Non-Cantorian Set Theory.

by H. Vic Dannon
vick@adnc.com
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Introduction:

In [1] we showed that in Cantor's set theory,

$$Card(0,1) = Card(rationals).$$

Since the rationals are countable,

$$Card(rationals) = \aleph_0^2 = \aleph_0,$$

and

$$\aleph_0 = Card(0,1) = 2^{\aleph_0}.$$

This disproves Cantor's claim that

$$\aleph_0 < 2^{\aleph_0},$$

and leads to a single infinity

$$\aleph_0 = 2^{\aleph_0}.$$

Will a non-Cantorian set theory allow for more infinities?

The existence of a non-Cantorian set theory was established in 1963 by Cohen's work on Cantor's Continuum Hypothesis that there is no set X with $\aleph_0 < CardX < 2^{\aleph_0}$.

Cohen proved that if the commonly accepted postulates of set theory are consistent, then the addition of the negation of the hypothesis does not result in inconsistency [2].

Cohen's result was interpreted to mean that there is a set theory where the negation of the Continuum Hypothesis holds. However, non-Cantorian cardinal numbers were not found, and the non-Cantorian set theory was never developed.

To develop a non Cantorian set theory, we will assume the negation of the Continuum Hypothesis, which is based on Cantor's claim that $\aleph_0 < 2^{\aleph_0}$. In [1] we disproved that claim, but here we will need to allow $\aleph_0 < 2^{\aleph_0}$ as an assumption. We aim to show that even with that disproved assumption, non Cantorian set theory does not exist.

To that end, we re-examine our proof of the Continuum Hypothesis in Cantor's set theory [3]. A close scrutiny of that proof reveals that rationals countability is equivalent to the Continuum Hypothesis.

1. Rationals countability $\equiv$ Continuum Hypothesis.

<u>*Theorem*</u> *Assume that* $\aleph_0 < 2^{\aleph_0}$. *Then*

$$\aleph_0^2 = \aleph_0 \Leftrightarrow \text{Continuum Hypothesis.}$$

Proof:

$(\Rightarrow)$

By [4, p.173], the sequence

$$\aleph_0 + \aleph_0^2 + \aleph_0^3 + ... + \aleph_0^n$$

sums up [4] to the series

$$\aleph_0 + \aleph_0^2 + \aleph_0^3 + ... + \aleph_0^n + ... = \sum_{n=1}^{\infty} \aleph_0^n .$$

By [4], the series has a well defined sum

$$\alpha \equiv \aleph_0 + \aleph_0^2 + \aleph_0^3 + \ldots + \aleph_0^n + \ldots,$$

which is a cardinal number.

By [4, p.174],

$$\alpha \geq \text{any component of the series.}$$

In particular, α is greater than the infinite product term

$$\alpha \geq \aleph_0 \cdot \aleph_0 \cdot \aleph_0 \cdot \ldots = \aleph_0^{1+1+1+\ldots}.$$

By [4, p.183],

$$\aleph_0^{1+1+1+\ldots} = \aleph_0^{\aleph_0}.$$

Therefore,

$$\alpha \geq \aleph_0^{\aleph_0}.$$

Or, using the product notation, by [5, p. 106], we obtain similarly,

$$\alpha \geq \prod_{n \in N} \aleph_0 = \aleph_0^{CardN} = \aleph_0^{\aleph_0}.$$

Since

$$\aleph_0^{\aleph_0} \geq 2^{\aleph_0},$$

by transitivity of cardinal inequalities [4, p. 147], we conclude that

$$\alpha \geq 2^{\aleph_0}.$$

Now suppose that the continuum hypothesis does not hold, and there is a set X with

$$\aleph_0 < CardX < 2^{\aleph_0}.$$

Since $\aleph_0^2 = \aleph_0$, then for any $n = 1,2,3...$,

$$\aleph_0 + \aleph_0^2 + \aleph_0^3 + ... + \aleph_0^n \leq CardX.$$

Tarski ([6], or [4, p.174]) proved that

If

$$m_1, m_2, ..m_n, \quad and \ m$$

are any cardinal numbers so that for any $n = 1,2,3....$,

$$m_1 + m_2 + ... + m_n \leq m,$$

then,

$$m_1 + m_2 + ... + m_n + ... \leq m.$$

By Tarski result

$$\aleph_0 + \aleph_0^2 + \aleph_0^3 + ... + \aleph_0^n + ... \leq CardX.$$

Since $CardX < 2^{\aleph_0}$, by transitivity of cardinal inequalities [4, p. 147],

$$\underbrace{\aleph_0 + \aleph_0^2 + \aleph_0^3 + ... + \aleph_0^n + ...}_{\alpha} < 2^{\aleph_0}$$

That is,

$$\alpha < 2^{\aleph_0},$$

which contradicts $\alpha \geq 2^{\aleph_0}$.

Therefore, there is no set X with $\aleph_0 < CardX < 2^{\aleph_0}$, and the continuum Hypothesis holds.

This completes the proof that $\aleph_0^2 = \aleph_0 \Rightarrow$ Continuum Hypothesis.$\square$

$(\Leftarrow)$

By [4, p.155], *For any cardinals* m, n,, m_1, and n_1,

$$m < n, \textit{ and } m_1 < n_1 \Rightarrow mm_1 < nn_1.$$

Since

$$\aleph_0 < 2^{\aleph_0},$$

we have

$$\aleph_0^2 < (2^{\aleph_0})^2.$$

But

$$(2^{\aleph_0})^2 = 2^{2\aleph_0} = 2^{\aleph_0}.$$

Hence,

$$\aleph_0^2 < 2^{\aleph_0}.$$

Therefore,

$$\aleph_0 < \aleph_0^2 \Rightarrow \aleph_0 < \aleph_0^2 < 2^{\aleph_0}.$$

Namely, if $\aleph_0 < \aleph_0^2$, the rationals serve as a set X which cardinality is between $CardN$,and $CardR$, and the Continuum Hypothesis does not hold. That is,

Negation of $\aleph_0^2 = \aleph_0 \Rightarrow$ Negation of the Continuum Hypothesis.

This says,

Continuum Hypothesis $\Rightarrow \aleph_0^2 = \aleph_0. \square$

In conclusion, the countability of the rationals is equivalent to the Continuum Hypothesis, and the uncountability of the rationals is equivalent to the negation of the Continuum Hypothesis.

Since the rationals are countable, non-Cantorian set theory does not exist. Consequently, the interpretation of Cohen's work, that there is a set theory in which the Continuum Hypothesis does not hold, is erroneous.

2. Non Cantorian Set Theory does not exist

Under Cantor's claim that

$$\aleph_0 < 2^{\aleph_0},$$

Cohen proved that the continuum hypothesis is an independent axiom of set theory.

Since by our Theorem the continuum Hypothesis is equivalent to rationals' countability, then according to Cohen, rationals' countability is an independent axiom of set theory.

Could the rationals be assumed uncountable?

If you seek a clue to the answer in Cantor's Zig-Zag proof of Rationals' countability, you'll find that the Zig-Zag proof is flawed.

Cantor's Zig-Zag aims to avoid following through infinitely many infinite sequences. Is it possible that towards its end, the Zig-Zag can not avoid the infinitely many infinite sequences?

Cantor's mapping in his Zig Zag proof that the rational numbers are countable has to be one-one. But in [7] we pointed out that it is not one-one.

However, we exhibited in [7] a one-one mapping from the rationals into the natural numbers, and established the effective countability of the rationals.

Hence, by our Theorem above, the Continuum Hypothesis is a fact, as much as the countability of the rationals is.

The Continuum Hypothesis is not an independent axiom of set theory.

The negation of the Continuum Hypothesis is equivalent to the negation of rationals' countability.

A non-Cantorian set theory based on the negation of the Continuum Hypothesis was not found so far, and will never be found, because it does not exist.

References

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[3] Dannon, H. Vic, *Hilbert's 1st Problem: Cantor's Continuum Hypothesis*. in ABSTRACTS of Papers Presented to the American Mathematical Society. Vol. 26, Number 2, issue 140, p. 405; in Gauge Institute Journal Vol.1 No 1, February 2005; Posted to www.gauge-institute.org

[4] Sierpinski, Waclaw, *Cardinal and Ordinal Numbers.* Warszawa, 1958 (or 2nd edition)

[5] Levy, Azriel, *Basic Set Theory*. Dover, 2002.

[6] Tarski, A., Axiomatic and algebraic aspects on two theorems on sums of cardinals. Fund. Math. 35 (1948), p.79-104.

[7] Dannon, H. Vic, *Rationals Countability and Cantor's Proof*, Gauge Institute Journal Volume 2, No 1, February 2006. Posted to www.gauge-institute.org as *Cantor's Proof of Rationals Countability*.