# Delta Function and Optical Catastrophe Models

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## Abstract

We propose models for the propagation of intense electromagnetic ultrashort pulses.

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**Contents**

# Introduction

Intense electromagnetic pulses can serve as bullets that demolish matter. Such pulses have been used as scalpels in surgery [W-4], and as shells by the military to shoot down missiles [W-5].

While dissipation is not a concern for surgical pulses, it requires that the military pulses be very powerful pulses in order for them to be effective on a remote target.

In [Dan10], we showed that the Fundamental Optical Soliton can be considered a Delta Function, and here we will show that the model that serves to describe its propagation, guarantees no dissipation.

The Delta Function is not the limit of a Delta sequence as presented in Engineering, and in Physics, and its singularity does not disappear when it is presented as a Generalized Functional in Mathematics.

We have shown that the Delta Function is a Hyper-real Function defined on the hyper-real line, an infinite dimensional line that has room for infinitesimals, and their reciprocals, the infinite hyper-reals.

In the next sections, we sum up the main points necessary for the definition, and the application of the Delta function

# 1.

# Hyper-real Line

Each real number $\alpha$ can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of infinitesimal hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the infinite hyper-reals.
4. The infinite hyper-reals are greater than any real number, yet strictly smaller than infinity.
5. The infinite hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a non-constant hyper-real.
7. The Hyper-reals are the totality of constant hyper-reals, a family of infinitesimals, a family of infinitesimals with

negative sign, a family of infinite hyper-reals, a family of infinite hyper-reals with negative sign, and non-constant hyper-reals.

8. The hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.

9. That line includes the real numbers separated by the non-constant hyper-reals. Each real number is the center of an interval of hyper-reals, that includes no other real number.

10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.

11. Zero is not an infinitesimal, because zero is not strictly greater than zero.

12. We do not add infinity to the hyper-real line.

13. The infinitesimals, the infinitesimals with negative signs, the infinite hyper-reals, and the infinite hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.

14. The hyper-real line is embedded in $\mathbb{R}^{\infty}$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the hyper-real onto the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal hyper-reals, or

to the infinite hyper-reals, or to the non-constant hyper-reals.

16. No neighbourhood of a hyper-real is homeomorphic to an $\mathbb{R}^n$ ball. Therefore, the hyper-real line is not a manifold.

17. The hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

# 2.

# Hyper-Real Integral

In [Dan3], we defined the integral of a Hyper-real Function.

Let $f(x)$ be a hyper-real function on the interval $[a,b]$.

The interval may not be bounded.

$f(x)$ may take infinite hyper-real values, and need not be bounded.

At each

$$a \le x \le b\,,$$

there is a rectangle with base $[x - \frac{dx}{2}, x + \frac{dx}{2}]$, height $f(x)$, and area

$$f(x)dx\,.$$

We form the **Integration Sum** of all the areas for the $x$'s that start at $x = a$, and end at $x = b$,

$$\sum_{x \in [a,b]} f(x)dx\,.$$

If for any infinitesimal $dx$, the Integration Sum has the same hyper-real value, then $f(x)$ is integrable over the interval $[a,b]$.

Then, we call the Integration Sum the integral of $f(x)$ from $x = a$, to $x = b$, and denote it by

$$\int\limits_{x=a}^{x=b} f(x)dx\,.$$

If the hyper-real is infinite, then it is the integral over $[a,b]$,

If the hyper-real is finite,

$$\int_{x=a}^{x=b} f(x)dx = \text{ real part of the hyper-real}. \square$$

## 2.1 The countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers,

$Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = .... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty.$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[a,b]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(x)dx$.

The Lower Integral is the Integration Sum where $f(x)$ is replaced by its lowest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

**2.2**

$$\sum_{x\in[a,b]}\left(\inf_{x-\frac{dx}{2}\le t\le x+\frac{dx}{2}} f(t)\right)dx$$

The Upper Integral is the Integration Sum where $f(x)$ is replaced by its largest value on each interval $[x-\frac{dx}{2}, x+\frac{dx}{2}]$

**2.3**

$$\sum_{x\in[a,b]}\left(\sup_{x-\frac{dx}{2}\le t\le x+\frac{dx}{2}} f(t)\right)dx$$

If the integral is a finite hyper-real, we have

**2.4** *A hyper-real function has a finite integral if and only if its upper integral and its lower integral are finite, and differ by an infinitesimal.*

# 3.

# Delta Function

In [Dan5], we defined the Delta Function, and established its properties

1. The Delta Function is a hyper-real function defined from the hyper-real line into the set of two hyper-reals $\left\{0, \frac{1}{dx}\right\}$. The hyper-real $0$ is the sequence $\left\langle 0,0,0,... \right\rangle$. The infinite hyper-real $\frac{1}{dx}$ depends on our choice of $dx$.
2. We will usually choose the family of infinitesimals that is spanned by the sequences $\left\langle \frac{1}{n} \right\rangle, \left\langle \frac{1}{n^2} \right\rangle, \left\langle \frac{1}{n^3} \right\rangle, ...$ It is a semigroup with respect to vector addition, and includes all the scalar multiples of the generating sequences that are non-zero. That is, the family includes infinitesimals with negative sign. Therefore, $\frac{1}{dx}$ will mean the sequence $\left\langle n \right\rangle$. Alternatively, we may choose the family spanned by the sequences $\left\langle \frac{1}{2^n} \right\rangle, \left\langle \frac{1}{3^n} \right\rangle, \left\langle \frac{1}{4^n} \right\rangle, ...$ Then, $\frac{1}{dx}$ will mean the sequence $\left\langle 2^n \right\rangle$. Once we determined the basic infinitesimal

$dx$, we will use it in the Infinite Riemann Sum that defines an Integral in Infinitesimal Calculus.

3. The Delta Function is strictly smaller than $\infty$

4. We define, $\delta(x) \equiv \frac{1}{dx} \chi_{\left[-\frac{dx}{2}, \frac{dx}{2}\right]}(x)$,

   where $\chi_{\left[-\frac{dx}{2}, \frac{dx}{2}\right]}(x) = \begin{cases} 1, x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right] \\ 0, \text{otherwise} \end{cases}$.

5. Hence,

   - for $x < 0$, $\delta(x) = 0$
   - at $x = -\frac{dx}{2}$, $\delta(x)$ jumps from $0$ to $\frac{1}{dx}$,
   - for $x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right]$, $\delta(x) = \frac{1}{dx}$.
   - at $x = 0$, $\delta(0) = \frac{1}{dx}$
   - at $x = \frac{dx}{2}$, $\delta(x)$ drops from $\frac{1}{dx}$ to $0$.
   - for $x > 0$, $\delta(x) = 0$.
   - $x\delta(x) = 0$

6. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle \chi_{\left[-\frac{1}{2}, \frac{1}{2}\right]}(x), 2\chi_{\left[-\frac{1}{4}, \frac{1}{4}\right]}(x), 3\chi_{\left[-\frac{1}{6}, \frac{1}{6}\right]}(x) ... \right\rangle$

7. If $dx = \left\langle \frac{2}{n} \right\rangle$, $\delta(x) = \left\langle \frac{1}{2\cosh^2 x}, \frac{2}{2\cosh^2 2x}, \frac{3}{2\cosh^2 3x}, ... \right\rangle$

8. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle e^{-x}\chi_{[0,\infty)}, 2e^{-2x}\chi_{[0,\infty)}, 3e^{-3x}\chi_{[0,\infty)}, ... \right\rangle$

**9.** $$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1.$$

# 4.

# The Fourier Transform

In [Dan6], we defined the Fourier Transform and established its properties

1. $\mathcal{F}\left\{\delta(x)\right\} = 1$

2. $\delta(x)$ = *the inverse Fourier Transform of the unit function* $1$

$$= \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x} d\omega$$

$$= \int_{\nu=-\infty}^{\nu=\infty} e^{2\pi i x} d\nu\,, \quad \omega = 2\pi\nu$$

3. $\left.\frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x} d\omega\right|_{x=0} = \frac{1}{dx} =$ an infinite hyper-real

$$\left.\int_{\omega=-\infty}^{\omega=\infty} e^{i\omega x} d\omega\right|_{x\neq 0} = 0$$

4. Fourier Integral Theorem

$$f(x) = \frac{1}{2\pi}\int_{k=-\infty}^{k=\infty}\left(\int_{\xi=-\infty}^{\xi=\infty} f(\xi)e^{-ik\xi} d\xi\right)e^{ikx} dk$$

does not hold in the Calculus of Limits, under any conditions.

5. Fourier Integral Theorem in Infinitesimal Calculus

*If* $f(x)$ *is hyper-real function,*

*Then,*

- *the Fourier Integral Theorem holds.*
- $\int_{x=-\infty}^{x=\infty} f(x)e^{-i\alpha x}dx$ ***converges to*** $F(\alpha)$
- $\frac{1}{2\pi}\int_{\alpha=-\infty}^{\alpha=\infty} F(\alpha)e^{-i\alpha x}d\alpha$ ***converges to*** $f(x)$

# 5.

# The Laplace Transform

In [Dan7], we have shown that

1. The Delta function $\delta(t)$, for $t \geq 0$ is represented by Sequence

   $\delta_n(t) = ne^{-nt}\chi_{_{[0,\infty)}}(t)$.

2. If $i_n = \frac{1}{n}$, $\quad \delta(x) = \left\langle e^{-t}\chi_{_{[0,\infty)}}(t), 2e^{-2t}\chi_{_{[0,\infty)}}(t), 3e^{-3t}\chi_{_{[0,\infty)}}(t), \ldots \right\rangle$

3. $\mathcal{L}\left\{\delta(t)\right\} = 1$

4. $\delta(t) =$ *the inverse Laplace Transform of the unit function* $1$

   $$= \frac{1}{2\pi i}\int_{s=-i\infty}^{s=i\infty} e^{st}ds\,.$$

5. $\frac{1}{2\pi i}\int_{s=\gamma-i\infty}^{s=\gamma+i\infty} e^{st}ds\Bigg|_{t=0} = \frac{1}{dt} =$ an infinite hyper-real

   $$\int_{s=\gamma-i\infty}^{s=\gamma+i\infty} e^{st}ds\Bigg|_{t\neq 0} = 0\,.$$

6. Laplace Integral Theorem

   *If* $f(t)$ *is hyper-real function,*

   *Then,*

- *the Laplace Integral Theorem holds.*

$$f(t) = \frac{1}{2\pi i} \int_{s=\gamma-i\infty}^{s=\gamma+i\infty} e^{st} \left( \int_{\tau=0}^{\tau=\infty} e^{-s\tau} f(\tau) d\tau \right) ds$$

- $\int_{t=0}^{t=\infty} e^{-st} f(t) dt$, ***converges to*** $F(s)$
- $\frac{1}{2\pi i} \int_{s=\gamma-i\infty}^{s=\gamma+i\infty} e^{st} F(s) ds$ ***converges to*** $f(t)$.

# 6.

# Optical Catastrophe Model for the Free Electron Laser Pulses

In the Free electron Laser [W-5], an electron beam accelerated to almost light speed, is forced into a wiggling sinusoidal path by a transverse magnetic field with alternating poles.

The radiation is coherent, in wavelengths ranging from microwaves to X-rays.

Since X-rays penetrate mirrors, the Free Electron Laser operates without mirrors, hence without an oscillator, and the pulse has to be created in a single pass of the beam.

The emitted power increases as coherence increases, but the pulse does not shape as a Soliton, and as it propagates in the $z$ direction, it is subject to broadening, and loss.

We shall approximate it by a Gaussian Pulse.

Thus, [e.g. Ghatak], its power is proportional to

$$\frac{\sigma_t^2}{\left(\sigma_t^4 + \beta_2^2 z^2\right)^{\frac{1}{2}}} e^{\frac{-2\sigma_t^2}{\sigma_t^4 + \beta_2^2 z^2}(t-\beta_1 z)^2}$$

where

$\sigma_t$ is the width of the pulse,

$\beta_1 = \partial_\omega \beta\Big|_{\substack{\omega=\omega_0 \\ E_0=0}}$ is the dispersion coefficient in the air

$\beta_2 = \partial_\omega^2 \beta\Big|_{\substack{\omega=\omega_0 \\ E_0=0}}$ is the second dispersion coefficient in the air.

Thus, the pulse power decays with the distance $z$.

Furthermore, the radiation from the electron beam disperses on the surface of a radially expanding cylinder.

# 7.

# Optical Catastrophe Model for Optical Solitons

The Solitons intensity and width qualifies them to be considered as Delta function.

We shall model the propagation of the Soliton in the air by the Nonlinear Schrödinger equation with the nonlinearity, and dispersion removed, forced by the Delta function.

$$\partial_{ct} G(x,t) + \partial_x G(x,t) = \delta(x)\delta(t)$$

**7.1** Let $G(x,t)$ be hyper-real in $x$, and in $t$, and have Principal Value Derivatives with respect to $x$, and $t$, that satisfy the wave equation

$$\frac{1}{c}\partial_t G(x,t) + \partial_x G(x,t) = \delta(x)\delta(t),$$

with

$$G(x,0) = 0,$$

Then,

$$G(x,t) = c\delta(ct - x)$$

*Proof*:

In the following, all functions are Hyper-Real.

The Laplace Transform of the hyper-real $G(x,t)$ exists with respect to $t$:

$$\mathcal{L}\left\{G(x,t)\right\} = \int_{t=0}^{t=\infty} e^{-st}G(x,t)dt \equiv g(x,s)$$

$$\begin{aligned}
\mathcal{L}\left\{\partial_t G(x,t)\right\} &= \int_{t=0}^{t=\infty} e^{-st}\partial_t G(x,t)dt \\
&= e^{-st}G(x,t)\Big|_{t=0}^{t=\infty} - \int_{t=0}^{t=\infty} \partial_t \left\{e^{-st}\right\}G(x,t)dt \\
&= -\underbrace{G(x,0)}_{=0} + s\int_{t=0}^{t=\infty} e^{-st}G(x,t)dt \\
&= sg(x,s),
\end{aligned}$$

$$\mathcal{L}\left\{\delta(t)\right\} = 1,$$

Thus, the hyper-real equation $\frac{1}{c}\partial_t G(x,t) + \partial_x G(x,t) = \delta(x)\delta(t)$, Laplace Transforms into

$$\frac{1}{c}sg(x,s) + \partial_x g(x,s) = \delta(x).$$

The Fourier Transform of $g(x,s)$ with respect to $x$ is

$$\mathcal{F}\left\{g(x,s)\right\} = \int_{x=-\infty}^{x=\infty} e^{-i\omega x}g(x,s)dx \equiv \tilde{g}(\omega,s)$$

The Inverse Fourier Transform is

$$g(x,s) = \frac{1}{2\pi} \int_{\nu=-\infty}^{\nu=\infty} e^{i\omega x} \tilde{g}(\omega,s) d\omega$$

Hence,

$$\partial_x g(x,s) = \frac{1}{2\pi} \int_{\nu=-\infty}^{\nu=\infty} \partial_x \left\{ e^{i\omega x} \right\} \tilde{g}(\omega,s) d\omega$$

$$= \frac{1}{2\pi} \int_{\nu=-\infty}^{\nu=\infty} e^{i\omega x} i\omega \tilde{g}(\omega,s) d\omega \,,$$

Therefore, the Fourier Transform of $\partial_x g(x,s)$ with respect to $x$ is

$$\mathcal{F}\left\{ \partial_x g(x,s) \right\} = i\omega \tilde{g}(\omega,s) .$$

Since $\mathcal{F}\left\{ \delta(x) \right\} = 1$, the equation $\frac{1}{c} s g(x,s) + \partial_x g(x,s) = \delta(x)$,

Fourier Transforms into

$$\frac{1}{c} s \tilde{g}(\omega,s) + i\omega \tilde{g}(\omega,s) = 1 .$$

Therefore,

$$\tilde{g}(\omega,s) = \frac{c}{s + i\omega c} .$$

By the Laplace Integral Theorem, The hyper-real $\tilde{g}(\omega,s)$ may be Inverse Laplace Transformed into a hyper-real function $\hat{G}(\omega,t)$:

$$\hat{G}(\omega,t) = \mathcal{L}^{-1}\left\{ \frac{c}{s + i\omega c} \right\},$$

From Laplace Transform Tables,

$$= c e^{-i\omega c t}$$

Hence,

$$\hat{G}(\omega, t) = ce^{-i\omega ct}.$$

By the Fourier Integral Theorem, The hyper-real $\hat{G}(\omega, t)$ may be Inverse Fourier Transformed into the hyper-real function $G(x, t)$:

$$\begin{aligned} G(x, t) &= c\delta(x - ct) \\ &= c\delta(ct - x) \\ &= \delta(t - \frac{x}{c}). \end{aligned}$$

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