$$

$p=$ prime.

$$T_{1/p} = \frac{1}{p-2}-\frac{1}{p+2}+\frac{1}{3p-2}-\frac{1}{3p+2}+\frac{1}{5p-2}-\frac{1}{5p+2}+... = \frac{\pi}{p}\cot(\tfrac{1}{p}\pi)$$

Proof: By $\mathbf{7_{tan}}$ based on $\pi\tan(\pi z)\frac{1}{z-\frac{1}{p}}$.

16.3

$$\boxed{\begin{aligned}\pi = p\sin(\tfrac{1}{p}\pi)\Big\{&\frac{1}{1}+\frac{1}{p-1}\\ &-\frac{1}{p+1}-\frac{1}{2p-1}\\ &+\frac{1}{2p+1}+\frac{1}{3p-1}-...\Big\}\end{aligned}}$$

$$S_{1/p} = 1+\frac{1}{p-1}-\frac{1}{p+1}-\frac{1}{2p-1}+\frac{1}{2p+1}+\frac{1}{3p-1}-... = \frac{\pi}{p\sin(\frac{1}{p}\pi)}$$

Proof: By $\mathbf{7_{sin}}$ based on $\pi\dfrac{1}{\sin(\pi z)}\dfrac{1}{z+\frac{1}{p}}$,

26.4

$$\boxed{\begin{aligned}\pi = 2p\cos(\tfrac{1}{p}\pi)\Big\{&\frac{1}{p-2}+\frac{1}{p+2}\\ &-\frac{1}{3p-2}-\frac{1}{3p+2}\\ &+\frac{1}{5p-2}+\frac{1}{5p+2}+...\Big\}\end{aligned}}$$

$p =$ prime.

$$C_{1/p} = \frac{1}{p-2}+\frac{1}{p+2}-\frac{1}{3p-2}-\frac{1}{3p+2}+\frac{1}{5p-2}+\frac{1}{5p+2}-... = \frac{\pi}{2p\cos(\frac{1}{p}\pi)}$$

Proof: By $\mathbf{7_{cos}}$ based on $\pi\dfrac{1}{\cos(\pi z)}\dfrac{1}{z-\frac{1}{p}}$.

27.

27.1

$$\pi = 3\underbrace{\tan(\tfrac{1}{3}\pi)}_{\sqrt{3}}\left(\frac{1}{1}-\frac{1}{2} + \frac{1}{4}-\frac{1}{5} + \frac{1}{7}-\frac{1}{8}+...\right)$$

$$R_{1/3} = \frac{1}{1}-\frac{1}{2}+\frac{1}{4}-\frac{1}{5}+\frac{1}{7}-\frac{1}{8}+\frac{1}{10}-\frac{1}{11}+... = \frac{\pi}{3\sqrt{3}}$$

Proof: By $\mathbf{7_{cot}}$ based on $\pi\cot(\pi z)\dfrac{1}{z+\frac{1}{3}}$.□

27.2

$$\pi = 6\underbrace{\cot(\tfrac{1}{3}\pi)}_{1/\sqrt{3}}\left(\frac{1}{1}-\frac{1}{5} + \frac{1}{7}-\frac{1}{11} + \frac{1}{13}-\frac{1}{17}+...\right)$$

$$T_{1/3} = \frac{1}{1}-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\frac{1}{17}+\frac{1}{19}-\frac{1}{23}+... = \frac{\pi}{2\sqrt{3}}$$

Proof: By $\mathbf{7_{tan}}$ based on $\pi\tan(\pi z)\dfrac{1}{z-\frac{1}{3}}$.□

27.3
$$\boxed{\pi = 3\underbrace{\sin(\tfrac{1}{3}\pi)}_{\sqrt{3}/2}\left(\frac{1}{1}+\frac{1}{2} -\frac{1}{4}-\frac{1}{5} +\frac{1}{7}+\frac{1}{8}+\ldots\right)}$$

$$S_{1/3} = \frac{1}{1}+\frac{1}{2}-\frac{1}{4}-\frac{1}{5}+\frac{1}{7}+\frac{1}{8}-\frac{1}{10}-\frac{1}{11}+\ldots = 2\frac{\pi}{3\sqrt{3}}$$

Proof: By $\mathbf{7_{sin}}$ based on $\pi\dfrac{1}{\sin(\pi z)}\dfrac{1}{z+\frac{1}{3}}$.□

27.4
$$\boxed{\pi = 6\underbrace{\cos(\tfrac{1}{3}\pi)}_{1/2}\left(\frac{1}{1}+\frac{1}{5} -\frac{1}{7}-\frac{1}{11} +\frac{1}{13}+\frac{1}{17}+\ldots\right)}$$

$$C_{1/3} = \frac{1}{1}+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\frac{1}{13}+\frac{1}{17}-\frac{1}{19}-\frac{1}{23}+\ldots = \frac{\pi}{3}$$

Proof: By $\mathbf{7_{cos}}$ based on $\pi\dfrac{1}{\cos(\pi z)}\dfrac{1}{z-\frac{1}{3}}$.□

28.

28.1

$$\pi = p^n \tan(\tfrac{l}{p^n}\pi)\left\{\frac{1}{l} - \frac{1}{p^n - l} + \frac{1}{p^n + l} - \frac{1}{2p^n - l} + \frac{1}{2p^n + l} - \frac{1}{3p^n - l} + \dots\right\}$$

$$R_{l/p^n} = \frac{1}{l} - \frac{1}{p^n - l} + \frac{1}{p^n + l} - \frac{1}{2p^n - l} + \frac{1}{2p^n + l} - \frac{1}{3p^n - l} + \dots = \frac{\pi}{p^n}\cot(\tfrac{l}{p^n}\pi)$$

$l < \frac{1}{2}p^n$, l and p^n have no common factor, $n = 1,2,3,\dots$

__Proof__: By $\mathbf{7}_{\mathbf{cot}}$ based on $\pi\cot(\pi z)\dfrac{1}{z + \frac{l}{p^n}}$.□

28.2

$$\pi = 2p^n \cot(\tfrac{l}{p^n}\pi)\left\{\frac{1}{p^n - 2l} - \frac{1}{p^n + 2l} + \frac{1}{3p^n - 2l} - \frac{1}{3p^n + 2l} + \frac{1}{5p^n - 2l} - \frac{1}{5p^n + 2l} + \dots\right\}$$

$$T_{l/p^n} = \frac{1}{p^n - 2l} - \frac{1}{p^n + 2l} + \frac{1}{3p^n - 2l} - \frac{1}{3p^n + 2l} + \frac{1}{5p^n - 2l} - \frac{1}{5p^n + 2l} + \dots = \frac{\pi}{2p^n}\tan(\tfrac{l}{p^n}\pi)$$

$l < \frac{1}{2}p^n$, l and p^n have no common factor, $n = 1, 2, 3, \ldots$

Proof: By $\mathbf{7_{tan}}$ based on $\pi \tan(\pi z)\dfrac{1}{z - \frac{l}{p^n}}$.□

28.3

$$\boxed{\pi = p^n \sin(\tfrac{l}{p^n}\pi)\left\{\frac{1}{l} + \frac{1}{p^n - l} - \frac{1}{p^n + l} - \frac{1}{2p^n - l} + \frac{1}{2p^n + l} + \frac{1}{3p^n - l} - \ldots\right\}}$$

$$S_{l/p^n} = \frac{1}{l} + \frac{1}{p^n - l} - \frac{1}{p^n + l} - \frac{1}{2p^n - l} + \frac{1}{2p^n + l} + \frac{1}{3p^n - l} - \ldots = \frac{\pi}{p^n}\frac{1}{\sin(\frac{l}{p^n}\pi)}$$

Proof: By $\mathbf{7_{sin}}$ based on $\pi \dfrac{1}{\sin(\pi z)}\dfrac{1}{z + \frac{l}{p^n}}$,

28.4

$$\boxed{\pi = 2p^n \cos(\tfrac{l}{p^n}\pi)\left\{\frac{1}{p^n - 2l} + \frac{1}{p^n + 2l} - \frac{1}{3p^n - 2l} - \frac{1}{3p^n + 2l} + \frac{1}{5p^n - 2l} + \frac{1}{5p^n + 2l} + \ldots\right\}}$$

$$C_{l/p^n} = \frac{1}{p^n - 2l} + \frac{1}{p^n + 2l} - \frac{1}{3p^n - 2l} - \frac{1}{3p^n + 2l} + \frac{1}{5p^n - 2l} + \frac{1}{5p^n + 2l} + ... = \frac{\pi}{2p^n \cos(\frac{l}{p^n}\pi)}$$

$l < \frac{1}{2}p^n$, l and p^n have no common factor, $n = 1,2,3,...$

Proof: By $\mathbf{7_{cos}}$ based on $\pi \frac{1}{\cos(\pi z)} \frac{1}{z - \frac{l}{p^n}}$.□

29.

29.1

$$\boxed{\pi = 9\tan(\tfrac{1}{9}\pi)\left(\frac{1}{1}-\frac{1}{8} + \frac{1}{10}-\frac{1}{17} + \frac{1}{19}-\frac{1}{26}+...\right)}$$

$$R_{1/9} = \frac{1}{1}-\frac{1}{8} + \frac{1}{10}-\frac{1}{17} + \frac{1}{19}-\frac{1}{26}+... = \frac{\pi}{9}\cot(\tfrac{1}{9}\pi)$$

Proof: By $\mathbf{7_{cot}}$ based on $\pi\cot(\pi z)\dfrac{1}{z+\frac{1}{9}}$.□

29.1

$$\boxed{\pi = 18\cot(\tfrac{1}{9}\pi)\left(\frac{1}{7}-\frac{1}{11} + \frac{1}{25}-\frac{1}{29} + \frac{1}{43}-\frac{1}{47}+...\right)}$$

$$T_{1/9} = \frac{1}{7}-\frac{1}{11} + \frac{1}{25}-\frac{1}{29} + \frac{1}{43}-\frac{1}{47}+... = \frac{\pi}{18}\tan(\tfrac{1}{9}\pi)$$

Proof: By $\mathbf{7_{tan}}$ based on $\pi\tan(\pi z)\dfrac{1}{z-\frac{1}{9}}$.□

29.3
$$\boxed{\pi = 9\sin(\tfrac{1}{9}\pi)\left(\frac{1}{1}+\frac{1}{8} -\frac{1}{10}-\frac{1}{17} +\frac{1}{19}+\frac{1}{26}+...\right)}$$

$$S_{1/8} = \frac{1}{1}+\frac{1}{8} -\frac{1}{10}-\frac{1}{17} +\frac{1}{19}+\frac{1}{26}-... = \frac{\pi}{9}\frac{1}{\sin(\frac{1}{9}\pi)}$$

$\sin(\frac{1}{9}\pi)$ **solves the cubic** $4x^3 - 3x + \frac{1}{2}\sqrt{3} = 0$

Proof: By $\mathbf{7_{sin}}$ **based on** $\pi\dfrac{1}{\sin(\pi z)}\dfrac{1}{z+\frac{1}{9}}$.□

$$\sin 3A = -4\sin^3 A + 3\sin A \Rightarrow$$

$$\underbrace{\sin(\tfrac{1}{3}\pi)}_{\frac{1}{2}\sqrt{3}} = -4\sin^3(\tfrac{1}{9}\pi) + 3\sin(\tfrac{1}{9}\pi) \Rightarrow$$

$\sin(\frac{1}{9}\pi)$ **solves the cubic** $4x^3 - 3x + \frac{1}{2}\sqrt{3} = 0$.□

29.4
$$\boxed{\pi = 18\cos(\tfrac{1}{9}\pi)\left(\frac{1}{7}+\frac{1}{11} -\frac{1}{25}-\frac{1}{29} +\frac{1}{43}+\frac{1}{47}+...\right)}$$

$$C_{1/9} = \frac{1}{7} - \frac{1}{11} + \frac{1}{25} - \frac{1}{29} + \frac{1}{43} - \frac{1}{47} + ... = \frac{\pi}{18\cos(\frac{1}{9}\pi)}$$

Proof: By $\mathbf{7_{cos}}$ based on $\pi \frac{1}{\cos(\pi z)} \frac{1}{z - \frac{1}{9}}$.$\square$

30.

30.1

$$\boxed{\pi = 9\tan(\tfrac{2}{9}\pi)\left(\frac{1}{2}-\frac{1}{7} + \frac{1}{11}-\frac{1}{16} + \frac{1}{20}-\frac{1}{25}+...\right)}$$

$$R_{2/9} = \frac{1}{2}-\frac{1}{7} + \frac{1}{11}-\frac{1}{16} + \frac{1}{20}-\frac{1}{25}+... = \frac{\pi}{9}\cot(\tfrac{2}{9}\pi)$$

Proof By $\mathbf{7_{cot}}$ based on $\pi\cot(\pi z)\dfrac{1}{z+\frac{2}{9}}$,

30.2

$$\boxed{\pi = 18\cot(\tfrac{2}{9}\pi)\left(\frac{1}{5}-\frac{1}{13} + \frac{1}{23}-\frac{1}{31} + \frac{1}{41}-\frac{1}{49}+...\right)}$$

$$T_{2/9} = \frac{1}{5}-\frac{1}{13} + \frac{1}{23}-\frac{1}{31} + \frac{1}{41}-\frac{1}{49}+... = \frac{\pi}{18}\tan(\tfrac{2}{9}\pi)$$

Proof By $\mathbf{7_{tan}}$ based on $\pi\tan(\pi z)\dfrac{1}{z-\frac{2}{9}}$,

30.3

$$\boxed{\pi = 9\sin(\tfrac{2}{9}\pi)\left(\frac{1}{2}-\frac{1}{7} + \frac{1}{11}-\frac{1}{16} + \frac{1}{20}-\frac{1}{25}+...\right)}$$

$$S_{2/9} = \frac{1}{2}+\frac{1}{7} - \frac{1}{11}-\frac{1}{16} + \frac{1}{20}+\frac{1}{25}-... = \frac{\pi}{9\sin(\tfrac{2}{9}\pi)}$$

$\sin(\tfrac{2}{9}\pi)$ **solves the cubic** $4x^3 - 3x + \tfrac{1}{2}\sqrt{3} = 0$

Proof: By $\mathbf{7_{sin}}$ based on $\pi\dfrac{1}{\sin(\pi z)}\dfrac{1}{z+\frac{2}{9}}$.□

$$\sin 3A = -4\sin^3 A + 3\sin A \Rightarrow$$

$$\underbrace{\sin(\tfrac{2}{3}\pi)}_{\frac{1}{2}\sqrt{3}} = -4\sin^3(\tfrac{2}{9}\pi) + 3\sin(\tfrac{2}{9}\pi) \Rightarrow$$

$\sin(\tfrac{2}{9}\pi)$ **solves the cubic** $4x^3 - 3x + \tfrac{1}{2}\sqrt{3} = 0$.□

30.4

$$\boxed{\pi = 18\cos(\tfrac{2}{9}\pi)\left(\frac{1}{5}-\frac{1}{13} + \frac{1}{23}-\frac{1}{31} + \frac{1}{41}-\frac{1}{49}+...\right)}$$

$$C_{2/9} = \frac{1}{5} - \frac{1}{13} + \frac{1}{23} - \frac{1}{31} + \frac{1}{41} - \frac{1}{49} + ... = \frac{\pi}{18\cos(\frac{2}{9}\pi)}$$

Proof By $\mathbf{7}_{\mathbf{cos}}$ based on $\pi\dfrac{1}{\cos(\pi z)}\dfrac{1}{z-\frac{2}{9}}$,

31.

31.1

$$\boxed{\pi = 9\tan(\tfrac{4}{9}\pi)\left(\frac{1}{4} - \frac{1}{5} + \frac{1}{13} - \frac{1}{14} + \frac{1}{22} - \frac{1}{23} + \dots\right)}$$

$$R_{4/9} = \frac{1}{4} - \frac{1}{5} + \frac{1}{13} - \frac{1}{14} + \frac{1}{22} - \frac{1}{23} + \dots = \frac{\pi}{9}\cot(\tfrac{4}{9}\pi)$$

Proof By $\mathbf{7_{cot}}$ based on $\pi\cot(\pi z)\dfrac{1}{z+\frac{4}{9}}$.□

$$\cot(\tfrac{4}{9}\pi) = \frac{\cos(\frac{4}{9}\pi)}{\sin(\frac{4}{9}\pi)}$$

$$= \frac{\sqrt{\frac{1}{2}[1+\cos(\frac{8}{9}\pi)]}}{\sqrt{\frac{1}{2}[1-\cos(\frac{8}{9}\pi)]}}$$

$$= \frac{\sqrt{1-\cos(\frac{1}{9}\pi)}}{\sqrt{1+\cos(\frac{1}{9}\pi)}}.\square$$

31.2

$$\boxed{\pi = 18\tan(\tfrac{4}{9}\pi)\left(\frac{1}{1} - \frac{1}{17} + \frac{1}{19} - \frac{1}{35} + \frac{1}{37} - \frac{1}{53} + \ldots\right)}$$

$$T_{4/9} = \frac{1}{1} - \frac{1}{17} + \frac{1}{19} - \frac{1}{35} + \frac{1}{37} - \frac{1}{53} + \ldots = \frac{\pi}{18}\tan(\tfrac{4}{9}\pi)$$

Proof By $\mathbf{7}_{\mathbf{tan}}$ based on $\pi\tan(\pi z)\dfrac{1}{z - \frac{4}{9}}$.□

31.3

$$\boxed{\pi = 9\sin(\tfrac{4}{9}\pi)\left(\frac{1}{4} + \frac{1}{5} - \frac{1}{13} - \frac{1}{14} + \frac{1}{22} + \frac{1}{23} - \ldots\right)}$$

$$S_{4/9} = \frac{1}{4} + \frac{1}{5} - \frac{1}{13} - \frac{1}{14} + \frac{1}{22} + \frac{1}{23} - \ldots = \frac{\pi}{9\sin(\frac{4}{9}\pi)}$$

$\sin(\tfrac{4}{9}\pi)$ solves the cubic $4x^3 - 3x - \tfrac{1}{2}\sqrt{3} = 0$

Proof By $\mathbf{7}_{\mathbf{sin}}$ based on $\pi\dfrac{1}{\sin(\pi z)}\dfrac{1}{z + \frac{4}{9}}$.□

$$\sin 3A = -4\sin^3 A + 3\sin A \Rightarrow$$

$$\underbrace{\sin(3\tfrac{4}{9}\pi)}_{-\frac{1}{2}\sqrt{3}} = -4\sin^3(\tfrac{4}{9}\pi) + 3\sin(\tfrac{4}{9}\pi) \Rightarrow$$

$\sin(\tfrac{4}{9}\pi)$ **solves the cubic** $4x^3 - 3x - \tfrac{1}{2}\sqrt{3} = 0.\square$

31.4

$$\boxed{\pi = 18\cos(\tfrac{4}{9}\pi)\left(\frac{1}{1} - \frac{1}{17} + \frac{1}{19} - \frac{1}{35} + \frac{1}{37} - \frac{1}{53} + \dots\right)}$$

$$C_{4/9} = \frac{1}{1} + \frac{1}{17} - \frac{1}{19} - \frac{1}{35} + \frac{1}{37} + \frac{1}{53} - \dots = \frac{\pi}{18\cos(\tfrac{4}{9}\pi)}$$

Proof By $\mathbf{7}_{\mathbf{cos}}$ based on $\pi\dfrac{1}{\cos(\pi z)}\dfrac{1}{z-\frac{4}{9}}.\square$

32.

32.1

$$\pi = 6\underbrace{\tan(\tfrac{1}{6}\pi)}_{1/\sqrt{3}}\left(\frac{1}{1}-\frac{1}{5} + \frac{1}{7}-\frac{1}{11} + \frac{1}{13}-\frac{1}{17}+...\right)$$

$$R_{1/6} = \frac{1}{1}-\frac{1}{5} + \frac{1}{7}-\frac{1}{11} + \frac{1}{13}-\frac{1}{17}+... = \frac{\pi}{2\sqrt{3}}$$

Proof: By $\mathbf{7_{cot}}$ based on $\pi\cot(\pi z)\dfrac{1}{z+\frac{1}{6}}$. □

32.2

$$\pi = 3\underbrace{\cot(\tfrac{1}{6}\pi)}_{\sqrt{3}}\left(\frac{1}{1}-\frac{1}{2} + \frac{1}{4}-\frac{1}{5} + \frac{1}{7}-\frac{1}{8}+...\right)$$

$$T_{1/6} = \frac{1}{1}-\frac{1}{2} + \frac{1}{4}-\frac{1}{5} + \frac{1}{7}-\frac{1}{8}+... = \frac{\pi}{3\sqrt{3}}$$

Proof: By $\mathbf{7_{tan}}$ based on $\pi\tan(\pi z)\dfrac{1}{z-\frac{1}{6}}$,

$$\pi = 12 \underbrace{\cot(\tfrac{1}{6}\pi)}_{\sqrt{3}} \left(\frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{20} + \frac{1}{28} - \frac{1}{32} + \ldots \right) . \square$$

32.3

$$\boxed{\pi = 6 \underbrace{\sin(\tfrac{1}{6}\pi)}_{1/2} \left(\frac{1}{1} + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \frac{1}{17} - \ldots \right)}$$

$$S_{1/6} = \frac{1}{1} + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \frac{1}{17} - \ldots = \frac{\pi}{3}$$

Proof: By $\mathbf{7_{sin}}$ based on $\pi \dfrac{1}{\sin(\pi z)} \dfrac{1}{z + \frac{1}{6}} . \square$

32.4

$$\sqrt{3} = 2 \frac{\frac{1}{1} - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \ldots}{\frac{1}{1} + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \ldots}$$

Proof: $\dfrac{R_{1/6}}{S_{1/6}} . \square$

32.5
$$\boxed{\pi = 3\underbrace{\cos(\tfrac{1}{6}\pi)}_{\sqrt{3}/2}\left(\frac{1}{1}+\frac{1}{2} -\frac{1}{4}-\frac{1}{5} +\frac{1}{7}+\frac{1}{8}+...\right)}$$

$$C_{1/6} = \frac{1}{1}+\frac{1}{2} -\frac{1}{4}-\frac{1}{5} +\frac{1}{7}+\frac{1}{8}-... = \frac{2\pi}{3\sqrt{3}}$$

Proof: By $\mathbf{7_{cos}}$ based on $\pi\frac{1}{\cos(\pi z)}\frac{1}{z-\frac{1}{6}}$.□

33.

33.1

$$\boxed{\pi = 18\tan(\tfrac{1}{18}\pi)\left(\frac{1}{1}-\frac{1}{17} + \frac{1}{19}-\frac{1}{35} + \frac{1}{37}-\frac{1}{53}+\dots\right)}$$

$$R_{1/18} = \frac{1}{1}-\frac{1}{17} + \frac{1}{19}-\frac{1}{35} + \frac{1}{37}-\frac{1}{53}+\dots = \frac{\pi}{18}\cot(\tfrac{1}{18}\pi)$$

Proof: By $\mathbf{7_{cot}}$ based on $\pi\cot(\pi z)\dfrac{1}{z+\frac{1}{18}}$

33.2

$$\boxed{\pi = 9\cot(\tfrac{1}{18}\pi)\left(\frac{1}{4}-\frac{1}{5} + \frac{1}{13}-\frac{1}{14} + \frac{1}{22}-\frac{1}{23}+\dots\right)}$$

$$T_{1/18} = \frac{1}{4}-\frac{1}{5} + \frac{1}{13}-\frac{1}{14} + \frac{1}{22}-\frac{1}{23}+\dots = \frac{\pi}{18}\cot(\tfrac{1}{18}\pi)$$

Proof: By $\mathbf{7_{tan}}$ based on $\pi\tan(\pi z)\dfrac{1}{z-\frac{1}{18}}$

33.3

$$\boxed{\pi = 18\sin(\tfrac{1}{18}\pi)\left(\frac{1}{1}-\frac{1}{17} + \frac{1}{19}-\frac{1}{35} + \frac{1}{37}-\frac{1}{53}+...\right)}$$

$$S_{1/18} = \frac{1}{1}+\frac{1}{17} - \frac{1}{19}-\frac{1}{35} + \frac{1}{37}+\frac{1}{53}-... = \frac{\pi}{18\sin(\frac{1}{18}\pi)}$$

$$\sin(\tfrac{1}{18}\pi) = \sqrt{\tfrac{1}{2}[1-\cos(\tfrac{1}{9}\pi)]} = \sqrt{\tfrac{1}{2}[1-\sin(\tfrac{4}{9}\pi)]}$$

And $\sin(\frac{4}{9}\pi)$ **solves the cubic** $4x^3-3x-\frac{1}{2}\sqrt{3}=0$

Proof: By $\mathbf{7_{sin}}$ based on $\pi\dfrac{1}{\sin(\pi z)}\dfrac{1}{z+\frac{1}{18}}$.□

33.4

$$\boxed{\pi = 9\cos(\tfrac{1}{18}\pi)\left(\frac{1}{4}-\frac{1}{5} + \frac{1}{13}-\frac{1}{14} + \frac{1}{22}-\frac{1}{23}+...\right)}$$

$$C_{1/18} = \frac{1}{4}-\frac{1}{5} + \frac{1}{13}-\frac{1}{14} + \frac{1}{22}-\frac{1}{23}+... = \frac{\pi}{18\cos(\frac{1}{18}\pi)}$$

Proof: By $\mathbf{7_{cos}}$ based on $\pi \frac{1}{\cos(\pi z)} \frac{1}{z - \frac{1}{18}}$.□

References

[Dan] H. Vic Dannon, "π Series" Gauge Institute Journal of Math and Physics, Volume 18. No 1. Feb. 2022

[Euler] Leonardi Euleri, Opera Omnia, "Introductio in Analysin Infinitorum", Tomus Primus, Series Prima, Opera Mathematica, Volumen Octavum, #166, #172, #174, #175, #176, #177, #178, #179, Lipsia et Berolini, 1922

[Euler-2] "Euler-Introduction to Analysis of the Infinite", Book One, Springer, 1988.

[Spiegel], Murray Spiegel, "Mathematical Handbook", Schaum's Outline, McGraw-Hill, 1968.

[Spiegel-2], Murray Spiegel, "Complex Variables", Schaum's Outline, McGraw-Hill, 1964.