The Neutrino Mixing Matrix

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March, 2014

Abstract: We establish with Chi-Squared Goodness-of-Fitness-Test that the Neutrino Mixing Matrix,

$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{12}s_{23} - s_{12}c_{23}s_{13} & c_{23}c_{13} \end{pmatrix}$$

where $c_{ij} = \cos\theta_{ij}$, and $s_{ij} = \sin\theta_{ij}$, can be approximated to a high degree accuracy by a matrix that depends only on θ_{12}.

$$\left|U_{e1}\right| \approx \sqrt{1-\sin^2\theta_{12}}\sqrt{1-\sin^6\theta_{12}}$$

$$\left|U_{e2}\right| \approx \sin\theta_{12}\sqrt{1-\sin^6\theta_{12}}$$

$$\left|U_{e3}\right| \approx \sin^3\theta_{12}$$

$$\left|U_{\mu 1}\right| \approx \sin\theta_{12}\sqrt{1-\sin\theta_{12}\sqrt{1-\sin^2\theta_{12}}} + \sin^{\frac{7}{2}}\theta_{12}[1-\sin^2\theta_{12}]^{\frac{3}{4}}$$

$$\left|U_{\mu 2}\right| \approx \sqrt{1-\sin^2\theta_{12}}\sqrt{1-\sin\theta_{12}\sqrt{1-\sin^2\theta_{12}}} - \sin^{\frac{9}{2}}\theta_{12}[1-\sin^2\theta_{12}]^{\frac{1}{4}}$$

$$\left|U_{\mu 3}\right| \approx \sqrt{\sin\theta_{12}}[1-\sin^2\theta_{12}]^{\frac{1}{4}}\sqrt{1-\sin^6\theta_{12}}$$

$$\left|U_{\tau 1}\right| \approx [\sin\theta_{12}]^{\frac{3}{2}}[1-\sin^2\theta_{12}]^{\frac{1}{4}} - \sqrt{1-\sin^2\theta_{12}}\sqrt{1-\sin\theta_{12}\sqrt{1-\sin^2\theta_{12}}}\sin^3\theta_{12}$$

$$\left|U_{\tau 2}\right| \approx [1-\sin^2\theta_{12}]^{\frac{3}{4}}\sqrt{\sin\theta_{12}} - \sin^4\theta_{12}\sqrt{1-\sin\theta_{12}\sqrt{1-\sin^2\theta_{12}}}$$

$$\left|U_{\tau 3}\right| \approx \sqrt{1-\sin\theta_{12}\sqrt{1-\sin^2\theta_{12}}}\sqrt{1-\sin^6\theta_{12}}$$

Keywords: Neutrino Mixing Matrix, Solar Neutrinos, Atmospheric Neutrinos, Reactor Neutrinos, Neutrino Mixing angles, Neutrinos Oscillations,

PACS:14.60.Pq; 12.15.Ff; 12.15.Hh;

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0.

The Neutrino Mixing Matrix

In 1957, and 1958, Pontecorvo proposed that deficit in Solar Neutrinos may be due to oscillations between solar electron-Neutrinos ν_e, and muon-Neutrinos ν_μ. He assumed that the Solar neutrino flavor vector $\begin{bmatrix}\nu_e\\ \nu_\mu\end{bmatrix}$ is a rotation of the neutrino mass-eigen states vector $\begin{bmatrix}\nu_1\\ \nu_2\end{bmatrix}$ through a mixing angle θ_{12}.

That is,

$$\begin{pmatrix}\cos\theta_{12} & \sin\theta_{12}\\ -\sin\theta_{12} & \cos\theta_{12}\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix}=\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix}$$

In 1962, Maki, Nagawata, and Sakata [MNS] proposed that the Atmospheric Neutrino flavor vector $\begin{bmatrix}\nu_\mu\\ \nu_\tau\end{bmatrix}$ is a rotation of the neutrino mass-eigen states vector $\begin{bmatrix}\nu_2\\ \nu_3\end{bmatrix}$ through a mixing angle θ_{23}. And that Reactor Neutrinos flavor vector $\begin{bmatrix}\nu_e\\ \nu_\tau\end{bmatrix}$, at short distances, is a rotation of the neutrino mass-eigen states vector

$\begin{bmatrix} \nu_1 \\ \nu_3 \end{bmatrix}$ through a mixing angle θ_{13}.

$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$ rotates into $\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}$ by the Mixing Matrix $\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}$.

$$\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}.$$

Based on the three mixing angles, $\theta_{12}, \theta_{23}, \theta_{31}$, the Particle Data Group defines the Neutrino Mixing Matrix [PDG, p.192] as

$$\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13} & -c_{12}s_{23} - s_{12}c_{23}s_{13} & c_{23}c_{13} \end{pmatrix}$$

where

$$c_{ij} = \cos\theta_{ij},$$

$$s_{ij} = \sin\theta_{ij},$$

This matrix equals the product of three rotation matrices,

$$\begin{pmatrix} 1 & & \\ & c_{23} & s_{23} \\ & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & & s_{13} \\ & 1 & \\ -s_{13} & & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1 \end{pmatrix}.$$

The Solar Neutrino rotation of $\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$ is about $\begin{pmatrix} 0 \\ 0 \\ \nu_3 \end{pmatrix}$, the Atmospheric Neutrino rotation is about $\begin{pmatrix} 0 \\ \nu_2 \\ 0 \end{pmatrix}$,

The Reactor Neutrino rotation is about $\begin{pmatrix} \nu_1 \\ 0 \\ 0 \end{pmatrix}$.

But the three rotations do not commute. In fact,

$$\begin{pmatrix} c_{13} & & s_{13} \\ & 1 & \\ -s_{13} & & c_{13} \end{pmatrix}\begin{pmatrix} 1 & & \\ & c_{23} & s_{23} \\ & -s_{23} & c_{23} \end{pmatrix}\begin{pmatrix} c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} c_{12}c_{13} + s_{12}s_{23}s_{13} & s_{12}c_{13} - c_{12}s_{23}s_{13} & s_{13}c_{23} \\ -s_{12}c_{23} & c_{12}c_{23} & s_{23} \\ -c_{12}s_{13} + s_{12}s_{23}s_{13} & -c_{12}c_{13}s_{23} - s_{12}s_{13} & c_{23}c_{13} \end{pmatrix}$$

while

$$\begin{pmatrix} c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & c_{23} & s_{23} \\ & -s_{23} & c_{23} \end{pmatrix}\begin{pmatrix} c_{13} & & s_{13} \\ & 1 & \\ -s_{13} & & c_{13} \end{pmatrix} =$$

$$= \begin{pmatrix} c_{12}c_{31} - s_{12}s_{23}s_{31} & s_{12}c_{23} & s_{31}c_{12} + s_{12}s_{23}c_{31} \\ -s_{12}c_{31} - c_{12}s_{23}s_{31} & c_{12}c_{23} & -s_{12}s_{31} + c_{12}s_{23}c_{31} \\ -c_{23}s_{31} & -s_{23} & c_{23}c_{31} \end{pmatrix}.$$

That is,

The decomposition of the Neutrino Mixing Matrix into simple rotations depends on the order of multiplication of the rotations

1.

Computing the Neutrino Mixing Matrix

From [PDG, p.193]

$$\sin^2\theta_{12} = 0.306^{+0.018}_{-0.015} \Rightarrow \overline{\sin\theta_{12}} \approx 0.553172667$$

$$\Rightarrow \overline{\cos\theta_{12}} \approx 0.833066624$$

$$\sin^2\theta_{23} = 0.42^{+0.08}_{-0.03} \Rightarrow \overline{\sin\theta_{23}} \approx 0.648074069$$

$$\Rightarrow \overline{\cos\theta_{23}} \approx 0.76157731$$

$$\sin^2\theta_{31} = 0.0251^{+0.0034}_{-0.0034} \Rightarrow \overline{\sin\theta_{31}} \approx 0.158429795$$

$$\Rightarrow \overline{\cos\theta_{31}} \approx 0.987370244$$

$$U_{e1} = \cos\theta_{12}\cos\theta_{31} \approx 0.822545196$$

$$U_{e2} = \sin\theta_{12}\cos\theta_{31} \approx 0.546186231$$

$$U_{e3} = \sin\theta_{31} \approx 0.158429795$$

$$U_{\mu 1} = \sin\theta_{12}\cos\theta_{23} + \cos\theta_{12}\sin\theta_{23}\sin\theta_{31} \approx 0.506818235$$

$$U_{\mu 2} = \cos\theta_{12}\cos\theta_{23} - \sin\theta_{12}\sin\theta_{23}\sin\theta_{31} \approx 0.577648053$$

$$U_{\mu 3} = \sin\theta_{23}\cos\theta_{31} \approx 0.639889051$$

$$U_{\tau 1} = \sin\theta_{12}\sin\theta_{23} - \cos\theta_{12}\cos\theta_{23}\sin\theta_{31} \approx 0.257981927$$

$$\left|U_{\tau 2}\right| = \cos\theta_{12}\sin\theta_{23} + \sin\theta_{12}\cos\theta_{23}\sin\theta_{31} \approx 0.606632774$$

$$U_{\tau 3} = \cos\theta_{23}\cos\theta_{31} \approx 0.751963937$$

We list the transition rates

	predicted	observed PDG 2012
$\left\|U_{e1}\right\|$		0.822545196
$\left\|U_{e2}\right\|$		0.546186231
$\left\|U_{e3}\right\|$		0.158429795
$\left\|U_{\mu 1}\right\|$		0.506818235
$\left\|U_{\mu 2}\right\|$		0.577648053
$\left\|U_{\mu 3}\right\|$		0.639889051
$\left\|U_{\tau 1}\right\|$		0.257981927
$\left\|U_{\tau 2}\right\|$		0.606632774
$\left\|U_{\tau 3}\right\|$		0.751963937

We propose an *approximation* for the Neutrino Mixing Matrix that depends only on θ_{12}, and test its validity with a Chi-Squared Goodness-of-Fitness Statistical Test.

2.

Predicting θ_{23}, and θ_{31} from θ_{12}

From [PDG, p.193]

$$\sin^2\theta_{12} = 0.306^{+0.018}_{-0.015} \Rightarrow \overline{\sin\theta_{12}} \approx 0.55317 \Rightarrow \overline{\theta_{12}} \approx 0.58617 \text{ Rad}$$

$$\sin^2\theta_{23} = 0.42^{+0.08}_{-0.03} \Rightarrow \overline{\sin\theta_{23}} \approx 0.64807 \Rightarrow \overline{\theta_{23}} \approx 0.70505 \text{ Rad}$$

$$\sin^2\theta_{31} = 0.0251^{+0.0034}_{-0.0034} \Rightarrow \overline{\sin\theta_{23}} \approx 0.15843 \Rightarrow \overline{\theta_{23}} \approx 0.15910 \text{ Rad}$$

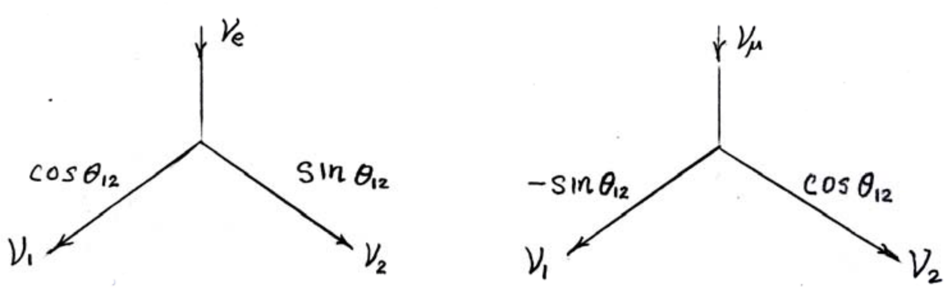
We will assume

$$\boxed{\sin^2\theta_{23} \approx \sin\theta_{12}\cos\theta_{12}}$$

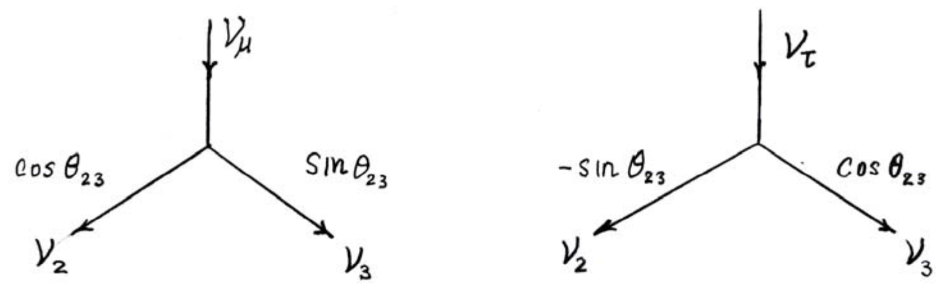
$$\boxed{\sin\theta_{31} \approx \sin^3\theta_{12}}$$

A plausibility argument for these assumptions may follow from the transitions due to the rotations.

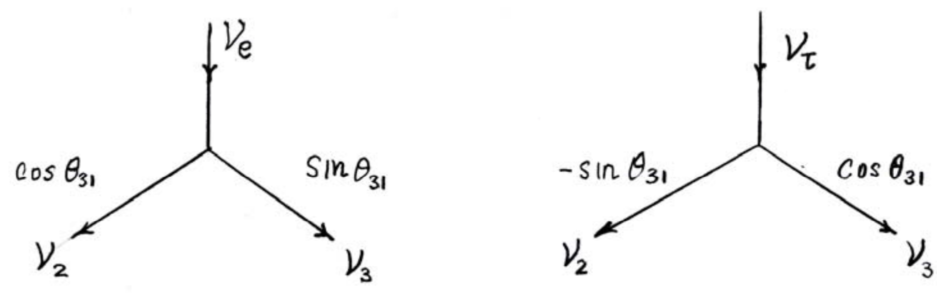
The Solar Neutrino rotation has the transitions



The Atmospheric neutrino rotation has the transitions



The Reactor Neutrino rotation has the transition



Substituting the relations above in the Neutrino Mixing matrix, yields an approximation to the Mixing Matrix that depends only on the Solar Mixing angle θ_{12}.

The validity of that approximation to the Neutron Mixing Matrix will be established by a Chi-Squared Goodness-of-Fitness Statistical test.

3.

Computing the Neutrino Mixing Matrix in θ_{12}

From [PDG, p.193],

$$\sin^2\theta_{12} = 0.306^{+0.018}_{-0.015} \quad \Rightarrow \overline{\sin\theta_{12}} \approx 0.553172667$$

$$\Rightarrow \overline{\cos\theta_{12}} \approx 0.833066624$$

Therefore,

$$\sin\theta_{23} \approx \sqrt{\sin\theta_{12}\cos\theta_{12}} \approx 0.678844375$$

$$\cos\theta_{23} = \sqrt{1-\sin^2\theta_{23}} \approx 0.734282176$$

$$\sin\theta_{31} \approx \sin^3\theta_{12} \approx 0.169270836$$

$$\cos\theta_{31} = \sqrt{1-\sin^2\theta_{31}} \approx 0.985569573$$

$$\left|U_{e1}\right| = \cos\theta_{12}\cos\theta_{31} \approx \sqrt{1-\sin^2\theta_{12}}\sqrt{1-\sin^6\theta_{12}}$$

$$\approx 0.821045117$$

$$\left|U_{e2}\right| = \sin\theta_{12}\cos\theta_{31} \approx \sin\theta_{12}\sqrt{1-\sin^6\theta_{12}}$$

$$\approx 0.545190149$$

$$\left|U_{e3}\right| = \sin\theta_{31} \approx \sin^3\theta_{12}$$

$$\approx 0.169270836$$

$$\left|U_{\mu 1}\right| = \sin\theta_{12}\cos\theta_{23} + \cos\theta_{12}\sin\theta_{23}\sin\theta_{31}$$

$$\approx \sin\theta_{12}\sqrt{1-\sin\theta_{12}\sqrt{1-\sin^2\theta_{12}}} + \sin^{\frac{7}{2}}\theta_{12}[1-\sin^2\theta_{12}]^{\frac{3}{4}}$$

$$\approx 0.50191131$$

$$\left|U_{\mu 2}\right| = \cos\theta_{12}\cos\theta_{23} - \sin\theta_{12}\sin\theta_{23}\sin\theta_{31}$$

$$\approx \sqrt{1-\sin^2\theta_{12}}\sqrt{1-\sin\theta_{12}\sqrt{1-\sin^2\theta_{12}}} - \sin^{\frac{9}{2}}\theta_{12}[1-\sin^2\theta_{12}]^{\frac{1}{4}}$$

$$\approx 0.634277836$$

$$\left|U_{\mu 3}\right| = \sin\theta_{23}\cos\theta_{31} \approx \sqrt{\sin\theta_{12}}[1-\sin^2\theta_{12}]^{\frac{1}{4}}\sqrt{1-\sin^6\theta_{12}}$$

$$\approx 0.66904836$$

$$\left|U_{\tau 1}\right| = \sin\theta_{12}\sin\theta_{23} - \cos\theta_{12}\cos\theta_{23}\sin\theta_{31}$$

$$\approx [\sin\theta_{12}]^{\frac{3}{2}}[1-\sin^2\theta_{12}]^{\frac{1}{4}} - \sqrt{1-\sin^2\theta_{12}}\sqrt{1-\sin\theta_{12}\sqrt{1-\sin^2\theta_{12}}}\sin^3\theta_{12}$$

$$\approx 0.271974171$$

$$\left|U_{\tau 2}\right| = \cos\theta_{12}\sin\theta_{23} + \sin\theta_{12}\cos\theta_{23}\sin\theta_{31}$$

$$\approx [1-\sin^2\theta_{12}]^{\frac{3}{4}}\sqrt{\sin\theta_{12}} - \sin^4\theta_{12}\sqrt{1-\sin\theta_{12}\sqrt{1-\sin^2\theta_{12}}}$$

$$\approx 0.634277838$$

$$\left|U_{\tau 3}\right| = \cos\theta_{23}\cos\theta_{31} \approx \sqrt{1-\sin\theta_{12}\sqrt{1-\sin^2\theta_{12}}}\sqrt{1-\sin^6\theta_{12}}$$

$$\approx 0.72368617$$

We list the transition rates

	predicted	observed PDG 2012
$\lvert U_{e1} \rvert$	0.821045117	0.822545196
$\lvert U_{e2} \rvert$	0.545190149	0.546186231
$\lvert U_{e3} \rvert$	0.169270836	0.158429795
$\lvert U_{\mu 1} \rvert$	0.50191131	0.506818235
$\lvert U_{\mu 2} \rvert$	0.634277836	0.577648053
$\lvert U_{\mu 3} \rvert$	0.66904836	0.639889051
$\lvert U_{\tau 1} \rvert$	0.271974171	0.257981927
$\lvert U_{\tau 2} \rvert$	0.634277838	0.606632774
$\lvert U_{\tau 3} \rvert$	0.72368617	0.751963937

We test the validity of our approximation with a Chi-Squared Goodness-of-Fitness Statistical Test

4.

Chi-Squared Goodness-of-Fit-Test of the Approximation to the Neutrino Mixing Matrix

4.1 The Null Hypothesis:

The predicted matrix is valid approximation to the Neutrino Mixing matrix $U(i,j) = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}$.

We aim to show that the values expected for the Neutrino Mixing Matrix fit well the observed values of the Neutrino Mixing Matrix

4.2 The Chi-Square Statistic for the Mixing Matrix

$$\chi^2_{\text{computed}}(\alpha,\nu) = \sum_{i,j} \frac{\left[U_{\text{observed}}(i,j) - U_{\text{expected}}(i,j)\right]^2}{U_{\text{expected}}(i,j)},$$

where α is the test level of confidence, and ν is the number of degrees of freedom.

4.3 The Number of Degrees of Freedom

Here, the number of degrees of freedom is $\nu = 9 - 1 = 8$.

At least 5 degrees of freedom are recommended for Chi-Squared Goodness-of-Fit testing. Greater confidence of the test, requires greater ν. As ν increases, the skewed Chi-Squared distribution gets close to the symmetric normal distribution.

4.4 The Chi-Square Test

	predicted	observed	$\frac{\lvert \text{error} \rvert^2}{\text{predicted}}$
$\lvert U_{e1} \rvert$	0.821045117	0.822545196	$2.740698361 \times 10^{-6} < 0.000003$
$\lvert U_{e2} \rvert$	0.545190149	0.546186231	$1.81987762 \times 10^{-6} < 0.000002$
$\lvert U_{e3} \rvert$	0.169270836	0.158429795	$6.943202547 \times 10^{-4} < 0.000695$
$\lvert U_{\mu 1} \rvert$	0.50191131	0.506818235	$4.797244548 \times 10^{-5} < 0.000048$
$\lvert U_{\mu 2} \rvert$	0.634277836	0.577648053	$5.056037182 \times 10^{-3} < 0.005057$
$\lvert U_{\mu 3} \rvert$	0.66904836	0.639889051	$1.270857762 \times 10^{-3} < 0.001271$
$\lvert U_{\tau 1} \rvert$	0.271974171	0.257981927	$7.198584021 \times 10^{-4} < 0.000720$
$\lvert U_{\tau 2} \rvert$	0.634277838	0.606632774	$1.204912923 \times 10^{-3} < 0.001205$
$\lvert U_{\tau 3} \rvert$	0.72368617	0.751963937	$1.104943912 \times 10^{-3} < 0.001105$

Therefore,

$$\begin{aligned} {\chi^2}_{\text{computed}}(8) &< 0.010106 \\ &= \chi^2(\alpha, 8) \end{aligned}$$

We need to find the confidence level α at which $\chi^2(\alpha, 8)$ is 0.010106.

4.5 The Confidence Level

By [Abramowitz], the Chi-Squared distribution is the Cumulative Probability Function

$$Q(\chi^2;\nu) = \frac{1}{2^{\nu/2}\Gamma(\nu / 2)} \int_{t=\chi^2}^{\infty} e^{-t/2} t^{\nu/2-1} dt .$$

To use the MATHEMATICA Gamma function

$$\text{Gamma}[a, z] = \int_{u=z}^{u=\infty} e^{-u} u^{a-1} du ,$$

we identify $a = \frac{\nu}{2}$, $u = \frac{t}{2}$, $z = \frac{\chi^2}{2}$. Then,

$$Q(\chi^2;\nu) = \frac{1}{\Gamma(\nu / 2)} \int_{u=\chi^2/2}^{\infty} e^{-u} u^{\nu/2-1} du$$

$$= \frac{1}{\Gamma(\nu / 2)} \text{Gamma}[\nu/2, \chi^2 / 2]$$

For $\nu = 8$, and $\chi^2 = 0.01$, we have $\Gamma(8 / 2) = (4-1)!$, and we obtain

$$Q(0.010106;8) = \text{ScientificForm}[\frac{1}{3!}\text{Gamma}[4,0.005053]]$$

$$= 9.99999999972946 \times 10^{-1}$$

This means that our approximation to the Neutrino Mixing Matrix gives the correct transition rates with confidence of

$$99.9999999972946\%$$

The uncertainty in the transition rates is

$$P(\chi^2, 8) = P(0.005053, 8) = 2.7054 \times 10^{-11} < \frac{3}{10^{11}}$$

In Conclusion**,** we have shown that our approximation for the Neutrino Mixing Matrix, that depends only on the solar mixing angle θ_{12} can be used with extremely high level of confidence.

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