Archimedes Series

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Abstract: For any number $\theta > 0$, the Archimedes Spiral

$$\rho(\theta) = \frac{1}{2\pi}\theta$$

leads to a Power Series in $\arctan\theta$ that we name after Archimedes

The Archimedes Series for 2π is the Taylor Series of $\tan\alpha(2\pi)$

$$\boxed{\begin{aligned}\pi = \tfrac{1}{2}\{&\alpha(2\pi) + \tfrac{1}{3}\alpha^3(2\pi) + \tfrac{2}{15}\alpha^5(2\pi) + \tfrac{17}{315}\alpha^7(2\pi) + \tfrac{62}{2835}\alpha^9(2\pi) + \\ &+ \tfrac{(819)(691)}{3^6 5^2 7(11)(91)}\alpha^{11}(2\pi) + \tfrac{5461}{3^5 5^2 7(11)(13)}\alpha^{13}(2\pi) + ... + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!}B_n\alpha^{2n-1}(2\pi) + ...\}\end{aligned}}$$

where $\boxed{\alpha(2\pi) = \arctan(2\pi) \approx 1.412965137....}$

and B_n = Bernoulli Numbers.

The Archimedes Series for e is the Taylor Series of $\tan\alpha(e)$

$$\boxed{\begin{aligned}e = &\alpha(e) + \tfrac{1}{3}\alpha^3(e) + \tfrac{2}{15}\alpha^5(e) + \tfrac{17}{315}\alpha^7(e) + \tfrac{62}{2835}\alpha^9(e) + \\ &+ \tfrac{(819)(691)}{3^6 5^2 7(11)(91)}\alpha^{11}(e) + \tfrac{5461}{3^5 5^2 7(11)(13)}\alpha^{13}(e) + ... + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!}B_n\alpha^{2n-1}(e) + ...\end{aligned}}$$

where $\boxed{\alpha(e) = \arctan(e) \approx 1.218282905...}$

The Archimedes Series for θ is the Taylor Series of $\tan\alpha(\theta)$

$$\theta = \alpha(\theta) + \tfrac{1}{3}\alpha^3(\theta) + \tfrac{2}{15}\alpha^5(\theta) + \tfrac{17}{315}\alpha^7(\theta) + \tfrac{62}{2835}\alpha^9(\theta) + $$
$$+ \tfrac{(819)(691)}{3^6 5^2 7(11)(91)}\alpha^{11}(\theta) + \tfrac{5461}{3^5 5^2 7(11)(13)}\alpha^{13}(\theta) + ... + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!}B_n\alpha^{2n-1}(\theta) + ...$$

where $\alpha(\theta) = \arctan(\theta)$

The Archimedes Series for 1 is the Taylor Series of $\tan\alpha(1)$

$$1 = \tfrac{\pi}{4} + \tfrac{1}{3}(\tfrac{\pi}{4})^3 + \tfrac{2}{15}(\tfrac{\pi}{4})^5 + \tfrac{17}{315}(\tfrac{\pi}{4})^7 + \tfrac{62}{2835}(\tfrac{\pi}{4})^9 + $$
$$+ \tfrac{(819)(691)}{3^6 5^2 7(11)(91)}(\tfrac{\pi}{4})^{11} + \tfrac{5461}{3^5 5^2 7(11)(13)}(\tfrac{\pi}{4})^{13} + ... + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!}B_n(\tfrac{\pi}{4})^{2n-1} + ...$$

The Archimedes Series for $\sqrt{3}$ is the Taylor Series of $\tan\alpha(\sqrt{3})$

$$\sqrt{3} = \tfrac{\pi}{3} + \tfrac{1}{3}(\tfrac{\pi}{3})^3 + \tfrac{2}{15}(\tfrac{\pi}{3})^5 + \tfrac{17}{315}(\tfrac{\pi}{3})^7 + \tfrac{62}{2835}(\tfrac{\pi}{3})^9 + $$
$$+ \tfrac{(819)(691)}{3^6 5^2 7(11)(91)}(\tfrac{\pi}{3})^{11} + \tfrac{5461}{3^5 5^2 7(11)(13)}(\tfrac{\pi}{3})^{13} + ... + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!}B_n(\tfrac{\pi}{3})^{2n-1} + ...$$

The Archimedes Series for $\frac{1}{\sqrt{3}}$ is the Taylor Series of $\tan\alpha(\frac{1}{\sqrt{3}})$

$$\tfrac{1}{\sqrt{3}} = \tfrac{\pi}{6} + \tfrac{1}{3}(\tfrac{\pi}{6})^3 + \tfrac{2}{15}(\tfrac{\pi}{6})^5 + \tfrac{17}{315}(\tfrac{\pi}{6})^7 + \tfrac{62}{2835}(\tfrac{\pi}{6})^9 + $$
$$+ \tfrac{(819)(691)}{3^6 5^2 7(11)(91)}(\tfrac{\pi}{6})^{11} + \tfrac{5461}{3^5 5^2 7(11)(13)}(\tfrac{\pi}{6})^{13} + ... + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!}B_n(\tfrac{\pi}{6})^{2n-1} + ...$$

Therefore,

$$\sqrt{3} = \frac{1}{\frac{\pi}{6} + \frac{1}{3}\left(\frac{\pi}{6}\right)^3 + \frac{2}{15}\left(\frac{\pi}{6}\right)^5 + \frac{17}{315}\left(\frac{\pi}{6}\right)^7 + \frac{62}{2835}\left(\frac{\pi}{6}\right)^9 + ...}$$

Hence,

$$3 = \frac{\frac{\pi}{3} + \frac{1}{3}\left(\frac{\pi}{3}\right)^3 + \frac{2}{15}\left(\frac{\pi}{3}\right)^5 + \frac{17}{315}\left(\frac{\pi}{3}\right)^7 + \frac{62}{2835}\left(\frac{\pi}{3}\right)^9 + ...}{\frac{\pi}{6} + \frac{1}{3}\left(\frac{\pi}{6}\right)^3 + \frac{2}{15}\left(\frac{\pi}{6}\right)^5 + \frac{17}{315}\left(\frac{\pi}{6}\right)^7 + \frac{62}{2835}\left(\frac{\pi}{6}\right)^9 + ...}$$

And

$$1 = \left\{\frac{\pi}{6} + \frac{1}{3}\left(\frac{\pi}{6}\right)^3 + \frac{2}{15}\left(\frac{\pi}{6}\right)^5 + \frac{17}{315}\left(\frac{\pi}{6}\right)^7 + \frac{62}{2835}\left(\frac{\pi}{6}\right)^9 + ...\right\} \times$$

$$\times \left\{\frac{\pi}{3} + \frac{1}{3}\left(\frac{\pi}{3}\right)^3 + \frac{2}{15}\left(\frac{\pi}{3}\right)^5 + \frac{17}{315}\left(\frac{\pi}{3}\right)^7 + \frac{62}{2835}\left(\frac{\pi}{3}\right)^9 + ...\right\}$$

1.

Archimedes Series for π

1.1 Archimedes Series for 2π is the Taylor Series of $\tan\alpha(2\pi)$

$$\pi = \frac{1}{2}\{\alpha(2\pi) + \frac{1}{3}\alpha^3(2\pi) + \frac{2}{15}\alpha^5(2\pi) + \frac{17}{315}\alpha^7(2\pi) + \frac{62}{2835}\alpha^9(2\pi) +$$

$$+\frac{(819)(691)}{3^6 5^2 7(11)(91)}\alpha^{11}(2\pi) + \frac{5461}{3^5 5^2 7(11)(13)}\alpha^{13}(2\pi) + ... + \frac{2^{2n}(2^{2n}-1)}{(2n)!}B_n \alpha^{2n-1}(2\pi) + ...\}$$

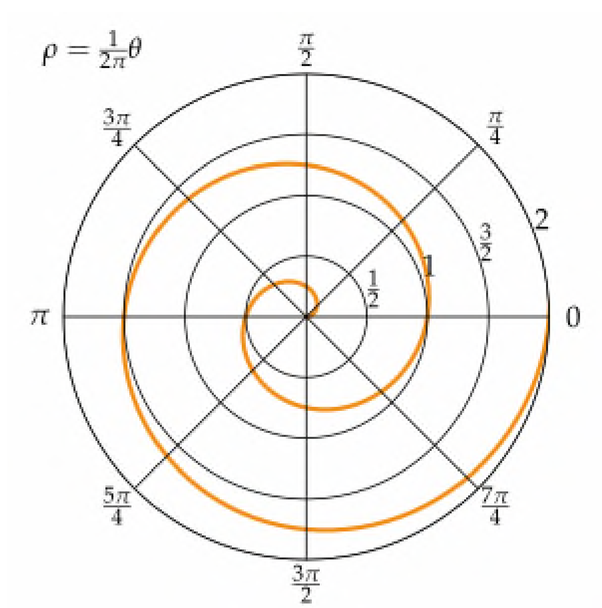
where $\alpha = \arctan(2\pi) \approx 1.412965137....$

and B_n = Bernoulli Numbers

Proof: Archimedes Spiral[1] in polar coordinates (ρ, θ) is given by

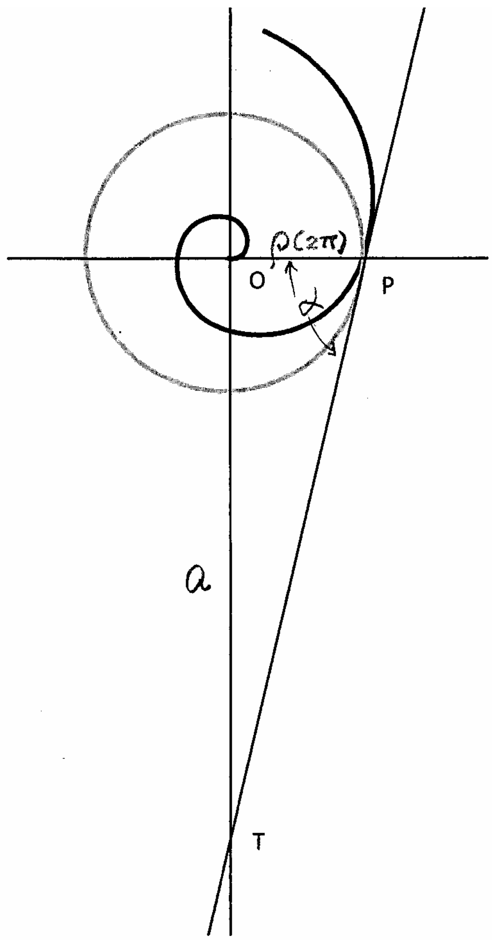
$$\rho(\theta) = \frac{1}{2\pi}\theta. \quad \text{So } \rho(2\pi) = 1$$

Hence, the circumference of the circle of radius $\rho(2\pi)$ is 2π.



[1]https://en.wikipedia.org/wiki/Archimedean_spiral#:~:text=The%20Archimedean%20spiral%20(also%20known,rotates%20with%20constant%20angular%20velocity.

The tangent to the spiral at $\theta = 2\pi$, at the point $P(2\pi)$, intersects the y axis at the point $T(2\pi)$.



Archimedes proved[2] that

a ***equals the circumference of the circle of radius*** $\rho(2\pi)$.

That is,

$$a = 2\pi \underbrace{\rho(2\pi)}_{1} = 2\pi .$$

Now, for the Archimedes Angle $\alpha = \alpha(2\pi)$ at P, we have

$$\tan \alpha(2\pi) = \frac{a}{\rho(2\pi)} = \frac{2\pi}{1} = 2\pi$$

[2] Archimedes, "On Spirals", Cambridge University Press.

Therefore,

$$\boxed{\alpha(2\pi) = \arctan(2\pi) \approx 1.412965137...}$$

It follows that

$$\pi = \tfrac{1}{2}\tan\alpha(2\pi).$$

Since $\alpha(2\pi) < \frac{1}{2}\pi$, we can expand $\tan\alpha(2\pi)$ in its Taylor Series[3].

The Archimedes Series for 2π is the Taylor Series of $\tan\alpha(2\pi)$

$$\boxed{\begin{aligned}\pi &= \tfrac{1}{2}\{\alpha(2\pi) + \tfrac{1}{3}\alpha^3(2\pi) + \tfrac{2}{15}\alpha^5(2\pi) + \tfrac{17}{315}\alpha^7(2\pi) + \tfrac{62}{2835}\alpha^9(2\pi) + \\ &+ \tfrac{(819)(691)}{3^6 5^2 7(11)(91)}\alpha^{11}(2\pi) + \tfrac{5461}{3^5 5^2 7(11)(13)}\alpha^{13}(2\pi) + ... + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!}B_n\alpha^{2n-1}(2\pi) + ...\}\end{aligned}}$$

where

$$B_n = \text{Bernoulli Numbers}$$

1.2 The Convergence Speed of the Series for π

$$B_1 = \tfrac{1}{6} \Rightarrow \quad \tfrac{1}{2}\alpha(2\pi) \approx 0.706482568$$

$$B_2 = \tfrac{1}{30} \Rightarrow \quad \tfrac{1}{6}\alpha^3(2\pi) \approx 0.470157196$$

$$B_3 = \tfrac{1}{42} \Rightarrow \quad \tfrac{1}{15}\alpha^5(2\pi) \approx 0.375461985$$

$$B_4 = \tfrac{1}{30} \Rightarrow \quad \tfrac{17}{630}\alpha^7(2\pi) \approx 0.303409025$$

$$B_5 = \tfrac{5}{66} \Rightarrow \quad \tfrac{31}{2835}\alpha^9(2\pi) \approx 0.24546617$$

$$B_6 = \tfrac{691}{2730} \Rightarrow \quad \tfrac{(819)(691)}{3^6 5^2 7(11)(91)}\alpha^{11}(2\pi) \approx 0.198613243$$

$$B_7 = \tfrac{7}{6} \Rightarrow \quad \tfrac{5461}{3^5 5^2 7(11)(13)}\alpha^{13}(2\pi) \approx 0.080352726$$

[3] Murray Spiegel, "Mathematical Handbook", Schaum's, p.111, #20.23

$$S_7 \approx 2.379942913$$

And the error in S_7 is

$$\left|\pi - S_7\right| \approx \left|3.141592654 - 2.379942913\right| \approx 0.76164974$$

In comparison, for the Leibniz Series of reciprocals for π,

$$S_7 = 4\{1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \tfrac{1}{13}\} \approx 3.283738484$$

And the error in S_7 is

$$\left|\pi - S_7\right| \approx \left|3.141592654 - 3.283738484\right| \approx 0.14214583$$

2.

Archimedes Series for e

At an angle $\theta =$ Euler's constant e,

$$\rho(e) = \tfrac{1}{2\pi} e\,,$$

$$2\pi\rho(e) = e$$

e ***equals the circumference of the circle of radius*** $\rho(e)$.

Now, we define the Archimedes Angle at e

$$\boxed{\alpha(e) = \arctan(e) \approx 1.218282905...}$$

It follows that

$$e = \tan\alpha(e).$$

Since $\alpha(e) < \frac{1}{2}\pi$, we can expand $\tan\alpha(e)$ in its Taylor Series.

The Archimedes Series for e **is the Taylor Series of** $\tan\alpha(e)$

$$e = \alpha(e) + \tfrac{1}{3}\alpha^3(e) + \tfrac{2}{15}\alpha^5(e) + \tfrac{17}{315}\alpha^7(e) + \tfrac{62}{2835}\alpha^9(e) +$$

$$+ \tfrac{(819)(691)}{3^6 5^2 7(11)(91)}\alpha^{11}(e) + \tfrac{5461}{3^5 5^2 7(11)(13)}\alpha^{13}(e) + ... + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!}B_n\alpha^{2n-1}(e) + ...$$

where

$$B_n = \text{Bernoulli Numbers}$$

3.

Archimedes Series for $\theta > 0$

At an angle $\theta > 0$

$$\rho(\theta) = \frac{1}{2\pi}\theta,$$

$$2\pi\rho(\theta) = \theta$$

θ ***equals the circumference of the circle of radius*** $\rho(\theta)$.

We define the Archimedes Angle at θ

$$\boxed{\alpha(\theta) = \arctan(\theta)}$$

Then,

$$\theta = \tan\alpha(\theta).$$

For $\theta > 0$, and $\tan\theta < \infty$,

$$\arctan\theta < \frac{\pi}{2}.$$

Then,

The Archimedes Series for θ **is the Taylor Series of** $\tan\alpha(\theta)$

$$\boxed{\begin{aligned}\theta &= \alpha(\theta) + \tfrac{1}{3}\alpha^3(\theta) + \tfrac{2}{15}\alpha^5(\theta) + \tfrac{17}{315}\alpha^7(\theta) + \tfrac{62}{2835}\alpha^9(\theta) + \\ &+ \tfrac{(819)(691)}{3^6 5^2 7(11)(91)}\alpha^{11}(\theta) + \tfrac{5461}{3^5 5^2 7(11)(13)}\alpha^{13}(\theta) + ... + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!}B_n\alpha^{2n-1}(\theta) + ...\end{aligned}}$$

where

$$B_n = \text{Bernoulli Numbers}$$

For $\theta = 1$, the Archimedes Angle at 1 is

$$\alpha(1) = \arctan(1) = \frac{\pi}{4}.$$

Then, the Archimedes Series for $\tan\alpha(1) = 1$ is

$$1 = \frac{\pi}{4} + \frac{1}{3}\left(\frac{\pi}{4}\right)^3 + \frac{2}{15}\left(\frac{\pi}{4}\right)^5 + \frac{17}{315}\left(\frac{\pi}{4}\right)^7 + \frac{62}{2835}\left(\frac{\pi}{4}\right)^9 +$$
$$+ \frac{(819)(691)}{3^6 5^2 7(11)(91)}\left(\frac{\pi}{4}\right)^{11} + \frac{5461}{3^5 5^2 7(11)(13)}\left(\frac{\pi}{4}\right)^{13} + \dots + \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_n \left(\frac{\pi}{4}\right)^{2n-1} + \dots$$

For $\theta = \sqrt{3}$, the Archimedes Angle at $\sqrt{3}$ is

$$\alpha(\sqrt{3}) = \arctan(\sqrt{3}) = \frac{\pi}{3}.$$

Then, the Archimedes Series for $\tan\alpha(\sqrt{3}) = \sqrt{3}$ is

$$\sqrt{3} = \frac{\pi}{3} + \frac{1}{3}\left(\frac{\pi}{3}\right)^3 + \frac{2}{15}\left(\frac{\pi}{3}\right)^5 + \frac{17}{315}\left(\frac{\pi}{3}\right)^7 + \frac{62}{2835}\left(\frac{\pi}{3}\right)^9 +$$
$$+ \frac{(819)(691)}{3^6 5^2 7(11)(91)}\left(\frac{\pi}{3}\right)^{11} + \frac{5461}{3^5 5^2 7(11)(13)}\left(\frac{\pi}{3}\right)^{13} + \dots + \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_n \left(\frac{\pi}{3}\right)^{2n-1} + \dots$$

For $\theta = \frac{1}{\sqrt{3}}$, the Archimedes Angle at $\frac{1}{\sqrt{3}}$ is

$$\alpha\left(\frac{1}{\sqrt{3}}\right) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$$

Then, the Archimedes Series for $\tan\alpha\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}$ is

$$\frac{1}{\sqrt{3}} = \frac{\pi}{6} + \frac{1}{3}\left(\frac{\pi}{6}\right)^3 + \frac{2}{15}\left(\frac{\pi}{6}\right)^5 + \frac{17}{315}\left(\frac{\pi}{6}\right)^7 + \frac{62}{2835}\left(\frac{\pi}{6}\right)^9 +$$
$$+ \frac{(819)(691)}{3^6 5^2 7(11)(91)}\left(\frac{\pi}{6}\right)^{11} + \frac{5461}{3^5 5^2 7(11)(13)}\left(\frac{\pi}{6}\right)^{13} + \dots + \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_n \left(\frac{\pi}{6}\right)^{2n-1} + \dots$$

Therefore,

$$\sqrt{3} = \frac{1}{\frac{\pi}{6} + \frac{1}{3}\left(\frac{\pi}{6}\right)^3 + \frac{2}{15}\left(\frac{\pi}{6}\right)^5 + \frac{17}{315}\left(\frac{\pi}{6}\right)^7 + \frac{62}{2835}\left(\frac{\pi}{6}\right)^9 + \dots}$$

Hence,

$$3 = \frac{\frac{\pi}{3} + \frac{1}{3}\left(\frac{\pi}{3}\right)^3 + \frac{2}{15}\left(\frac{\pi}{3}\right)^5 + \frac{17}{315}\left(\frac{\pi}{3}\right)^7 + \frac{62}{2835}\left(\frac{\pi}{3}\right)^9 + ...}{\frac{\pi}{6} + \frac{1}{3}\left(\frac{\pi}{6}\right)^3 + \frac{2}{15}\left(\frac{\pi}{6}\right)^5 + \frac{17}{315}\left(\frac{\pi}{6}\right)^7 + \frac{62}{2835}\left(\frac{\pi}{6}\right)^9 + ...}$$

And

$$1 = \left\{\frac{\pi}{6} + \frac{1}{3}\left(\frac{\pi}{6}\right)^3 + \frac{2}{15}\left(\frac{\pi}{6}\right)^5 + \frac{17}{315}\left(\frac{\pi}{6}\right)^7 + \frac{62}{2835}\left(\frac{\pi}{6}\right)^9 + ...\right\} \times$$

$$\times \left\{\frac{\pi}{3} + \frac{1}{3}\left(\frac{\pi}{3}\right)^3 + \frac{2}{15}\left(\frac{\pi}{3}\right)^5 + \frac{17}{315}\left(\frac{\pi}{3}\right)^7 + \frac{62}{2835}\left(\frac{\pi}{3}\right)^9 + ...\right\}$$

References

Archimedes, "On Spirals", Cambridge University Press.

Oyvind Hammer, "The Perfect Shape, Spiral Stories", Springer 2016

Murray Spiegel, "Mathematical Handbook", Schaum's, p.111, #20.23

https://en.wikipedia.org/wiki/Archimedean_spiral#:~:text=The%20Archimedean%20spiral%20(also%20known,rotates%20with%20constant%20angular%20velocity.