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Pi, e, and Gamma are Transcendental Numbers

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Abstract: The partial sums of expansion in fractions of

$$\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + ...)$$

and Euler's Constant

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + ...$$

and Euler's constant γ, are quotients of integers with denominators that include products of increasingly many primes.

The sum of each series has a numerator which is infinite sum of integers. and a denominator that includes a product of infinitely many primes.

$$2, 3, 5, 7, 11, 13, 17, 19, 23, ...$$

We show that such sums are transcendental numbers.

<u>**Prime Numbers Criteria for Transcendental Numbers**</u>

If a Number has expansion in a series of fractions, and infinitely many primes are factors of the common denominator of the series' sum, Then, the number is transcendental.

By this criteria, the following numbers are transcendental

$$\boxed{\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + ...)}$$

All the powers of π

$$\boxed{\pi^k, \quad k = 1,2,3,...}$$

$$\boxed{e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + ...},$$

All the powers of e

$$\boxed{e^k, \quad k = 1,2,3,...}$$

$$\boxed{\begin{aligned}\gamma = 1&\frac{1}{2\cdot 3}\\ &+ 2\left(\frac{1}{4\cdot 5} + \frac{1}{6\cdot 7}\right)\\ &+ 3\left(\frac{1}{8\cdot 9} + \frac{1}{10\cdot 11} + \frac{1}{12\cdot 13} + \frac{1}{14\cdot 15}\right) + ...\end{aligned}}$$

All the powers of γ

$$\boxed{\gamma^k, \quad k = 1,2,3,...}$$

$$\boxed{\zeta(l) = 1 + \frac{1}{2^l} + \frac{1}{3^l} + \frac{1}{4^l} + ..., l = 2,3,4,...}$$

All the powers of $\zeta(k)$

$$\boxed{(\zeta(l))^k, \quad 1 = 2,3,4,....,k, = 1,2,3,...}$$

The Catalan Constant

$$\boxed{C = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + ...}$$

All the powers of C

$$\boxed{C^k, \quad k = 1,2,3,...}$$

The numbers,

$$\boxed{\pi + e} \qquad \boxed{\pi + \gamma} \qquad \boxed{e + \gamma}$$

$$\boxed{\pi^k + e^l + \gamma^m, \quad k,l,m,\text{=integers 1,2,3,...}}$$

$$\boxed{\pi e} \qquad \boxed{\pi\gamma} \qquad \boxed{e\gamma} \qquad \boxed{e\pi\gamma}$$

The Criteria does not apply if the series expansion is in transcendental numbers that themselves have expansion in fractions.

For instance, we have shown[1]

$$\boxed{\begin{aligned}1 &= \tfrac{\pi}{4} + \tfrac{1}{3}\left(\tfrac{\pi}{4}\right)^3 + \tfrac{2}{15}\left(\tfrac{\pi}{4}\right)^5 + \tfrac{17}{315}\left(\tfrac{\pi}{4}\right)^7 + \tfrac{62}{2835}\left(\tfrac{\pi}{4}\right)^9 + \\ &+ \tfrac{(819)(691)}{3^6 5^2 7(11)(91)}\left(\tfrac{\pi}{4}\right)^{11} + \tfrac{5461}{3^5 5^2 7(11)(13)}\left(\tfrac{\pi}{4}\right)^{13} + ... + \tfrac{2^{2n}(2^{2n}-1)}{(2n)!} B_n \left(\tfrac{\pi}{4}\right)^{2n-1} + ...\end{aligned}}$$

Then, each partial sum is transcendental and involves powers of π. Each of these powers can be expanded in series with sums that have denominators that satisfy the transcendence criteria.

But the sum will include terms such as

$$\left(4(1 - \frac{1}{3} + \frac{1}{5} - ...)\right)^{\infty}$$

to which our criteria does not apply.

Similarly, we cannot tell about e^{π}

[1] Archimedes Series Gauge Institute Journal, Volume 18, No. 3, August 2022

$$e^{\pi} = 1 + \left(4(1 - \frac{1}{3} + \frac{1}{5} - ...)\right) + \frac{1}{2}\left(4(1 - \frac{1}{3} + \frac{1}{5} - ...)\right)^2 + \frac{1}{2\cdot 3}\left(4(1 - \frac{1}{3} + \frac{1}{5} - ...)\right)^3 + ...$$

Then, every partial sum is transcendental. And the sum is bounded by 23.15.

But our criteria need not apply to terms like

$$\left(4(1 - \frac{1}{3} + \frac{1}{5} - ...)\right)^{\infty}.$$

Similarly, we cannot tell about the exponential of γ

$$\exp(\gamma) = 1 + \gamma + \frac{1}{2}\gamma^2 + \frac{1}{2\cdot 3}\gamma^3 + ..,$$

Then, the sum includes terms such as

$$(\text{vacca expansion of } \gamma)^{\infty}.$$

And we cannot tell about the exponential of e,

$$\exp(e) = 1 + e + \frac{1}{2}e^2 + \frac{1}{2\cdot 3}e^3 + \frac{1}{2\cdot 3\cdot 4}e^4 + ...$$

Then, the sum includes terms such as

$$(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + ...)^{\infty}.$$

1.
The Prime Numbers Criteria for Transcendental Numbers

The partial sums of expansion in fractions of Leibniz π,

$$\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + ...)$$

and Euler's Constant

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + ...$$

and Euler's constant γ, are quotients with denominators that include products of increasingly many primes.

The sum of each series has a numerator which is infinite sum of integers, and denominator that includes a product of infinitely many primes.

$$2, 3, 5, 7, 11, 13, 17, 19, 23, ...$$

We show that such sums are transcendental numbers.

Prime Numbers Criteria for Transcendental Numbers

If a Number has expansion in a series of fractions, and infinitely many primes are factors of the common denominator of the series' sum, Then, the number is transcendental.

In contrast, the partial sums of the infinite series expansion of Integers, and rational numbers have denominators that do not include products with increasingly many prime numbers.

$$1 = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{5}) + (\frac{1}{5} - \frac{1}{6}) + (\frac{1}{6} - \frac{1}{7}) + ...$$

$$1 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \frac{1}{5 \cdot 6} + \frac{1}{6 \cdot 7} + ...$$

$$s_1 = \frac{1}{2}$$

$$s_2 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

$$s_3 = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}$$

$$s_4 = \frac{3}{4} + \frac{1}{4 \cdot 5} = \frac{4}{5}$$

$$s_5 = \frac{4}{5} + \frac{1}{5 \cdot 6} = \frac{5}{6}$$

$$s_6 = \frac{5}{6} + \frac{1}{6 \cdot 7} = \frac{6}{7}$$

$$\dots\dots\dots\dots$$

$$s_m = 1 - \frac{1}{m} \to 1, \quad m \to \infty$$

For any integer k,

$$k = \frac{k}{1 \cdot 2} + \frac{k}{2 \cdot 3} + \frac{k}{3 \cdot 4} + \frac{k}{4 \cdot 5} + \frac{k}{5 \cdot 6} + \frac{k}{6 \cdot 7} + ...$$

$$s_m = k(1 - \frac{1}{m}) \to k, \quad m \to \infty.$$

For any rational q,

$$q = \frac{q}{1 \cdot 2} + \frac{q}{2 \cdot 3} + \frac{q}{3 \cdot 4} + \frac{q}{4 \cdot 5} + \frac{q}{5 \cdot 6} + \frac{q}{6 \cdot 7} + ...$$

$$s_m = q(1 - \frac{1}{m}) \to q\,, \qquad m \to \infty$$

We show that π satisfies the criteria, and is therefore a transcendental number.

π is based on all the odd numbers, and is the simplest of the numbers to which the criteria applies.

That is why we discuss π in greater detail.

But our proof applies without change to numbers that have similar series expansion to π.

2.

π is a Transcendental Number

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \frac{1}{21} - \frac{1}{23} + \dots$$

The series is absolutely convergent, and can be summed in pairs

$$\frac{\pi}{4} = (1 - \frac{1}{3}) + (\frac{1}{5} - \frac{1}{7}) + (\frac{1}{9} - \frac{1}{11}) + (\frac{1}{13} - \frac{1}{15}) + (\frac{1}{17} - \frac{1}{19}) + (\frac{1}{21} - \frac{1}{23}) + \dots$$

This summation is faster, and the partial sums are increasing monotonically to $\frac{\pi}{8}$.

$$\frac{\pi}{8} = \frac{1}{1\cdot 3} + \frac{1}{5\cdot 7} + \frac{1}{9\cdot 11} + \frac{1}{13\cdot 15} + \frac{1}{17\cdot 19} + \frac{1}{21\cdot 23} + \dots.$$

$$s_1 = \frac{1}{3}$$

The common denominator is the prime 3.

$$s_2 = \frac{1}{3} + \frac{1}{5\cdot 7} = \frac{5\cdot 7 + 3}{3\cdot 5\cdot 7}$$

$$= \frac{38}{3\cdot 5\cdot 7}$$

The common denominator is the product of the primes $3\cdot 5\cdot 7$

$$s_3 = \frac{38}{3\cdot 5\cdot 7} + \frac{1}{9\cdot 11} = \frac{38\cdot 3\cdot 11 + 5\cdot 7}{3^2\cdot 5\cdot 7\cdot 11}$$

$$= \frac{1289}{3^2\cdot 5\cdot 7\cdot 11}$$

The common denominator is the product of the primes

$$3^2 \cdot 5 \cdot 7 \cdot 11$$

$$s_4 = \frac{1289}{3^2 \cdot 5 \cdot 7 \cdot 11} + \frac{1}{13 \cdot 15} = \frac{1289 \cdot 13 + 3 \cdot 7 \cdot 11}{3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13} = \frac{17088}{3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13}$$

$$= \frac{5696}{3 \cdot 5 \cdot 7 \cdot 11 \cdot 13}$$

The common denominator is the product of the primes

$$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$$

$$s_5 = \frac{17,088}{3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13} + \frac{1}{17 \cdot 19} = \frac{17,088 \cdot 17 \cdot 19 + 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13}{3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19}$$

$$= \frac{5,564,469}{3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19}$$

$$= \frac{1,854,823}{3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19}$$

The common denominator is the product of the primes

$$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$$

$$s_6 = \frac{5,564,469}{3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19} + \frac{1}{21 \cdot 23}$$

$$= \frac{5,564,469 \cdot 23 + 3 \cdot 5 \cdot 11 \cdot 13 \cdot 17 \cdot 19}{3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23}$$

$$= \frac{128,675,622}{3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23}$$

$$= \frac{42,891,874}{3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23}$$

The common denominator is the product of the primes

$$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$$

....

That trend continues so that the denominator of the sum of the series includes a product of infinitely many prime numbers

$$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53 \cdot .....$$

The partial sums converge to

$$\frac{\text{infinite sum of integers}}{\text{product that includes infintely many primes } 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot ....},$$

which is π.

We first show that:

π **is Irrational**.

Because if there were integers

a, and b

with no common factor

so that

$$a\pi + b = 0,$$

$$b\frac{1}{\pi} = -a$$

Therefore we have

$$b\frac{\text{product that includes infinitely many primes}}{\text{infinite sum of integers}} = -a$$

Then,

since a, and b have no common factor,

and since the infinite sum of integers has no common factor with any of the primes in the denominator.

Therefore,

the integer a has to be dividable by any prime in the denominator.

And there is no such integer a.

Therefore, π is irrational.□

Similarly,

$$\pi = \frac{\text{infinite sum of integers}}{\text{product that includes infinitely many primes}}$$

is not a quadratic irrational.

Because if there were three integers

$$a, b, c$$

with no common factors, and

$$a\pi^2 + b\pi + c = 0,$$

Then,

$$\frac{1}{\pi}\left(b + c\frac{1}{\pi}\right) = -a$$

$$\frac{\text{product that includes infinitely many primes}}{\text{infinite sum of integers}} \times (b + c\frac{1}{\pi}) = -a$$

To divide a,

$$b + c\frac{1}{\pi}$$

has to be an integer d.

$$b + c\frac{1}{\pi} = d$$

Then,

$$(b - d)\pi + c = 0.$$

But π is irrational.

Therefore,

$$b + c\frac{1}{\pi}$$

does not divide a.

Since the

product of infinitely many primes in the denominator,

and the

infinite sum of integers in the numerator

have no common factor,

It must be that

the integer a is dividable by infinitely many primes

And there is no such integer a.

Therefore, π is not a quadratic irrational.□

Similarly, it follows that π **is not a cubic irrational**:

Because if there are four integers

$$a, b, c, d$$

with no common factors, and

$$a\pi^3 + b\pi^2 + c\pi + d = 0,$$

Then,

$$\frac{1}{\pi}\left(b + c\frac{1}{\pi} + d\frac{1}{\pi^2}\right) = -a$$

To divide a,

$$b + c\frac{1}{\pi} + d\frac{1}{\pi^2}$$

has to be an integer f. Then,

$$b + c\frac{1}{\pi} + d\frac{1}{\pi^2} = f$$

$$(b - f)\pi^2 + c\pi + d = 0$$

But π is not a quadratic irrational.

Therefore,

$$b + c\frac{1}{\pi} + d\frac{1}{\pi^2}$$

does not divide a.

Since the

product of infinitely many primes in the denominator,

and the

infinite sum of integers in the numerator

have no common factor,

It must be that

the integer a is dividable by infinitely many primes

And there is no such integer a.

Therefore, π is not a cubic irrational.□

We will show that

from the assumption that π is Not an n-powered Irrational number, it follows that

$$\pi = \frac{\text{infinite sum of integers}}{\text{product that includes infinitely many primes}}$$

is Not an $(n+1)$-powered Irrational Number

Suppose <u>there are No</u> $n+1$ <u>integers</u>

$$a_n, a_{n-1}, \ldots\ldots a_2, a_1, a_0$$

with no common factors,

and so that

$$a_n\pi^n + a_{n-1}\pi^{n-1} + \ldots\ldots a_2\pi^2 + a_1\pi + a_0 = 0$$

But there are $n+2$ integers

$$b_{n+1}, b_n, \ldots\ldots b_2, b_1, b_0$$

with no common factors,

and so that

$$b_{n+1}\pi^{n+1} + b_n\pi^n + b_{n-1}\pi^{n-1} + \ldots\ldots + b_1\pi + b_0 = 0$$

$$b_{n+1} + b_n\frac{1}{\pi} + b_{n-1}\frac{1}{\pi^2}\ldots\ldots + b_1\frac{1}{\pi^n} + b_0\frac{1}{\pi^{n+1}} = 0$$

$$\frac{1}{\pi}\left(b_n + b_{n-1}\frac{1}{\pi}\ldots\ldots + b_1\frac{1}{\pi^{n-1}} + b_0\frac{1}{\pi^n}\right) = -b_{n+1}$$

To divide the integer b_{n+1},

$$b_n + b_{n-1}\frac{1}{\pi}\ldots\ldots + b_1\frac{1}{\pi^{n-1}} + b_0\frac{1}{\pi^n}$$

has to be an integer c. But then π would be n-powered irrational.

Since the

product of infinitely many primes in the denominator,

and the

infinite sum of integers in the numerator

have no common factor,

It must be that

the integer b_{n+1} is dividable by infinitely many primes

And there is no such integer.

Therefore, By induction, for any n,

$$\pi = \frac{\text{infinite sum of integers}}{\text{product that includes infinitely many primes}}$$

is Not an n-powered Irrational Number

Thus, π is transcendental,

Consequently, **for any** $k = 1, 2, 3, \ldots$ π^k **is transcendental**.

3.

e is a Transcendental Number

$$e-2=\frac{1}{2}+\frac{1}{2\cdot 3}+\frac{1}{\underbrace{2^3\cdot 3}_{1/4!}}+\frac{1}{\underbrace{2^3\cdot 3\cdot 5}_{1/5!}}+\frac{1}{\underbrace{2^4\cdot 3^2\cdot 5}_{1/6!}}+\frac{1}{\underbrace{2^4\cdot 3^2\cdot 5\cdot 7}_{1/7!}}$$

$$+\frac{1}{\underbrace{2^7\cdot 3^2\cdot 5\cdot 7}_{1/8!}}+\frac{1}{\underbrace{2^7\cdot 3^4\cdot 5\cdot 7}_{1/9!}}+\frac{1}{\underbrace{2^8\cdot 3^4\cdot 5^2\cdot 7}_{1/10!}}+\frac{1}{\underbrace{2^8\cdot 3^4\cdot 5^2\cdot 7\cdot 11}_{1/11!}}$$

$$+\frac{1}{\underbrace{2^{10}\cdot 3^5\cdot 5^2\cdot 7\cdot 11}_{1/12!}}+\frac{1}{\underbrace{2^{10}\cdot 3^5\cdot 5^2\cdot 7\cdot 11\cdot 13}_{1/13!}}+\frac{1}{\underbrace{2^{11}\cdot 3^5\cdot 5^2\cdot 7^2\cdot 11\cdot 13}_{1/14!}}$$

$$+\frac{1}{\underbrace{2^{11}\cdot 3^6\cdot 5^3\cdot 7^2\cdot 11\cdot 13}_{1/15!}}+\frac{1}{\underbrace{2^{15}\cdot 3^6\cdot 5^3\cdot 7^2\cdot 11\cdot 13}_{1/16!}}$$

$$+\frac{1}{\underbrace{2^{15}\cdot 3^6\cdot 5^3\cdot 7^2\cdot 11\cdot 13\cdot 17}_{1/17!}}+\frac{1}{\underbrace{2^{16}\cdot 3^8\cdot 5^3\cdot 7^2\cdot 11\cdot 13\cdot 17}_{1/18!}}$$

$$+\frac{1}{\underbrace{2^{16}\cdot 3^8\cdot 5^3\cdot 7^2\cdot 11\cdot 13\cdot 17\cdot 19}_{1/19!}}+\frac{1}{\underbrace{2^{18}\cdot 3^8\cdot 5^4\cdot 7^2\cdot 11\cdot 13\cdot 17\cdot 19}_{1/20!}}$$

$$\frac{1}{\underbrace{2^{18}\cdot 3^9\cdot 5^4\cdot 7^3\cdot 11\cdot 13\cdot 17\cdot 19}_{1/21!}}+\frac{1}{\underbrace{2^{19}\cdot 3^9\cdot 5^4\cdot 7^3\cdot 11^2\cdot 13\cdot 17\cdot 19}_{1/22!}}$$

$$+\frac{1}{\underbrace{2^{19}\cdot 3^9\cdot 5^4\cdot 7^3\cdot 11^2\cdot 13\cdot 17\cdot 19\cdot 23}_{1/23!}}$$

$$s_2=\frac{1}{2}$$

$$s_3 = \frac{1}{2} + \frac{1}{2\cdot 3} = \frac{2}{3}$$

$$s_4 = \frac{2}{3} + \frac{1}{2^3\cdot 3} = \frac{2\cdot 2^3 + 1}{2^3\cdot 3} = \frac{17}{2^3\cdot 3}$$

$$s_5 = \frac{17}{2^3\cdot 3} + \frac{1}{2^3\cdot 3\cdot 5} = \frac{17\cdot 5 + 1}{2^3\cdot 3\cdot 5} = \frac{86}{2^3\cdot 3\cdot 5} = \frac{43}{2^2\cdot 3\cdot 5}$$

$$s_6 = \frac{43}{2^2\cdot 3\cdot 5} + \frac{1}{2^4\cdot 3^2\cdot 5} = \frac{43\cdot 2^2\cdot 3 + 1}{2^4\cdot 3^2\cdot 5} = \frac{517}{2^4\cdot 3^2\cdot 5}$$

$$s_7 = \frac{517}{2^4\cdot 3^2\cdot 5} + \frac{1}{2^4\cdot 3^2\cdot 5\cdot 7} = \frac{517\cdot 7 + 1}{2^4\cdot 3^2\cdot 5\cdot 7}$$

$$= \frac{3620}{2^4\cdot 3^2\cdot 5\cdot 7} = \frac{181}{2^2\cdot 3^2\cdot 7}$$

$$s_8 = \frac{181}{2^2\cdot 3^2\cdot 7} + \frac{1}{2^7\cdot 3^2\cdot 5\cdot 7} = \frac{181\cdot 2^5\cdot 5 + 1}{2^7\cdot 3^2\cdot 5\cdot 7} = \frac{28,961}{2^7\cdot 3^2\cdot 5\cdot 7}$$

$$s_9 = \frac{28,961}{2^7\cdot 3^2\cdot 5\cdot 7} + \frac{1}{2^7\cdot 3^4\cdot 5\cdot 7} = \frac{(28,961)\cdot 3^2 + 1}{2^7\cdot 3^4\cdot 5\cdot 7} = \frac{260,650}{2^7\cdot 3^4\cdot 5\cdot 7}$$

$$= \frac{26,065}{2^6\cdot 3^4\cdot 7}$$

$$s_{10} = \frac{26,065}{2^6\cdot 3^4\cdot 7} + \frac{1}{2^8\cdot 3^4\cdot 5^2\cdot 7} = \frac{(26,065)\cdot 2^2 + 1}{2^8\cdot 3^4\cdot 5^2\cdot 7} = \frac{104,261}{2^8\cdot 3^4\cdot 5^2\cdot 7}$$

$$s_{11} = \frac{104,261}{2^8\cdot 3^4\cdot 5^2\cdot 7} + \frac{1}{2^8\cdot 3^4\cdot 5^2\cdot 7\cdot 11} = \frac{(104,261)\cdot 10 + 1}{2^8\cdot 3^5\cdot 5^2\cdot 7}$$

$$= \frac{1,042,611}{2^8\cdot 3^4\cdot 5^2\cdot 7} = \frac{347,547}{2^8\cdot 3^3\cdot 5^2\cdot 7}$$

$$s_{12} = \frac{347,547}{2^8 \cdot 3^3 \cdot 5^2 \cdot 7} + \frac{1}{2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11} = \frac{(347,547) \cdot 2^2 \cdot 3^2 + 1}{2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11}$$

$$= \frac{12,511,693}{2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11}$$

$$s_{13} = \frac{12,511,693}{2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11} + \frac{1}{2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13} = \frac{(12,511,693) \cdot 13 + 1}{2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13}$$

$$= \frac{161,652,010}{2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13} = \frac{16165201}{2^9 \cdot 3^5 \cdot 5 \cdot 7 \cdot 11}$$

$$s_{14} = \frac{16,165,201}{2^9 \cdot 3^5 \cdot 5 \cdot 7 \cdot 11} + \frac{1}{2^{11} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13} = \frac{16,165,201 \cdot 2^2 \cdot 13 + 1}{2^{11} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13}$$

$$= \frac{845,790,453}{2^{11} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13} = \frac{93,976,717}{2^{11} \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13}$$

$$s_{15} = \frac{93,976,717}{2^{11} \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13} + \frac{1}{2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13}$$

$$= \frac{93,976,717 \cdot 3^3 \cdot 5 + 1}{2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13} = \frac{1,057,238,066}{2^9 \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13}$$

$$s_{16} = \frac{1,057,238,066}{2^9 \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13} + \frac{1}{2^{15} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13}$$

$$= \frac{1,057,238,066 \cdot 2^6 + 1}{2^{15} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13}$$

....

That trend continues so that the denominator of the sum of the series includes a product of infinitely many prime numbers

The partial sums converge to

$$\frac{\text{infinite sum of integers}}{\text{product that includes infinitely many primes}},$$

which is e.

By our criteria, e is a transcendental number.

Consequently, **for any** $k = 1,2,3,...$ e^k **is transcendental**.

4.

γ is a Transcendental Number

By Vacca[2]

$$\gamma = 1\left(\frac{1}{2}-\frac{1}{3}\right) + 2\left(\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}\right) + 3\left(\frac{1}{8}-\frac{1}{9}+\frac{1}{10}-\frac{1}{11}+\frac{1}{12}-\frac{1}{13}+\frac{1}{14}-\frac{1}{15}\right) + \dots + m\left(\frac{1}{2^m}-\frac{1}{2^m+1}+\dots+\frac{1}{2^{m+1}-2}-\frac{1}{2^{m+1}-1}\right) + \dots$$

That is,

$$\gamma = 1\frac{1}{2\cdot 3} + 2\frac{1}{4\cdot 5} + 2\frac{1}{6\cdot 7} + 3\frac{1}{8\cdot 9} + 3\frac{1}{10\cdot 11} + 3\frac{1}{12\cdot 13} + 3\frac{1}{14\cdot 15} + \dots + m\frac{1}{2^m(2^m+1)} + \dots + m\frac{1}{(2^{m+1}-2)(2^{m+1}-1)} + \dots$$

$$\gamma = 0.57721\ 56649\ 01532\ \dots$$

$$s_1 = \frac{1}{2\cdot 3}$$

[2] Steven R. Finch, "Mathematical Constants", Cambridge U Press, 2003, p.31

The denominator has the primes product

$$2 \cdot 3.$$

$$s_2 = \frac{1}{2 \cdot 3} + 2\frac{1}{4 \cdot 5} = \frac{5}{2 \cdot 3 \cdot 5} + \frac{3}{2 \cdot 3 \cdot 5} = \frac{4}{3 \cdot 5}$$

The denominator has the primes product

$$3 \cdot 5.$$

$$s_3 = \frac{4}{3 \cdot 5} + 2\frac{1}{6 \cdot 7} = \frac{4 \cdot 7}{3 \cdot 5 \cdot 7} + \frac{5}{3 \cdot 5 \cdot 7} = \frac{11}{5 \cdot 7}$$

The denominator has the primes product

$$5 \cdot 7.$$

$$s_4 = \frac{11}{3 \cdot 5} + 3\frac{1}{8 \cdot 9} = \frac{11 \cdot 8}{3 \cdot 5 \cdot 8} + \frac{5}{3 \cdot 5 \cdot 8} = \frac{31}{5 \cdot 8}$$

The denominator has the primes product

$$2^3 \cdot 5$$

$$s_5 = \frac{31}{5 \cdot 8} + 3\frac{1}{10 \cdot 11} = \frac{31 \cdot 11}{5 \cdot 8 \cdot 11} + \frac{3 \cdot 4}{2 \cdot 5 \cdot 11 \cdot 4} = \frac{353}{5 \cdot 8 \cdot 11}$$

The denominator has the primes product

$$2^3 \cdot 5 \cdot 11.$$

$$s_6 = \frac{353}{5 \cdot 8 \cdot 11} + 3\frac{1}{12 \cdot 13} = \frac{353 \cdot 13}{5 \cdot 8 \cdot 11 \cdot 13} + \frac{2 \cdot 5 \cdot 11}{5 \cdot 8 \cdot 11 \cdot 13} = \frac{4699}{5 \cdot 8 \cdot 11 \cdot 13}$$

The denominator has the primes product

$$2^3 \cdot 5 \cdot 11 \cdot 13.$$

$$s_7 = \frac{4699}{5\cdot 8\cdot 11\cdot 13} + 3\frac{1}{14\cdot 15} = \frac{4699\cdot 7}{5\cdot 7\cdot 8\cdot 11\cdot 13} + \frac{4\cdot 11\cdot 13}{5\cdot 2\cdot 7\cdot 4\cdot 11\cdot 13}$$
$$= \frac{33465}{5\cdot 7\cdot 8\cdot 11\cdot 13}$$
$$= \frac{6693}{7\cdot 8\cdot 11\cdot 13}$$

The denominator has the primes product

$$2^3\cdot 7\cdot 11\cdot 13.$$

$$s_8 = \frac{6693}{7\cdot 8\cdot 11\cdot 13} + 4\frac{1}{16\cdot 17} = \frac{6693\cdot 2\cdot 17}{2\cdot 7\cdot 8\cdot 11\cdot 13\cdot 17} + \frac{2\cdot 7\cdot 11\cdot 13}{7\cdot 8\cdot 11\cdot 13\cdot 17}$$
$$= \frac{229564}{7\cdot 8\cdot 11\cdot 13\cdot 17}$$
$$= \frac{57391}{7\cdot 2\cdot 11\cdot 13\cdot 17}$$

The denominator has the primes product

$$2\cdot 7\cdot 11\cdot 13\cdot 17.$$

$$s_9 = \frac{57391}{2\cdot 7\cdot 11\cdot 13} + 4\frac{1}{18\cdot 19} = \frac{57391\cdot 9\cdot 19}{2\cdot 7\cdot 9\cdot 11\cdot 13\cdot 19} + \frac{4\cdot 7\cdot 11\cdot 13}{2\cdot 7\cdot 9\cdot 11\cdot 13\cdot 19}$$
$$= \frac{9817865}{2\cdot 7\cdot 9\cdot 11\cdot 13\cdot 19}$$

The denominator has the primes product

$$2\cdot 3^2\cdot 7\cdot 11\cdot 13\cdot 19.$$

…………………………………………………………..

That trend continues so that the denominator of the sum of the series includes a product of infinitely many prime numbers

The partial sums converge to

$$\frac{\text{infinite sum of integers}}{\text{product that includes infinitely many primes } 3\cdot 5\cdot 7\cdot \ldots.},$$

which is γ.

By our Criteria, γ is transcendental number.

Consequently, **for any** $k = 1,2,3,...$ γ^k **is transcendental**.

5.
Catalan's Constant C is a Transcendental Number

The sum of the expansion of Catalan's Constant C in fractions

$$C = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + ...$$

has a denominator that includes the product of infinitely many squared primes.

Thus, by our Criteria for transcendental number, C is a transcendental number.

Thus, for any $k = 1, 2, 3, \ldots$, C^k **are transcendental numbers**

6.
$\zeta(n)$ is a Transcendental Number

For each $n = 2, 3, 4, ..$,

$$\zeta(n) = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + ...$$

has a denominator that includes the product of infinitely many n-powered primes.

Thus, by our Transcendental Number Criteria, $\zeta(n)$ is a transcendental number.

Thus, for each $n = 2, 3, 4, ..$, and for any $k = 1, 2, 3, ...$,

$(\zeta(n))^k$ **are transcendental numbers**

7. πe, $\pi\gamma$, $e\gamma$, $\pi e\gamma$, $\pi + e$, $\pi + \gamma$, $e + \gamma$, $\pi^k + e^l + \gamma^m$, $k, l, m,$=1,2,3,.. are Transcendental Numbers

Each of the numbers

$$\pi e \quad \pi\gamma \quad e\gamma \quad e\pi\gamma$$

$$\pi + e \quad \pi + \gamma \quad e + \gamma$$

$$\pi^k + e^l + \gamma^m, \quad k, l, m, =1,2,3,..$$

has expansion in fractions which denominator contains product of infinitely many primes.

Therefore, by our Transcendental Number Criteria, each of these numbers is a transcendental number.

8.
Transcendence of Exponentials

By the Gelfond-Schneider theorem

$$\text{algebraic } a \neq 0,1, \text{ algebraic irrational } b \Rightarrow$$

$$\Rightarrow a^b = e^{b \log a} \text{ is a transcendental number}$$

For instance,

$$\sqrt{2}^{\sqrt{3}} \text{ is transcendental number.}$$

That leaves out exponentiation of e.

The Criteria does not apply if the series expansion is in transcendental numbers that themselves have expansion in fractions.

For instance, we have shown[3]

$$1 = \frac{\pi}{4} + \frac{1}{3}\left(\frac{\pi}{4}\right)^3 + \frac{2}{15}\left(\frac{\pi}{4}\right)^5 + \frac{17}{315}\left(\frac{\pi}{4}\right)^7 + \frac{62}{2835}\left(\frac{\pi}{4}\right)^9 +$$

$$+ \frac{(819)(691)}{3^6 5^2 7(11)(91)}\left(\frac{\pi}{4}\right)^{11} + \frac{5461}{3^5 5^2 7(11)(13)}\left(\frac{\pi}{4}\right)^{13} + ... + \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_n \left(\frac{\pi}{4}\right)^{2n-1} + ...$$

Then, each partial sum is transcendental and involves powers of π. Each of these powers can be expanded in series with sums that have denominators that satisfy the transcendence criteria.

But the sum will include terms such as

$$\left(4(1 - \frac{1}{3} + \frac{1}{5} - ...)\right)^{\infty}$$

[3] Archimedes Series Gauge Institute Journal, Volume 18, No. 3, August 2022

to which our criteria does not apply.

Similarly, we cannot tell about e^{π}

$$e^{\pi} = 1 + \left(4(1 - \frac{1}{3} + \frac{1}{5} - ...)\right) + \frac{1}{2}\left(4(1 - \frac{1}{3} + \frac{1}{5} - ...)\right)^2 + \frac{1}{2 \cdot 3}\left(4(1 - \frac{1}{3} + \frac{1}{5} - ...)\right)^3 + ...$$

Then, every partial sum is transcendental. And the sum is bounded by 23.15.

But our criteria need not apply to terms like

$$\left(4(1 - \frac{1}{3} + \frac{1}{5} - ...)\right)^{\infty}.$$

Similarly, we cannot tell about the exponential of γ

$$\exp(\gamma) = 1 + \gamma + \frac{1}{2}\gamma^2 + \frac{1}{2 \cdot 3}\gamma^3 + ..,$$

Then, the sum includes terms such as

$$(\text{vacca expansion of } \gamma)^{\infty}.$$

And we cannot tell about the exponential of e,

$$\exp(e) = 1 + e + \frac{1}{2}e^2 + \frac{1}{2 \cdot 3}e^3 + \frac{1}{2 \cdot 3 \cdot 4}e^4 + ...$$

Then, the sum includes terms such as

$$(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + ...)^{\infty}.$$

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