# The College Mathematics Journal explains why Pi is a constant

H. Vic Dannon
vic0@comcast.net
June, 2022

A circle of radius $a$ has circumference

$$c(a) = \int_{\theta=0}^{\theta=360^\circ} a d\theta = a(360^\circ)$$

Therefore,

$$\frac{c(a)}{a} = 360^\circ = 2\pi \text{ radians} = \text{constant}$$

That is why $c(a) / a$ is a constant.

The circumference of the unit circle is

$$c(1) = 2\pi .$$

To determine $\pi$, Archimedes approximated the circle by two sequences of polygons. One sequence inscribed in the circle ascending to it, and another sequence enveloping the circle, descending to it. He found[1]

$$\frac{223}{71} < \pi < \frac{22}{7} .$$

The College Mathematics Journal[2] explains confusingly why $c(a) / a$ is a constant:

[1] Victor Katz, "A History of Mathematics", 3rd edition, Addison Wesley, 2009, p.101.
[2] May 2022, pp. 178-179

*In* $x, y$ *coordinates*[3]*, The circumference*

$$x^2 + y^2 = r^2$$

*can be divided into two semicircles*

$$y = \pm\sqrt{r^2 - x^2}\,,$$

*each with length* $C \,/\, 2$.

*...Consider only the upper one whose derivative is*

$$y'(x) = -\frac{x}{\sqrt{r^2 - x^2}}$$

*From* $Length = \int\limits_{x=a}^{x=b} \sqrt{1 + [y'(x)]^2}dx$*, one finds*

$$\frac{C}{2} = \int\limits_{x=-r}^{x=r} \sqrt{1 + \frac{x^2}{r^2 - x^2}}dx$$

$$= 2\int\limits_{x=0}^{x=r} \sqrt{1 + \frac{x^2}{r^2 - x^2}}dx$$

$$= 2\int\limits_{x=0}^{x=r} \sqrt{\frac{r^2}{r^2 - x^2}}dx$$

$$= 2\int\limits_{x=0}^{x=r} \sqrt{\frac{1}{1 - \frac{x^2}{r^2}}}dx$$

*This is an improper integral*[4]*. So let us write it as*

---

[3] The symmetry of the circle, mandates $r, \theta$ coordinates, on a circle of radius $a$, $ad\theta$ is the length of an arc $d\theta$. Eventually, they must transfer to polar coordinates.

[4] substituting $x = r\sin\theta$ earlier, would have saved them from the improper integral, and $\varepsilon$

$$\lim_{\varepsilon\to 0+} \int_{x=0}^{x=r-\varepsilon} \sqrt{\frac{1}{1-\frac{x^2}{r^2}}}dx$$

*Substituting*

$$x = r\sin\theta,$$

*one finds*

$$\frac{C}{2} = 2\lim_{\varepsilon\to 0+} \int_{\theta=0}^{\theta=\arcsin\frac{r-\varepsilon}{r}} \frac{r\cos\theta}{\cos\theta}\,d\theta$$

$$= 2r\lim_{\varepsilon\to 0+} \arcsin\frac{r-\varepsilon}{r}$$

$$= 2r\arcsin(1).$$

The College Mathematics Journal calls this

"answering deep questions about $\pi$"...