Riemannian Trigonometric Series

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Abstract: Riemann derived necessary conditions for the equality of a function to its Fourier Series, and claimed that these conditions are sufficient. We observe that they are not.

Riemann claimed that a Trigonometric Series is the convolution of its second primitive with the Dirichlet Kernel,

$$\cos(x-t)+\cos 2(x-t)+\cos 3(x-t)+\ldots.$$

But that infinite Series diverges to infinity at $x=t$, and cannot be integrated.

Riemann claimed that a function with infinitely many maxima or minima on any interval has diverging Fourier Coefficients. But his proof is incomplete, and the claim is unproven

Riemann presents examples of Fourier Series expansions of pathologically constructed functions. These examples support Fourier's claim that any function equals its Fourier Series.

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Function, Riemann (x) Series, Integral, Integration, Dirichlet Kernel,

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Contents

Introduction

In his paper "On the Representation of a Function by a Trigonometric Series", Riemann presents his integral for a bounded function, and reviews the Dirichlet Conditions sufficient for the equality of a function to its associated Fourier Series.

Riemann wrote

"The Integral

$$\frac{1}{2\pi}\int_{\alpha=-\pi}^{\alpha=\pi} f(\alpha)\frac{\sin\left((n+\frac{1}{2})(x-\alpha)\right)}{\sin\left(\frac{1}{2}(x-\alpha)\right)}d\alpha$$

approaches the value $f(x)$ infinitesimally closely when $n \to \infty$".

However, at $x = \alpha$, when $n \to \infty$,

$$\frac{\sin\left((n+\frac{1}{2})(x-\alpha)\right)}{\sin\left(\frac{1}{2}(x-\alpha)\right)} \to \infty,$$

$$f(\alpha)\frac{\sin\left((n+\frac{1}{2})(x-\alpha)\right)}{\sin\left(\frac{1}{2}(x-\alpha)\right)} \text{ is unbounded function,}$$

and the integral does not exist as a Riemann Integral.

This raises doubts regarding Riemann attempts there to

obtain necessary and sufficient conditions for a function to equal its associated Fourier Series,

represent a Trigonometric Series as a convolution with the Dirichlet Kernel

show that a function with infinitely many maxima and minima, has divergent Fourier coefficients

We follow through the paper to find which of its results hold.

1.

Riemann's 1st and 2nd Theorems

1.1 Necessary Condition for Convergence of a Trigonometric Series

If *for any* x, *the Trigonometric Series*

$$a_1 \sin x + a_2 \sin 2x + ... + \tfrac{1}{2}b_0 + b_1 \cos x + b_2 \cos 2x + ... =$$

$$= \underbrace{\tfrac{1}{2}b_0}_{A_0} + \underbrace{a_1 \sin x + b_1 \cos x}_{A_1(x)} + \underbrace{a_2 \sin 2x + b_2 \cos 2x}_{A_2(x)} + ...,$$

converges to a function

$$f(x).$$

Then *for any* x,

$$A_n(x) \to 0, \text{ as } n \to \infty$$

1.2 The first primitive of $f(x)$ *is the series*

$$C' + A_0 x - a_1 \cos x + b_1 \sin x - \tfrac{1}{2}a_2 \cos 2x + \tfrac{1}{2}b_2 \sin 2x + ...$$

1.3 The second primitive of $f(x)$ *is the series*

$$C + C'x + \tfrac{1}{2}A_0 x^2 \underbrace{-a_1 \sin x - b_1 \cos x}_{-A_1(x)} \underbrace{-\tfrac{1}{2^2}a_2 \sin 2x - \tfrac{1}{2^2}b_2 \cos 2x}_{-\frac{1}{4}A_2(x)} + ...$$

$$= C + C'x + \tfrac{1}{2}A_0x^2 - A_1(x) - \tfrac{1}{2^2}A_2(x) - \tfrac{1}{3^2}A_3(x)...$$

1.4 *The second primitive of $f(x)$ converges for any x to a continuous Integrable function*

$$F(x).$$

Proof: Convergence

Since for any x, $A_n(x) \to 0$, then for $k = n+1, n+2, ...$,

$$|A_k(x)| < \varepsilon,$$

and for the tail of the second primitive series we have

$$\left| -\frac{A_{n+1}(x)}{(n+1)^2} - \frac{A_{n+2}(x)}{(n+2)^2} - ... \right| < \varepsilon\left(\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + ... \right).$$

Since $\dfrac{1}{(n+1)^2} + \dfrac{1}{(n+2)^2} + ...$ is the tail of $1 + \tfrac{1}{2^2} + \tfrac{1}{3^2} + ...$, it is bounded by $\dfrac{1}{n}$, and we have

$$< \frac{\varepsilon}{n}.$$

Therefore, the tail of the second primitive series can be made arbitrarily small, and that series converges to $F(x)$.$\square$

Continuity

For any x, and for $\delta > 0$,

$$|F(x+\delta) - F(x)| \le C'\delta + b_0(x + \tfrac{1}{2}\delta)\delta +$$

$$+\left|A_1(x+\delta)-A_1(x)\right|+...+\tfrac{1}{n^2}\left|A_n(x+\delta)-A_n(x)\right|+\tfrac{1}{(n+1)^2}\left|A_{n+1}(x+\delta)-A_{n+1}(x)\right|+...$$

For $k = 1...n$,

$$\tfrac{1}{k^2}\left|A_k(x+\delta)-A_k(x)\right| = \left|a_k(\sin(kx+\delta)-\sin kx)+b_k(\cos(kx+\delta)-\cos kx)\right|$$

$$\leq \left|a_k\right|\left|\sin(kx+\delta)-\sin(kx)\right|+\left|b_k\right|\left|\cos(kx+\delta)-\cos(kx)\right|,$$

and applying the Intermediate Value Theorem to each term,

$$= \left|a_k\right|\left|\cos\xi_k\right|\delta+\left|b_k\right|\left|\sin\eta_k\right|\delta$$

$$\leq \left|a_k\right|\delta+\left|b_k\right|\delta.$$

For $k = n+1, n+2, ...$,

$$\tfrac{1}{k^2}\left|A_k(x+\delta)-A_k(x)\right| \leq \tfrac{1}{k^2}\left(\left|A_k(x+\delta)\right|+\left|A_k(x)\right|\right),$$

and by 1.1,

$$< \tfrac{2}{k^2}\varepsilon.$$

Therefore, $\left|F(x+\delta)-F(x)\right|$ is bounded by

$$\left\{\left|C'\right|+\left|b_0\right|(\left|x\right|+\tfrac{1}{2}\delta)+\left|a_1\right|+\left|b_1\right|+...+\left|a_n\right|+\left|b_n\right|\right\}\delta+2\varepsilon\left(\tfrac{1}{(n+1)^2}+\tfrac{1}{(n+2)^2}+...\right)$$

$$\leq \left\{\left|C'\right|+\left|b_0\right|(\left|x\right|+\tfrac{1}{2}\delta)+\left|a_1\right|+\left|b_1\right|+...+\left|a_n\right|+\left|b_n\right|\right\}\delta+2\varepsilon\tfrac{1}{n} \equiv \eta.$$

Consequently, given x, and arbitrarily small $\eta > 0$, there is n so that $\eta - 2\varepsilon\frac{1}{n} > 0$, and there is a quadratic equation solution, $\delta > 0$, so that $\left|F(x+\delta)-F(x)\right| < \eta$.□

1.5 ***If*** *the Trigonometric Series converges to* $f(x)$

Then *for infinitesimals* α, *and* β

$$\frac{F(x+\alpha+\beta)-F(x+\alpha-\beta)-F(x-\alpha+\beta)+F(x-\alpha-\beta)}{4\alpha\beta}=$$

$$=A_0+A_1\frac{\sin\alpha}{\alpha}\frac{\sin\beta}{\beta}+A_2\frac{\sin 2\alpha}{2\alpha}\frac{\sin 2\beta}{2\beta}+A_3\frac{\sin 3\alpha}{3\alpha}\frac{\sin 3\beta}{3\beta}+...$$

Proof:

The terms $C+C'x$ in the series for $F(x)$ yield zero.

The term $\frac{1}{2}b_0x^2$ yields

$$\tfrac{1}{2}b_0\frac{(x+\alpha+\beta)^2-(x+\alpha-\beta)^2-(x-\alpha+\beta)^2+(x-\alpha-\beta)^2}{4\alpha\beta}=$$

$$=\tfrac{1}{2}b_0\frac{2\alpha\beta+2\alpha\beta+2\alpha\beta+2\alpha\beta}{4\alpha\beta}=b_0$$

The term $-\frac{1}{k^2}a_k\sin(kx)$ yields

$$-\frac{1}{k^2}a_k\frac{\sin k(x+\alpha+\beta)-\sin k(x+\alpha-\beta)-\sin k(x-\alpha+\beta)+\sin k(x-\alpha-\beta)}{4\alpha\beta}$$

$$=-\frac{1}{k^2}a_k\frac{\left[\sin k(x+\alpha+\beta)+\sin k(x-\alpha-\beta)\right]-\left[\sin k(x+\alpha-\beta)+\sin k(x-\alpha+\beta)\right]}{4\alpha\beta}$$

$$=-\frac{1}{k^2}a_k\frac{\left[2\sin(kx)\cos k(\alpha+\beta)\right]-\left[2\sin(kx)\cos k(\alpha-\beta)\right]}{4\alpha\beta}$$

$$=-\frac{1}{k^2}a_k 2\sin(kx)\frac{\cos k(\alpha+\beta)-\cos k(\alpha-\beta)}{4\alpha\beta}$$

$$=-\frac{1}{k^2}a_k 2\sin(kx)\frac{-2\sin(k\alpha)\sin(k\beta)}{4\alpha\beta}$$

$$=a_k\sin(kx)\frac{\sin(k\alpha)}{k\alpha}\frac{\sin(k\beta)}{k\beta}$$

The term $-\frac{1}{k^2}b_k\cos(kx)$ yields

$$-\frac{1}{k^2} b_k \frac{\cos k(x+\alpha+\beta) - \cos k(x+\alpha-\beta) - \cos k(x-\alpha+\beta) + \cos k(x-\alpha-\beta)}{4\alpha\beta}$$

$$= -\frac{1}{k^2} b_k \frac{\left[\cos k(x+\alpha+\beta) + \cos k(x-\alpha-\beta)\right] - \left[\cos k(x+\alpha-\beta) + \cos k(x-\alpha+\beta)\right]}{4\alpha\beta}$$

$$= -\frac{1}{k^2} b_k \frac{\left[2\cos(kx)\cos k(\alpha+\beta)\right] - \left[2\cos(kx)\cos k(\alpha-\beta)\right]}{4\alpha\beta}$$

$$= -\frac{1}{k^2} b_k 2\cos(kx) \frac{\cos k(\alpha+\beta) - \cos k(\alpha-\beta)}{4\alpha\beta}$$

$$= -\frac{1}{k^2} b_k 2\cos(kx) \frac{-2\sin(k\alpha)\sin(k\beta)}{4\alpha\beta}$$

$$= b_k \cos(kx) \frac{\sin(k\alpha)}{k\alpha} \frac{\sin(k\beta)}{k\beta}$$

Therefore,

$$\frac{F(x+\alpha+\beta) - F(x+\alpha-\beta) - F(x-\alpha+\beta) + F(x-\alpha-\beta)}{4\alpha\beta} =$$

$$= \underbrace{b_0}_{A_0} + (\underbrace{a_1 \sin x + b_1 \cos x}_{A_1}) \frac{\sin\alpha}{\alpha} \frac{\sin\beta}{\beta} + (\underbrace{a_2 \sin 2x + b_2 \cos 2x}_{A_2}) \frac{\sin 2\alpha}{2\alpha} \frac{\sin 2\beta}{2\beta} + \ldots$$

$$= A_0 + A_1 \frac{\sin\alpha}{\alpha} \frac{\sin\beta}{\beta} + A_2 \frac{\sin 2\alpha}{2\alpha} \frac{\sin 2\beta}{2\beta} + A_3 \frac{\sin 3\alpha}{3\alpha} \frac{\sin 3\beta}{3\beta} + \ldots. \square$$

1.6 ***If*** *the Trigonometric Series converges to* $f(x)$

Then *for an infinitesimal* α*,*

$$\frac{F(x+2\alpha) - 2F(x) + F(x-2\alpha)}{4\alpha^2} = A_0 + A_1 \left(\frac{\sin\alpha}{\alpha}\right)^2 + A_2 \left(\frac{\sin 2\alpha}{2\alpha}\right)^2 + \ldots$$

1.7 ***Denote*** *the negative tails of the Trigonometric Series by*

$$\varepsilon_1 = -A_1 - A_2 - A_3 - A_4 - \ldots = A_0 - f(x)$$

$$\varepsilon_2 = \qquad - A_2 - A_3 - A_4 - \ldots = A_0 + A_1 - f(x)$$

$$\varepsilon_3 = \qquad\qquad - A_3 - A_4 - \ldots = A_0 + A_1 + A_3 - f(x)$$

....

$$\varepsilon_n = \qquad\qquad\qquad = A_0 + A_1 + A_3 + \ldots + A_{n-1} - f(x)$$

Then, $A_0 = f(x) + \varepsilon_1$

$$A_1 = \varepsilon_2 - \varepsilon_1$$

$$A_2 = \varepsilon_3 - \varepsilon_2$$

$$A_3 = \varepsilon_4 - \varepsilon_3$$

....

$$A_n = \varepsilon_{n+1} - \varepsilon_n$$

1.8 ***If*** *the Trigonometric Series converges to* $f(x)$

Then $\dfrac{F(x+2\alpha) - 2F(x) + F(x-2\alpha)}{4\alpha^2} - f(x) =$

$$= \varepsilon_1 \left\{ 1 - \left(\frac{\sin \alpha}{\alpha} \right)^2 \right\} + \varepsilon_2 \left\{ \left(\frac{\sin \alpha}{\alpha} \right)^2 - \left(\frac{\sin 2\alpha}{2\alpha} \right)^2 \right\} + \varepsilon_3 \left(\left(\frac{\sin 2\alpha}{2\alpha} \right)^2 - \left(\frac{\sin 3\alpha}{3\alpha} \right)^2 \right) + \ldots$$

Proof: By 1.6, and 1.7,

$$\frac{F(x+2\alpha)-2F(x)+F(x-2\alpha)}{4\alpha^2}=$$

$$=\underbrace{A_0}_{f(x)+\varepsilon_1}+\underbrace{A_1}_{\varepsilon_2-\varepsilon_1}\left(\frac{\sin\alpha}{\alpha}\right)^2+\underbrace{A_2}_{\varepsilon_3-\varepsilon_2}\left(\frac{\sin 2\alpha}{2\alpha}\right)^2+\underbrace{A_3}_{\varepsilon_4-\varepsilon_3}\left(\frac{\sin 3\alpha}{3\alpha}\right)^2+\dots\square$$

1.9 ***If*** *the Trigonometric Series converges to* $f(x)$

Then $\frac{F(x+2\alpha)-2F(x)+F(x-2\alpha)}{4\alpha^2}\to f(x)$, as $\alpha\to 0$.

Proof:

By 1.8, we aim to show that

$$\left|\varepsilon_1\right|\left|1-\left(\frac{\sin\alpha}{\alpha}\right)^2\right|+\left|\varepsilon_2\right|\left|\left(\frac{\sin\alpha}{\alpha}\right)^2-\left(\frac{\sin 2\alpha}{2\alpha}\right)^2\right|+\left|\varepsilon_3\right|\left|\left(\frac{\sin 2\alpha}{2\alpha}\right)^2-\left(\frac{\sin 3\alpha}{3\alpha}\right)^2\right|+\dots$$

vanishes as $\alpha\to 0$.

Although each of the terms vanishes as $\alpha\to 0$, there are infinitely many terms, and $\infty\times 0$ is undefined. Therefore, we have to separate the infinite series into parts.

Our first separation is based on the fact that the n$^{\text{th}}$ tail of the Trigonometric Series, ε_n of 1.7, vanishes as $n\to\infty$. Hence, for arbitrarily small $\delta>0$, there is M, so that for any $n>M$,

$$\delta>\left|\varepsilon_n\right|.$$

As $\alpha\to 0$, we have $\frac{\sin(k\alpha)}{k\alpha}\to 1$, and the partial sum till $n=M$,

$$\left|\varepsilon_1\right|\left|1-\left(\frac{\sin\alpha}{\alpha}\right)^2\right|+\left|\varepsilon_2\right|\left|\left(\frac{\sin\alpha}{\alpha}\right)^2-\left(\frac{\sin 2\alpha}{2\alpha}\right)^2\right|+...+\left|\varepsilon_M\right|\left|\left(\frac{\sin(M-1)\alpha}{(M-1)\alpha}\right)^2-\left(\frac{\sin M\alpha}{M\alpha}\right)^2\right|,$$

vanishes as $\alpha \to 0$.

We need to show that the tail with $n > M$, vanishes as $\alpha \to 0$.

That requires another separation:

For small enough $\alpha > 0$, there is $N > M$, so that

$$(N-1)\alpha < \pi\,,$$

and

$$N\alpha > \pi\,.$$

For $n > M$, we have $\delta > \left|\varepsilon_n\right|$, and

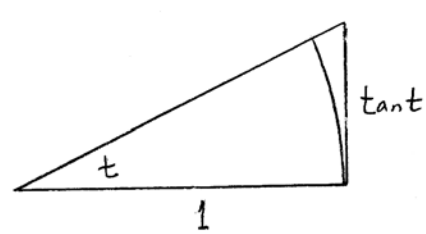
$$\left|\varepsilon_{M+1}\right|\left|\left(\frac{\sin M\alpha}{M\alpha}\right)^2-\left(\frac{\sin(M+1)\alpha}{(M+1)\alpha}\right)^2\right|+...+\left|\varepsilon_N\right|\left|\left(\frac{\sin(N-1)\alpha}{(N-1)\alpha}\right)^2-\left(\frac{\sin N\alpha}{N\alpha}\right)^2\right|$$

$$\leq \delta\left\{\left|\left(\frac{\sin M\alpha}{M\alpha}\right)^2-\left(\frac{\sin(M+1)\alpha}{(M+1)\alpha}\right)^2\right|+...+\left|\left(\frac{\sin(N-1)\alpha}{(N-1)\alpha}\right)^2-\left(\frac{\sin N\alpha}{N\alpha}\right)^2\right|\right\}$$

For $t > 0$, the function $\dfrac{\sin t}{t}$ has the derivative

$$\frac{\cos t}{t}-\frac{1}{t^2}\sin t=\frac{1}{t}\left(\cos t-\frac{\sin t}{t}\right),$$

which is negative because for small enough arc-length t,



$$t < \tan(t) = \frac{\sin(t)}{\cos(t)}$$

Therefore, the function $\dfrac{\sin t}{t}$ is decreasing.

Thus, for $\alpha > 0$,

$$\frac{\sin k\alpha}{k\alpha} - \frac{\sin(k+1)\alpha}{(k+1)\alpha} > 0\,,$$

and

$$\left(\frac{\sin k\alpha}{k\alpha}\right)^2 - \left(\frac{\sin(k+1)\alpha}{(k+1)\alpha}\right)^2 = \underbrace{\left(\frac{\sin k\alpha}{k\alpha} - \frac{\sin(k+1)\alpha}{(k+1)\alpha}\right)}_{>0}\left(\underbrace{\frac{\sin k\alpha}{k\alpha}}_{>0} + \underbrace{\frac{\sin(k+1)\alpha}{(k+1)\alpha}}_{>0}\right) > 0$$

Consequently,

$$\delta\left\{\left|\left(\frac{\sin M\alpha}{M\alpha}\right)^2 - \left(\frac{\sin(M+1)\alpha}{(M+1)\alpha}\right)^2\right| + ... + \left|\left(\frac{\sin(N-1)\alpha}{(N-1)\alpha}\right)^2 - \left(\frac{\sin N\alpha}{N\alpha}\right)^2\right|\right\}$$

$$= \delta\left\{\left(\frac{\sin M\alpha}{M\alpha}\right)^2 - \left(\frac{\sin(M+1)\alpha}{(M+1)\alpha}\right)^2 + ... + \left(\frac{\sin(N-1)\alpha}{(N-1)\alpha}\right)^2 - \left(\frac{\sin N\alpha}{N\alpha}\right)^2\right\}$$

$$= \delta\left\{\left(\frac{\sin M\alpha}{M\alpha}\right)^2 - \left(\frac{\sin N\alpha}{N\alpha}\right)^2\right\}$$

$$< \delta\left(\frac{\sin M\alpha}{M\alpha}\right)^2$$

Since $\frac{\sin(M\alpha)}{M\alpha} \to 1$, as $\alpha \to 0$, the sum is bounded by δ.

We are left with the terms with $n = N+1, N+2, ...$

Then, $\delta > \left|\varepsilon_n\right|$, and

$$|\varepsilon_{N+1}|\left|\left(\frac{\sin N\alpha}{N\alpha}\right)^2-\left(\frac{\sin(N+1)\alpha}{(N+1)\alpha}\right)^2\right|+|\varepsilon_{N+2}|\left|\left(\frac{\sin(N+1)\alpha}{(N+1)\alpha}\right)^2-\left(\frac{\sin(N+2)\alpha}{(N+2)\alpha}\right)^2\right|+\dots,$$

$$<\delta\left\{\left|\left(\frac{\sin N\alpha}{N\alpha}\right)^2-\left(\frac{\sin(N+1)\alpha}{(N+1)\alpha}\right)^2\right|+\left|\left(\frac{\sin(N+1)\alpha}{(N+1)\alpha}\right)^2-\left(\frac{\sin(N+2)\alpha}{(N+2)\alpha}\right)^2\right|+\dots\right\}$$

Since $\left(\frac{\sin k\alpha}{k\alpha}\right)^2-\left(\frac{\sin(k+1)\alpha}{(k+1)\alpha}\right)^2>0$, we can remove the absolute value sign, and we have

$$=\delta\left\{\left(\frac{\sin^2(N\alpha)}{(N\alpha)^2}-\frac{\sin^2[(N+1)\alpha]}{[(N+1)\alpha]^2}\right)+\left(\frac{\sin^2[(N+1)\alpha]}{[(N+1)\alpha]^2}-\frac{\sin^2[(N+2)\alpha]}{[(N+2)\alpha]^2}\right)+\dots\right\}.$$

Nevertheless, the telescopic series sum, $\delta\frac{\sin^2(N\alpha)}{(N\alpha)^2}$, has no limit for $N\to\infty$, and $\alpha\to 0$. Thus, we need another series separation:

Now, for $n>N$,

$$\frac{\sin^2[(n-1)\alpha]}{[(n-1)\alpha]^2}-\frac{\sin^2[n\alpha]}{[n\alpha]^2}=$$

$$=\left\{\frac{\sin^2[(n-1)\alpha]}{[(n-1)\alpha]^2}-\frac{\sin^2[(n-1)\alpha]}{[n\alpha]^2}\right\}+\left\{\frac{\sin^2[(n-1)\alpha]}{[n\alpha]^2}-\frac{\sin^2[n\alpha]}{[n\alpha]^2}\right\}$$

$$=\frac{\sin^2[(n-1)\alpha]}{\alpha^2}\left\{\frac{1}{[n-1]^2}-\frac{1}{n^2}\right\}+\frac{1}{n^2\alpha^2}\left\{\sin^2[(n-1)\alpha]-\sin^2[n\alpha]\right\}$$

The first term $\frac{\sin^2[(n-1)\alpha]}{\alpha^2}\left\{\frac{1}{[n-1]^2}-\frac{1}{n^2}\right\}$ generates the series

$$=\delta\frac{\sin^2(N\alpha)}{\alpha^2}\left\{\frac{1}{N^2}-\frac{1}{[N+1]^2}+\frac{1}{[N+1]^2}-\frac{1}{[N+2]^2}+\dots\right\}$$

$$= \delta \frac{\sin^2(N\alpha)}{N^2\alpha^2}$$

$$\leq \delta \frac{1}{N^2\alpha^2}$$

$$\leq \delta \frac{1}{\pi^2}$$

The second term is

$$\frac{1}{n^2\alpha^2}\left\{\sin^2[(n-1)\alpha] - \sin^2[n\alpha]\right\} =$$

$$= \frac{1}{n^2\alpha^2}\Big(\sin[(n-1)\alpha] - \sin[n\alpha]\Big)\Big(\sin[(n-1)\alpha] + \sin[n\alpha]\Big)$$

$$= \frac{1}{n^2\alpha^2}\Big(2\sin\tfrac{(2n-1)\alpha}{2}\cos\tfrac{\alpha}{2}\Big)\Big(2\cos\tfrac{(2n-1)\alpha}{2}\sin(-\tfrac{\alpha}{2})\Big)$$

$$= -\frac{1}{n^2\alpha^2}\sin[(2n-1)\alpha]\sin\alpha$$

$$= -\frac{\sin\alpha}{\alpha}\sin[(2n-1)\alpha]\frac{1}{n^2\alpha}$$

Thus, for infinitesimal α, the second term is bounded by $\dfrac{1}{n^2\alpha}$,

and generates the series

$$\delta\frac{1}{\alpha}\left\{\frac{1}{(N+1)^2} + \frac{1}{(N+2)^2} + ...\right\}$$

$$\leq \delta\frac{1}{\alpha}\frac{1}{(N+1)}$$

$$< \delta \frac{1}{\pi}$$

In summary, the Series is bounded by $\delta\left\{1 + \frac{1}{\pi^2} + \frac{1}{\pi}\right\}$,and we let $\delta \downarrow 0$.□

1.10 Riemann's 1st Theorem

If *the Trigonometric Series converges to* $f(x)$

Then

$$\frac{F(x+\alpha+\beta) - F(x+\alpha-\beta) - F(x-\alpha+\beta) + F(x-\alpha-\beta)}{4\alpha\beta} \to f(x),$$

for infinitesimals α, *and* β, *so that* $\frac{\alpha}{\beta}$, *and* $\frac{\beta}{\alpha}$ *are finite*.

Proof: By 1.9,

$$\frac{F(x+2\alpha_1) - 2F(x) + F(x-2\alpha_1)}{(2\alpha_1)^2} = f(x) + \delta_1$$

$$\frac{F(x+2\alpha_2) - 2F(x) + F(x-2\alpha_2)}{(2\alpha_2)^2} = f(x) + \delta_2$$

Hence,

$$F(x+2\alpha_1) - 2F(x) + F(x-2\alpha_1) = (2\alpha_1)^2 f(x) + (2\alpha_1)^2\delta_1$$

$$F(x+2\alpha_2) - 2F(x) + F(x-2\alpha_2) = (2\alpha_2)^2 f(x) + (2\alpha_2)^2\delta_1$$

Subtracting the second equation from the first,

$$\frac{F(x+2\alpha_1)-F(x+2\alpha_2)-F(x-2\alpha_2)+F(x-2\alpha_1)}{(2\alpha_1)^2-(2\alpha_2)^2}=$$

$$=f(x)+\frac{(2\alpha_1)^2}{(2\alpha_1)^2-(2\alpha_2)^2}\delta_1+\frac{(2\alpha_2)^2}{(2\alpha_1)^2-(2\alpha_2)^2}\delta_2$$

Denote

$$2\alpha_1=\alpha+\beta$$
$$2\alpha_2=\alpha-\beta$$

Then,

$$\frac{F(x+\alpha+\beta)-F(x+\alpha-\beta)-F(x-\alpha+\beta)+F(x-\alpha-\beta)}{4\alpha\beta}=$$

$$=f(x)+\frac{(\alpha+\beta)^2}{4\alpha\beta}\delta_1+\frac{(\alpha-\beta)^2}{4\alpha\beta}\delta_2$$

$$=f(x)+\frac{1}{4}\left(\frac{\alpha}{\beta}+2+\frac{\beta}{\alpha}\right)\delta_1+\frac{1}{4}\left(\frac{\alpha}{\beta}-2+\frac{\beta}{\alpha}\right)\delta_2$$

For infinitesimal α, and β, so that $\frac{\alpha}{\beta}$, and $\frac{\beta}{\alpha}$ are finite, the coefficients of δ_1, and δ_2 are bounded, and we let $\delta_1\downarrow 0$, and $\delta_2\downarrow 0$, to obtain the equality to $f(x)$.□

1.11 Riemann's 2nd Theorem

If *the Trigonometric Series converges to* $f(x)$

Then *For any* x, $\frac{F(x+2\alpha)-2F(x)+F(x-2\alpha)}{2\alpha}\to 0$, *as* $\alpha\to 0$, *Proof*: By 1.6,

$$\frac{F(x+2\alpha)-2F(x)+F(x-2\alpha)}{2\alpha}=2\alpha\left\{A_0+A_1\left(\frac{\sin\alpha}{\alpha}\right)^2+A_2\left(\frac{\sin 2\alpha}{2\alpha}\right)^2+...\right\}$$

We need to show that the series in $\{\ \}$ is bounded.

As in the proof of 1.9, we separate the series into three parts:

First, since the series converges to $f(x)$, for $\varepsilon>0$, there is M, so that for $n>M$, we have $\varepsilon>|A_n|$.

For infinitesimal α, the partial sum till M,

$$A_0+A_1\left(\frac{\sin\alpha}{\alpha}\right)^2+A_2\left(\frac{\sin(2\alpha)}{2\alpha}\right)^2+...+A_M\left(\frac{\sin(M\alpha)}{M\alpha}\right)^2,$$

has a limit $A_0+A_1+A_2+...+A_M$, and is bounded by a number Q.

Second, for small enough $\alpha>0$, there is $N>M$, so that

$$(N-1)\alpha<\pi,$$

and

$$N\alpha>\pi.$$

For the part of the series from $n=M+1$, to $n=N$, we have $|A_n|<\varepsilon$, and

$$A_{M+1}\left(\frac{\sin[(M+1)\alpha]}{(M+1)\alpha}\right)^2+A_{M+2}\left(\frac{\sin[(M+2)\alpha]}{(M+2)\alpha}\right)^2+...+A_N\left(\frac{\sin(N\alpha)}{N\alpha}\right)^2,$$

is bounded by

$$\varepsilon\left\{\left(\frac{\sin[(M+1)\alpha]}{(M+1)\alpha}\right)^2+\left(\frac{\sin[(M+2)\alpha]}{(M+2)\alpha}\right)^2+...+\left(\frac{\sin(N\alpha)}{N\alpha}\right)^2\right\}.$$

Since $\sin(t) \leq t$, the series from $n = M+1$, to $n = N$ is bounded by

$$\varepsilon \underbrace{\left\{1+1+...+1\right\}}_{N-M} < \varepsilon N < \varepsilon \frac{\pi}{\alpha}$$

<u>Third</u>, the tail of series for $n > N$,

$$A_{N+1}\left(\frac{\sin[(N+1)\alpha]}{(N+1)\alpha}\right)^2 + A_{N+2}\left(\frac{\sin[(N+2)\alpha]}{(N+2)\alpha}\right)^2 + \dots,$$

is bounded by

$$\varepsilon\left\{\frac{1}{(N+1)^2\alpha^2} + \frac{1}{(N+2)^2\alpha^2} + \frac{1}{(N+3)^2\alpha^2} + ...\right\}$$

$$< \varepsilon \frac{1}{N+1}\frac{1}{\alpha^2}$$

$$< \frac{\varepsilon}{\alpha}\frac{1}{\pi}.$$

Therefore,

$$2\alpha\left\{A_0 + A_1\left(\frac{\sin\alpha}{\alpha}\right)^2 + A_2\left(\frac{\sin 2\alpha}{2\alpha}\right)^2 + ...\right\}$$

is bounded by

$$2\alpha\left\{Q + \varepsilon\frac{\pi}{\alpha} + \varepsilon\frac{\alpha}{\pi}\right\} = 2\alpha Q + 2\varepsilon\left(\pi + \frac{1}{\pi}\right).$$

We let $\alpha \downarrow 0$, and $\varepsilon \downarrow 0$. $\square$

2.

Riemann's 3rd Theorem

2.1 ***If*** *the Trigonometric Series converges to* $f(x)$

$c > b$ *are arbitrary constants*

$\lambda(x)$ *is continuous on* $[b, c]$

Then $\mu^2 \int\limits_{x=b}^{x=c} F(x)\cos(\mu[x-a])\lambda(x)dx =$

$$= \mu^2 \int\limits_{x=b}^{x=c} \left(C + C'x + \tfrac{1}{2}A_0 x^2\right)\cos(\mu[x-a])\lambda(x)dx +$$

$$-\mu^2 \int\limits_{x=b}^{x=c} \underbrace{\tfrac{1}{2}\left[\left\{a_1 \sin a + b_1 \cos a\right\}\cos(\mu+1)(x-a) + \left\{a_1 \cos a + b_1 \sin a\right\}\sin(\mu+1)(x-a)\right]}_{B_{\mu+1}(x)}\lambda(x)dx$$

$$-\mu^2 \int\limits_{x=b}^{x=c} \underbrace{\tfrac{1}{2}\left[\left\{a_1 \sin a + b_1 \cos a\right\}\cos(\mu-1)(x-a) - \left\{a_1 \cos a + b_1 \sin a\right\}\sin(\mu-1)(x-a)\right]}_{B_{\mu-1}(x)}\lambda(x)dx$$

$$-\frac{\mu^2}{2^2} \int\limits_{x=b}^{x=c} \underbrace{\tfrac{1}{2}\left[\left\{a_2 \sin 2a + b_2 \cos 2a\right\}\cos(\mu+2)(x-a) + \left\{a_2 \cos 2a + b_2 \sin 2a\right\}\sin(\mu+2)(x-a)\right]}_{B_{\mu+2}(x)}\lambda(x)dx$$

$$-\frac{\mu^2}{2^2} \int\limits_{x=b}^{x=c} \underbrace{\tfrac{1}{2}\left[\left\{a_2 \sin 2a + b_2 \cos 2a\right\}\cos(\mu-2)(x-a) - \left\{a_2 \cos na + b_2 \sin 2a\right\}\sin(\mu-2)(x-a)\right]}_{B_{\mu-2}(x)}\lambda(x)dx$$

$- \dots\dots\dots$

$$\underbrace{\left\{a_n \sin nx + b_n \cos nx\right\}}_{A_n}\cos\mu(x-a) = B_{\mu+n}(x) + B_{\mu-n}(x)$$

$$B''_{\mu+n}(x) = -(\mu+n)^2 B_{\mu+n}(x).$$

$$B''_{\mu-n}(x) = -(\mu-n)^2 B_{\mu-n}(x).$$

$$B_{\mu+n}(x) \to 0, \text{ as } n \to \infty.$$

$$B_{\mu-n}(x) \to 0, \text{ as } n \to \infty.$$

Proof:

$$A_n \cos\mu(x-a) = a_n \sin nx \cos\mu(x-a) + b_n \cos nx \cos\mu(x-a)$$

$$= a_n \sin(nx - na + na)\cos\mu(x-a) + b_n \cos(nx - na + na)\cos\mu(x-a)$$

$$= a_n \left\{\sin n(x-a)\cos na + \cos n(x-a)\sin na\right\}\cos\mu(x-a)$$

$$+b_n \left\{\cos n(x-a)\cos na + \sin n(x-a)\sin na\right\}\cos\mu(x-a)$$

$$= \left[a_n \cos na + b_n \sin na\right]\sin n(x-a)\cos\mu(x-a)$$

$$+\left[a_n \sin na + b_n \cos na\right]\cos n(x-a)\cos\mu(x-a)$$

$$= \left[a_n \cos na + b_n \sin na\right]\tfrac{1}{2}\left\{\sin(\mu+n)(x-a) - \sin(\mu-n)(x-a)\right\}$$

$$+\left[a_n \sin na + b_n \cos na\right]\tfrac{1}{2}\left\{\cos(\mu+n)(x-a) + \cos(\mu-n)(x-a)\right\}$$

$$= \underbrace{\tfrac{1}{2}\left\{\left[a_n \cos na + b_n \sin na\right]\sin(\mu+n)(x-a) + \left[a_n \sin na + b_n \cos na\right]\cos(\mu+n)(x-a)\right\}}_{B_{\mu+n}(x)}$$

$$+\underbrace{\tfrac{1}{2}\left\{\left[a_n \sin na + b_n \cos na\right]\cos(\mu-n)(x-a) - \left[a_n \cos na + b_n \sin na\right]\sin(\mu-n)(x-a)\right\}}_{B_{\mu-n}(x)}$$

Therefore,

$$A_n \cos\mu(x-a) = B_{\mu+n}(x) + B_{\mu-n}(x),$$

$$B''_{\mu+n}(x) = -(\mu+n)^2 B_{\mu+n}(x)$$

$$B''_{\mu-n}(x) = -(\mu - n)^2 B_{\mu-n}(x).$$

and since $a_n \to 0$, and $b_n \to 0$,

$$B_{\mu+n}(x) \to 0, \text{ as } n \to \infty.$$

$$B_{\mu-n}(x) \to 0, \text{ as } n \to \infty.\square$$

2.2 ***If*** *the Trigonometric Series converges to* $f(x)$

$c > b$ *are arbitrary constants*

$\lambda(x)$ *has a continuous derivative* $\lambda'(x)$ *on* $[b,c]$

$\lambda(b) = \lambda(c) = 0$

$\lambda'(b) = \lambda'(c) = 0$

$\lambda''(x)$ *has finitely many maxima and minima*

Then $\mu^2 \int_{x=b}^{x=c} F(x)\cos(\mu[x-a])\lambda(x)dx =$

$$= \mu^2 \int_{x=b}^{x=c} \left(C + C'x + \tfrac{1}{2}A_0 x^2\right)\cos(\mu[x-a])\lambda(x)dx +$$

$$+\frac{\mu^2}{(\mu+1)^2}\int_{x=b}^{x=c} B_{\mu+1}(x)\lambda''(x)dx + \frac{\mu^2}{(\mu-1)^2}\int_{x=b}^{x=c} B_{\mu-1}(x)\lambda''(x)dx$$

$$+\frac{\mu^2}{2^2(\mu+2)^2}\int_{x=b}^{x=c} B_{\mu+2}(x)\lambda''(x)dx + \frac{\mu^2}{2^2(\mu-2)^2}\int_{x=b}^{x=c} B_{\mu-2}(x)\lambda''(x)dx$$

……………………………………………………………………..

$$\int_{x=b}^{x=c} B_{\mu+n}(x)\lambda''(x)dx \to 0, \text{ as } \mu+n\to\infty$$

$$\int_{x=b}^{x=c} B_{\mu-n}(x)\lambda''(x)dx \to 0, \text{ as } \mu-n\to\infty$$

Proof: $$-\frac{\mu^2}{n^2}\int_{x=b}^{x=c} A_n\cos(\mu[x-a])\lambda(x)dx =$$

$$= -\frac{\mu^2}{n^2}\int_{x=b}^{x=c} B_{\mu+n}(x)\lambda(x)dx - \frac{\mu^2}{n^2}\int_{x=b}^{x=c} B_{\mu-n}(x)\lambda(x)dx$$

Substituting

$$B''_{\mu+n}(x) = -(\mu+n)^2 B_{\mu+n}(x),$$

$$B''_{\mu-n}(x) = -(\mu-n)^2 B_{\mu-n}(x),$$

$$= \frac{\mu^2}{n^2(\mu+n)^2}\int_{x=b}^{x=c} B''_{\mu+n}(x)\lambda(x)dx + \frac{\mu^2}{n^2(\mu-n)^2}\int_{x=b}^{x=c} B''_{\mu-n}(x)\lambda(x)dx$$

Integrating by parts,

$$\int_{x=b}^{x=c} B''_{\mu+n}(x)\lambda(x)dx = \int_{x=b}^{x=c}\lambda(x)dB'_{\mu+n}(x)$$

$$= \underbrace{\Big[\lambda(x)B'_{\mu+n}(x)\Big]_{x=b}^{x=c}}_{0} - \int_{x=b}^{x=c} B'_{\mu+n}(x)\underbrace{d\lambda(x)}_{\lambda'(x)dx}$$

$$= -\int_{x=b}^{x=c} \lambda'(x) \underbrace{B'_{\mu+n}(x)dx}_{dB_{\mu+n}}$$

$$= -\left\{ \underbrace{\left[\lambda'(x)B_{\mu+n}(x)\right]_{x=b}^{x=c}}_{0} - \int_{x=b}^{x=c} B_{\mu+n}(x) \underbrace{d\lambda'(x)}_{\lambda''(x)dx} \right\}$$

$$= \int_{x=b}^{x=c} B_{\mu+n}(x)\lambda''(x)dx\,.$$

Hence, $-\dfrac{\mu^2}{n^2}\displaystyle\int_{x=b}^{x=c} A_n \cos(\mu[x-a])\lambda(x)dx =$

$$= \frac{\mu^2}{n^2(\mu+n)^2}\int_{x=b}^{x=c} B_{\mu+n}(x)\lambda''(x)dx + \frac{\mu^2}{n^2(\mu-n)^2}\int_{x=b}^{x=c} B_{\mu-n}(x)\lambda''(x)dx\,.\square$$

Now, since $\lambda''(x)$ has finitely many maxima and minima, then, by [Riemann] or [Dan1], it is integrable on $[b,c]$. Therefore, by the Riemann-Lebesgue Theorem, if $\mu \to \infty$, we have,

$$\int_{x=b}^{x=c} \lambda''(x)\sin\mu x dx \to 0,$$

and

$$\int_{x=b}^{x=c} \lambda''(x)\cos\mu x dx \to 0.$$

Consequently, if $\mu \to \infty$,

$$\int_{x=b}^{x=c} B_{\mu+n}(x)\lambda''(x)dx \to 0,$$

and

$$\int_{x=b}^{x=c} B_{\mu-n}(x)\lambda''(x)dx \to 0. \square$$

2.3 Riemann's 3rd Theorem

If *the Trigonometric Series converges to* $f(x)$

$c > b$ *are arbitrary constants*

$\lambda(x)$ *has a continuous derivative* $\lambda'(x)$ *on* $[b,c]$

$\lambda(b) = \lambda(c) = 0$

$\lambda'(b) = \lambda'(c) = 0$

$\lambda''(x)$ *has finitely many maxima and minima*

Then $\mu^2 \int_{x=b}^{x=c} F(x)\cos(\mu[x-a])\lambda(x)dx \to 0$, as $\mu \to \infty$.

Proof: In three parts

<u>*Part 1:*</u>

The first term in the expansion of $\mu^2 \int_{x=b}^{x=c} F(x)\cos(\mu[x-a])\lambda(x)dx$

in 2.2, that needs to vanish as $\mu \to \infty$ is

$$\mu^2 \int_{x=b}^{x=c} \left(C + C'x + \tfrac{1}{2} A_0 x^2 \right) \cos(\mu[x-a])\lambda(x)dx \,.$$

We confirm that $\mu^2 \int_{x=b}^{x=c} C'x\cos(\mu[x-a])\lambda(x)dx$ vanishes after two consecutive integrations by parts.

$$\mu^2 C' \int_{x=b}^{x=c} x \underbrace{\cos(\mu[x-a])}_{\frac{1}{\mu}\frac{d\left\{\sin(\mu[x-a])\right\}}{dx}} \lambda(x)dx = \mu C' \int_{x=b}^{x=c} x\lambda(x) d\left\{\sin(\mu[x-a])\right\}$$

$$= \mu C' \left\{ \underbrace{\left[x\lambda(x)\sin(\mu[x-a]) \right]_{x=b}^{x=c}}_{0} - \int_{x=b}^{x=c} \sin(\mu[x-a]) d\left\{ x\lambda(x) \right\} \right\}$$

$$= -\mu C' \left\{ \int_{x=b}^{x=c} \underbrace{\sin(\mu[x-a])}_{-\frac{1}{\mu}\frac{d\left\{\cos(\mu[x-a])\right\}}{dx}} \left\{ \lambda(x) + x\lambda'(x) \right\} dx \right\}$$

$$= C' \left\{ \int_{x=b}^{x=c} \left\{ \lambda(x) + x\lambda'(x) \right\} d\left\{ \cos(\mu[x-a]) \right\} \right\}$$

$$= C' \left\{ \underbrace{[\lambda(x) + x\lambda'(x)][\cos(\mu[x-a])]_{x=b}^{x=c}}_{0} - \int_{x=b}^{x=c} \cos(\mu[x-a]) d\left\{ \lambda(x) + x\lambda'(x) \right\} \right\}$$

$$= -C' \int_{x=b}^{x=c} \cos(\mu[x-a])[2\lambda'(x) + \lambda''(x)]dx \,.$$

Since $2\lambda'(x) + \lambda''(x)$ is integrable on $[b,c]$, by the Riemann-

Lebesgue Lemma, the last integral vanishes as $\mu \to \infty$.

The rest of the first term vanishes similarly.□

Part 2:

Next we show that as $\mu \to \infty$, the series

$$\frac{\mu^2}{1^2(\mu+1)^2}\int_{x=b}^{x=c} B_{\mu+1}(x)\lambda''(x)dx + \frac{\mu^2}{2^2(\mu+2)^2}\int_{x=b}^{x=c} B_{\mu+2}(x)\lambda''(x)dx + \ldots$$

vanishes.

By 2.2,

$$\int_{x=b}^{x=c} B_{\mu+n}(x)\lambda''(x)dx \to 0, \text{ as } \mu \to \infty.$$

Thus, for $\varepsilon > 0$, and for all $\mu > \mathrm{M}$

$$\left|\int_{x=b}^{x=c} B_{\mu+n}(x)\lambda''(x)dx\right| < \varepsilon.$$

Hence, the series is bounded by

$$\varepsilon\left\{\frac{\mu^2}{1^2(\mu+1)^2} + \frac{\mu^2}{2^2(\mu+2)^2} + \ldots\right\} = \varepsilon\left\{\frac{1}{(1+\frac{1}{\mu})^2} + \frac{1}{2^2(1+\frac{2}{\mu})^2} + \ldots\right\}$$

$$< \varepsilon\left\{\frac{1}{1^2} + \frac{1}{2^2} + \ldots\right\} = \varepsilon\frac{\pi^2}{6}$$

and we let $\varepsilon \downarrow 0$.□

Part 3:

We show that as $\mu \to \infty$, the series

$$\frac{\mu^2}{1^2(\mu-1)^2}\int_{x=b}^{x=c} B_{\mu-1}(x)\lambda''(x)dx + \frac{\mu^2}{2^2(\mu-2)^2}\int_{x=b}^{x=c} B_{\mu-2}(x)\lambda''(x)dx + ...$$

vanishes.

By 2.2,

$$\int_{x=b}^{x=c} B_{\mu-n}(x)\lambda''(x)dx \to 0, \text{ as } \mu \to \infty.$$

Thus, for $\varepsilon > 0$, and for all $\mu > \mathrm{M}$

$$\left|\int_{x=b}^{x=c} B_{\mu-n}(x)\lambda''(x)dx\right| < \varepsilon.$$

Hence, the series is bounded by

$$\varepsilon\left\{\frac{\mu^2}{1^2(\mu-1)^2} + \frac{\mu^2}{2^2(\mu-2)^2} + ...\right\} = \varepsilon\frac{1}{\mu}\left\{\frac{\frac{1}{\mu}}{(\frac{1}{\mu})^2(1-\frac{1}{\mu})^2} + \frac{\frac{1}{\mu}}{(\frac{2}{\mu})^2(1-\frac{2}{\mu})^2} + ...\right\}$$

The terms in { } sum up to the Lower Riemann Sum for the function

$$\frac{1}{x^2(1-x)^2},$$

over a partition with subintervals of length $\frac{1}{\mu}$,

Thus, the Series is bounded by

$$\varepsilon\frac{1}{\mu}\int_{x=-\infty}^{x=\infty}\frac{1}{x^2(1-x)^2}dx.$$

Decomposing the integrand,

$$\frac{1}{x^2(1-x)^2} = \frac{A+Bx}{x^2} + \frac{C+D(1-x)}{(1-x)^2}.$$

$$1 = A(1-x)^2 + Bx(1-x)^2 + Cx^2 + D(1-x)x^2$$

$$x = 0 \Rightarrow A = 1$$

$$x = 1 \Rightarrow C = 1$$

$$1 = (1-x)^2 + Bx(1-x)^2 + x^2 + D(1-x)x^2$$

$$x = 2 \Rightarrow \qquad 1 = 1 + 2B + 4 - 4D \Rightarrow \qquad -2B + 4D = 4$$

$$x = -1 \Rightarrow \qquad 1 = 4 - 4B + 1 + 2D \Rightarrow \qquad 4B - 2D = 4$$

$$B = 2$$

$$D = 2$$

$$\frac{1}{x^2(1-x)^2} = \frac{1}{x^2} + \frac{2}{x} + \frac{1}{(1-x)^2} + \frac{2}{1-x}.$$

$$\frac{1}{\mu}\int \frac{1}{x^2(1-x)^2}dx = \frac{1}{\mu}\left\{-\frac{1}{x} + \frac{1}{1-x} + 2\log x - 2\log(1-x)\right\} + Const.$$

$$= \frac{1}{\mu}\left\{-\frac{1}{x} + \frac{1}{1-x} + 2\log\frac{x}{1-x}\right\} + Const.$$

$$\frac{1}{\mu}\int_{x=-\mu}^{x=\mu} \frac{1}{x^2(1-x)^2}dx = \frac{1}{\mu}\left\{-\frac{1}{x} + \frac{1}{1-x} + 2\log\frac{x}{1-x}\right\}_{x=-\mu}^{x=-\frac{1}{\mu}}$$

$$+\frac{1}{\mu}\left\{-\frac{1}{x} + \frac{1}{1-x} + 2\log\frac{x}{1-x}\right\}_{x=\frac{1}{\mu}}^{x=1-\frac{1}{\mu}}$$

$$+\frac{1}{\mu}\left\{-\frac{1}{x}+\frac{1}{1-x}+2\log\frac{x}{1-x}\right\}_{x=1+\frac{1}{\mu}}^{x=\mu}$$

$$=\left\{1+\frac{1}{\mu+1}+\frac{2}{\mu}\log\frac{-1}{\mu+1}\right\}+\left\{\frac{1}{\mu^2}-\frac{1}{\mu}\frac{1}{1+\mu}-\frac{2}{\mu}\log\frac{-\mu}{1+\mu}\right\}$$

$$+\left\{\frac{1}{1-\mu}+1+2\frac{1}{\mu}\log(\mu-1)\right\}+\left\{1+\frac{1}{1-\mu}-2\frac{1}{\mu}\log\frac{1}{\mu-1}\right\}$$

$$+\left\{-\frac{1}{\mu^2}+\frac{1}{\mu}\frac{1}{1-\mu}+2\frac{1}{\mu}\log\frac{\mu}{1-\mu}\right\}+\left\{\frac{1}{\mu+1}+1-2\frac{1}{\mu}\log[-(\mu+1)]\right\}$$

$$=4+\frac{2}{\mu+1}+\frac{2}{1-\mu}-\frac{1}{\mu}\frac{1}{1+\mu}+\frac{1}{\mu}\frac{1}{1-\mu}$$

$$+\frac{2}{\mu}\left\{\log(\mu-1)-2\log(-1)-\log(\mu+1)\right\}$$

Letting $\mu\to\infty$,

$$\varepsilon\frac{1}{\mu}\int_{x=-\mu}^{x=\mu}\frac{1}{x^2(1-x)^2}\,dx\to 4\varepsilon.$$

And we let $\varepsilon\downarrow 0$. $\square$

3.

Unproven equality of $f(x)$ to its Trigonometric Series

Riemann claimed that the necessary conditions of his 1st and 3rd Theorem, are also sufficient for a periodic $f(x)$ to equal its Fourier Series.

His plausibility argument is fatally flawed, leaving his claim unproven.

We state Riemann's claim, and follow his attempted proof, till its breakdown.

3.1 *Unproven equality of* $f(x)$ *to its Trigonometric Series*

Let $f(x)$ *be periodic with period* 2π.

Then,

$$f(x) = \underbrace{\tfrac{1}{2}b_0}_{A_0} + \underbrace{a_1 \sin x + b_1 \cos x}_{A_1(x)} + \underbrace{a_2 \sin 2x + b_2 \cos 2x}_{A_2(x)} + \ldots$$

so that for each x, $A_n \to 0$, *as* $n \to \infty$.

$\Leftrightarrow$

There is a continuous $F(x)$ *so that*

(I) $$\frac{F(x+\alpha+\beta)-F(x+\alpha-\beta)-F(x-\alpha+\beta)+F(x-\alpha-\beta)}{4\alpha\beta} \to f(x),$$

for infinitesimals α*, and* β*, so that* $\frac{\alpha}{\beta}$*, and* $\frac{\beta}{\alpha}$ *are finite.*

(II) *for any* $c > b$*, there is* $\lambda(x)$ *with* $\lambda'(x)$ *continuous on* (b,c)

$\lambda(b) = \lambda(c) = 0$

$\lambda'(b) = \lambda'(c) = 0$

$\lambda''(x)$ *bounded and has finitely many maxima and minima*

and $\mu^2 \int_{x=b}^{x=c} F(x)\cos(\mu[x-a])\lambda(x)dx \to 0$*, as* $\mu \to \infty$.

Riemann's Non-Proof:

$(\Rightarrow)$ By the 1st Theorem, and the 3rd Theorem.□

$(\Leftarrow)$ By (I), there is a continuous $F(x)$ so that

$$\frac{F(x+\alpha+\beta)-F(x+\alpha-\beta)-F(x-\alpha+\beta)+F(x-\alpha-\beta)}{4\alpha\beta} \to f(x),$$

which is periodic with period 2π.

Thus,

$$F''(x) = f(x).$$

Take C', and A_0 so that

$$\Phi(x) \equiv F(x) - C'x - \tfrac{1}{2}A_0x^2$$

is periodic with period 2π, and form a trigonometric series

$$C - \underbrace{(a_1 \sin x + b_1 \cos x)}_{A_1(x)} - \tfrac{1}{2^2}\underbrace{(a_2 \sin 2x + b_2 \cos 2x)}_{A_2(x)} - \ldots,$$

where

$$C = \frac{1}{2\pi}\int_{t=-\pi}^{t=\pi} \Phi(t)dt\,,$$

and

$$-\frac{1}{n^2}A_n(x) = \frac{1}{\pi}\int_{t=-\pi}^{t=\pi} \Phi(t)\cos[n(x-t)]dt\,.$$

Integrating twice by parts,

$$= -\frac{1}{n^2\pi}\int_{t=-\pi}^{t=\pi} \Phi''(t)\cos[n(x-t)]dt$$

Now, for infinitesimals α, and β, so that $\frac{\alpha}{\beta}$, and $\frac{\beta}{\alpha}$ are finite,

$F(x)$ satisfies

$$\frac{F(x+\alpha+\beta) - F(x+\alpha-\beta) - F(x-\alpha+\beta) + F(x-\alpha-\beta)}{4\alpha\beta} \to f(x)$$

$C'x$ satisfies

$$\frac{C'}{4\alpha\beta}\left\{(x+\alpha+\beta) - (x+\alpha-\beta) - (x-\alpha+\beta) + (x-\alpha-\beta)\right\} = 0$$

$\frac{1}{2}A_0x^2$ satisfies

$$\frac{\frac{1}{2}A_0}{4\alpha\beta}\left\{(x+\alpha+\beta)^2 - (x+\alpha-\beta)^2 - (x-\alpha+\beta)^2 + (x-\alpha-\beta)^2\right\} = 0$$

Hence, for infinitesimals α, and β, so that $\frac{\alpha}{\beta}$, and $\frac{\beta}{\alpha}$ are finite,

$$\frac{\Phi(x+\alpha+\beta)-\Phi(x+\alpha-\beta)-\Phi(x-\alpha+\beta)+\Phi(x-\alpha-\beta)}{4\alpha\beta} \to f(x).$$

Thus,

$$\Phi''(x) = f(x).$$

Substituting this in the integral above, we obtain

$$-\frac{1}{n^2\pi}\int_{t=-\pi}^{t=\pi} f(t)\cos[n(x-t)]dt.$$

Therefore,

$$A_n(x) = \frac{1}{\pi}\int_{t=-\pi}^{t=\pi} f(t)\cos[n(x-t)]dt.$$

Hence,

$$A_0 + A_1(x) + A_2(x) + \ldots$$

is the Trigonometric Series associated with the periodic function $f(x)$. So far, no hint about its equality to $f(x)$.

By (II), for $[b,c] = [-\pi,\pi]$, and for $\lambda(x) = \begin{cases} 1, & x \in (-\pi,\pi) \\ 0, & \text{otherwise} \end{cases}$,

$\lambda(x)$ and $\lambda'(x)$ are continuous on (b,c),

$\lambda(b) = \lambda(c) = 0$,

$\lambda'(b) = \lambda'(c) = 0$,

$\lambda''(x)$ bounded and has finitely many maxima and minima,

Hence,

$$n^2 \int_{x=-\pi}^{x=\pi} F(x)\cos(n[x-a])\lambda(x)dx \to 0, \text{ as } n \to \infty.$$

That is,

$$n^2 \int_{x=-\pi}^{x=\pi} F(x)\cos(n[x-a])dx \to 0, \text{ as } n \to \infty.$$

By integration by parts, we confirm that

$$n^2 \int_{x=-\pi}^{x=\pi} (C'x + \tfrac{1}{2}A_0x^2)\cos(n[x-a])dx \to 0, \text{ as } n \to \infty.$$

Therefore,

$$n^2 \int_{x=-\pi}^{x=\pi} \underbrace{(F(x) - C'x - \tfrac{1}{2}A_0x^2)}_{\Phi(x)}\cos(n[x-a])dx \to 0, \text{ as } n \to \infty.$$

Namely,

$$\underbrace{n^2 \int_{t=-\pi}^{t=\pi} \Phi(t)\cos(n[t-x])dt}_{-A_n(x)} \to 0, \text{ as } n \to \infty.$$

Thus, $A_n(x) \to 0$, as $n \to \infty$, for each x.

Clearly, $A_n(x) \to 0$ is only a necessary condition for the convergence of the Trigonometric Series.

To establish that $f(x)$ is equal to the Trigonometric Series, Riemann needed to show that as $n \to \infty$, for each x,

$$A_0 + A_1(x) + ...A_n(x) - f(x) \to 0.$$

Riemann's concludes with

> "*It follows by Theorem 1 of the preceding section that the series*
>
> $$A_0 + A_1 + A_2 + ...$$
>
> *converges to the function* $f(x)$*, wherever it converges*".

Clearly, Riemann's 1st Theorem of 1.10 assumes that the function equals its trigonometric series, to obtain condition (I).

Applying Theorem 1 here, means assuming that the function equals its trigonometric series.

Thus, Riemann's proof is based on assuming its result.

His claim that conditions (I) and (II) are sufficient remains unproven.

4.

Unproven equality of Fourier Series to a Convolution of $F(t)$ with Dirichlet Kernel

Riemann attempted to represent the Trigonometric Series as a convolution of $F(t)$ with the Dirichlet Kernel.

He failed because the Dirichlet Kernel is the infinite Series

$$\cos(x-t)+\cos 2(x-t)+\cos 3(x-t)+...$$

that diverges to infinity at $x=t$.

4.1 *Unproven equality of Fourier Series to a convolution of $F(t)$ with the Dirichlet Kernel*

If *$f(x)$ is periodic with period 2π,*

There is a continuous $F(x)$ so that

(I) $$\frac{F(x+\alpha+\beta)-F(x+\alpha-\beta)-F(x-\alpha+\beta)+F(x-\alpha-\beta)}{4\alpha\beta}\to f(x),$$

for infinitesimals α, and β, so that $\dfrac{\alpha}{\beta}$, and $\dfrac{\beta}{\alpha}$ are finite.

(II) *for any* $c > b$, *there is* $\lambda(x)$ *with* $\lambda'(x)$ *continuous on* (b,c)

$\lambda(b) = \lambda(c) = 0$

$\lambda'(b) = \lambda'(c) = 0$

$\lambda''(x)$ *bounded and has finitely many maxima and minima*

and $\mu^2 \int_{x=b}^{x=c} F(x)\cos(\mu[x-a])\lambda(x)dx \to 0$, *as* $\mu \to \infty$.

(III) $\rho(t)$ *and* $\rho'(t)$ *are continuous on* (b,c) *so that*

$\rho(b) = \rho(c) = 0$,

$\rho'(b) = \rho'(c) = 0$,

$\rho''(t)$ *is bounded and has finitely many maxima and minima,*

and at a fixed point $b < x < c$,

$\rho(x) = 1$,

$\rho'(x) = 1$,

$\rho''(x) = 1$,

$\rho'''(x)$ *and* $\rho''''(x)$ *are finite and continuous*

Then, *as* $n \to \infty$,

$$\{A_0 + A_1(x) + ...A_n(x)\} - \frac{1}{2\pi}\int_{t=b}^{t=c} F(t)\frac{d^2}{dt^2}\left\{\frac{\sin\frac{(2n+1)(x-t)}{2}}{\sin\frac{x-t}{2}}\right\}\rho(t)dt \to 0.$$

Hence, *as* $n \to \infty$,

$$A_0 + A_1(x) + A_2(x) + ... \text{ converges to } f(x)$$

$$\Leftrightarrow$$

$$\frac{1}{2\pi}\int_{t=b}^{t=c} F(t)\frac{d^2}{dt^2}\left\{\frac{\sin\frac{(2n+1)(x-t)}{2}}{\sin\frac{x-t}{2}}\right\}\rho(t)dt \text{ converges to } f(x)$$

Riemann's Non-Proof:

Keeping the notations of the unproven 3.1,

$A_1(x) + A_2(x) + ...A_n(x) =$

$$= \frac{1}{\pi}\int_{t=-\pi}^{t=\pi} \underbrace{(F(t) - C't - \tfrac{1}{2}A_0 t^2)}_{\Phi(t)}\left\{\underbrace{-1^2\cos(x-t)}_{D_t^2\left\{\cos(x-t)\right\}} - ...\underbrace{-n^2\cos n(x-t)}_{D_t^2\left\{\cos n(x-t)\right\}}\right\}dt$$

$$= \frac{1}{\pi}\int_{t=-\pi}^{t=\pi} \Phi(t)D_t^2\left\{\cos(x-t) + ... + \cos n(x-t)\right\}dt$$

Since $\cos(x-t) + ... + \cos n(x-t) = \dfrac{\sin\frac{(2n+1)(x-t)}{2}}{2\sin\frac{x-t}{2}}$,

$$A_1(x) + ...A_n(x) = \frac{1}{2\pi}\int_{t=-\pi}^{t=\pi} \Phi(t)\frac{d^2}{dt^2}\left\{\frac{\sin\frac{(2n+1)(x-t)}{2}}{\sin\frac{x-t}{2}}\right\}dt.$$

Therefore,

$$\left\{A_1(x) + ...A_n(x)\right\} - \frac{1}{2\pi}\int_{t=b}^{t=c} \Phi(t)\frac{d^2}{dt^2}\left\{\frac{\sin\frac{(2n+1)(x-t)}{2}}{\sin\frac{x-t}{2}}\right\}\rho(t)dt =$$

$$= \frac{1}{2\pi}\int_{t=-\pi}^{t=\pi} \Phi(t)\frac{d^2}{dt^2}\left\{\frac{\sin\frac{(2n+1)(x-t)}{2}}{\sin\frac{x-t}{2}}\right\}dt - \frac{1}{2\pi}\int_{t=b}^{t=c} \Phi(t)\frac{d^2}{dt^2}\left\{\frac{\sin\frac{(2n+1)(x-t)}{2}}{\sin\frac{x-t}{2}}\right\}\rho(t)dt\,.$$

Denoting $\lambda(t) = \begin{cases} 1, & t \notin (b,c) \\ 1-\rho(t), & t \in (b,c) \end{cases}$, we have,

$$= \frac{1}{2\pi}\int_{t=-\pi}^{t=\pi} \underbrace{\Phi(t)}_{(F(t)-C't-\frac{1}{2}A_0t^2)} \frac{d^2}{dt^2}\left\{\frac{\sin\frac{(2n+1)(x-t)}{2}}{\sin\frac{x-t}{2}}\right\}\lambda(t)dt$$

Now, $\lambda(t)$, and $\lambda'(t)$ are continuous

$\lambda''(t)$ has finitely many maxima and minima

For $t = x$, $\lambda(x) = 1-\rho(x) = 0$

$\lambda'(x) = -\rho'(x) = 0$

$\lambda''(x) = -\rho''(x) = 0$

$\lambda'''(x)$, and $\lambda''''(x)$ are finite and continuous

Riemann claims that by his 3rd Theorem, as $n \to \infty$,

$$\frac{1}{2\pi}\int_{t=-\pi}^{t=\pi} (F(t) - C't - \tfrac{1}{2}A_0t^2)\frac{d^2}{dt^2}\left\{\frac{\sin\frac{(2n+1)(x-t)}{2}}{\sin\frac{x-t}{2}}\right\}\lambda(t)dt \to 0,$$

and claims further that by integration by parts

$$\frac{1}{2\pi}\int_{t=b}^{t=c} (C't + \tfrac{1}{2}A_0t^2)\frac{d^2}{dt^2}\left\{\frac{\sin\frac{(2n+1)(x-t)}{2}}{\sin\frac{x-t}{2}}\right\}\rho(t)dt \to A_0\,.$$

Thus, as $n \to \infty$,

$$\left\{A_0 + A_1(x) + ...A_n(x)\right\} - \frac{1}{2\pi}\int_{t=b}^{t=c} F(t)\frac{d^2}{dt^2}\left\{\frac{\sin\frac{(2n+1)(x-t)}{2}}{\sin\frac{x-t}{2}}\right\}\rho(t)dt \to 0.$$

Rather then fill in the details, we proceed to Riemann's purpose of this derivation, the representation of $f(x)$ by the Dirichlet Integral.

To start with, note that Riemann did not prove in 3.1 that as $n \to \infty$, $A_0 + A_1(x) + A_2(x) + ...$ equals $f(x)$.

Therefore, if the convolution $\frac{1}{2\pi}\int_{t=b}^{t=c} F(t)\frac{d^2}{dt^2}\left\{\frac{\sin\frac{(2n+1)(x-t)}{2}}{\sin\frac{x-t}{2}}\right\}\rho(t)dt$

converges, its limit need not be $f(x)$.

However, since

$$\frac{d^2}{dt^2}\left\{\frac{\sin\frac{(2n+1)(x-t)}{2}}{\sin\frac{x-t}{2}}\right\} = -\cos(x-t) - 2^2\cos 2(x-t) - ... - n^2\cos n(x-t),$$

then, as $n \to \infty$, we obtain the infinite series

$$-\cos(x-t) - 2^2\cos 2(x-t) - 3^2\cos 3(x-t) - ...$$

that diverges to infinity at $t = x$.

Therefore, the integral cannot be defined, and the question of its convergence is mute.

Riemann attempted to represent $f(x)$ by an integral that does not exist.

5.

Unproven Divergence of Fourier Coefficients

Riemann attempted to show that a function with infinitely many maxima or minima may have diverging Fourier Coefficients. But his proof is incomplete, and the claim is not proven.

5.1 *Unproven divergence of Fourier Coefficients*

$$\frac{d}{dx}\left\{x^{\nu}\cos\tfrac{1}{x}\right\}, \quad 0 \le x \le 2\pi\,, \quad 0 < \nu < \tfrac{1}{2},$$

is integrable,

but has infinitely many maxima and minima,

and diverging Fourier Coefficients

Riemann's Non-Proof:

Since $$\frac{d}{dx}\left\{x^{\nu}\cos\tfrac{1}{x}\right\} = \nu x^{\nu-1}\cos\tfrac{1}{x} + x^{\nu}(-\sin\tfrac{1}{x})(-\tfrac{1}{x^2}),$$

the Fourier Coefficient $\displaystyle\int_{x=0}^{x=2\pi}\frac{d}{dx}\left\{x^{\nu}\cos\tfrac{1}{x}\right\}\cos n(x-a)dx$ has the term

$$\int_{x=0}^{x=2\pi} x^{\nu}\,\tfrac{1}{x^2}\sin(\tfrac{1}{x})\cos n(x-a)dx\,.$$

Since $\sin(\frac{1}{x})\cos n(x-a) = \frac{1}{2}\sin\left[\frac{1}{x}+n(x-a)\right]+\frac{1}{2}\sin\left[\frac{1}{x}-n(x-a)\right]$,

the Fourier coefficient has the term

$$\int_{x=0}^{x=2\pi} x^{\nu-2}\sin\left[\frac{1}{x}+n(x-a)\right]dx\,.$$

The slowest change of the sign of $\sin\left[\frac{1}{x}+n(x-a)\right]$ is in the neighborhood of the point where

$$\frac{d}{dx}\left[\frac{1}{x}+n(x-a)\right]=0\,,$$

$$-\frac{1}{x^2}+n=0\,,$$

$$x=\frac{1}{\sqrt{n}}\,.$$

Expanding

$$y(x)=\frac{1}{x}+n(x-a)\,,$$

in a second order Taylor Polynomial about $x=\frac{1}{\sqrt{n}}$,

$$y(x)\approx \underbrace{y(\tfrac{1}{\sqrt{n}})}_{2\sqrt{n}-na}+\underbrace{y'(\tfrac{1}{\sqrt{n}})}_{0}(x-\tfrac{1}{\sqrt{n}})+\frac{1}{2}\underbrace{y''(\tfrac{1}{\sqrt{n}})}_{2n^{\frac{3}{2}}}(x-\tfrac{1}{\sqrt{n}})^2$$

$$=2\sqrt{n}-na+n^{\frac{3}{2}}(x-\tfrac{1}{\sqrt{n}})^2\,,$$

$$\frac{dy}{dx}=2n^{\frac{3}{2}}(x-\tfrac{1}{\sqrt{n}}),$$

$$(x-\tfrac{1}{\sqrt{n}})^2=\frac{y-(2\sqrt{n}-na)}{n^{\frac{3}{2}}}\,.$$

Therefore,

$x > \frac{1}{\sqrt{n}} \Rightarrow\ x - \frac{1}{\sqrt{n}} = \frac{\sqrt{y-(2\sqrt{n}-na)}}{n^{\frac{3}{4}}}$, and $\frac{dy}{dx} = 2n^{\frac{3}{4}}\sqrt{y-(2\sqrt{n}-na)}$

$x < \frac{1}{\sqrt{n}} \Rightarrow \qquad x - \frac{1}{\sqrt{n}} = -\frac{\sqrt{y-(2\sqrt{n}-na)}}{n^{\frac{3}{4}}}$, and

$\frac{dy}{dx} = -2n^{\frac{3}{4}}\sqrt{y-(2\sqrt{n}-na)}$

The contribution from the term $\int_{x=0}^{x=2\pi} x^{\nu-2}\sin\left[\frac{1}{x}+n(x-a)\right]dx$ in the interval $[0,\frac{2}{\sqrt{n}}]$ is

$$\int_{x=0}^{x=\frac{2}{\sqrt{n}}} x^{\nu-2}\sin y dx\,.$$

Substituting

$$x^{\nu-2} \sim \left(\frac{1}{\sqrt{n}}\right)^{\nu-2},$$

the integral is

$$\approx n^{1-\frac{\nu}{2}}\int_{x=0}^{x=\frac{2}{\sqrt{n}}}\sin y dx$$

$$= n^{1-\frac{\nu}{2}}\left(\int_{x=0}^{x=\frac{1}{\sqrt{n}}}\sin y dx + \int_{x=\frac{1}{\sqrt{n}}}^{x=\frac{2}{\sqrt{n}}}\sin y dx\right).$$

Changing the integration variable to y,

$x = 0 \Rightarrow \qquad y \approx 2\sqrt{n} - na + n^{\frac{3}{2}}(0-\frac{1}{\sqrt{n}})^2$

$\qquad = 3\sqrt{n} - na$

$x = \frac{1}{\sqrt{n}} \Rightarrow \qquad y = 2\sqrt{n} - na$

$$x = \frac{2}{\sqrt{n}} \Rightarrow \qquad y \approx 2\sqrt{n} - na + n^{\frac{3}{2}}\left(\frac{2}{\sqrt{n}} - \frac{1}{\sqrt{n}}\right)^2$$

$$= 3\sqrt{n} - na$$

In $[0, \frac{1}{\sqrt{n}}]$, $\quad x < \frac{1}{\sqrt{n}}$,

$$dx = -\frac{dy}{2n^{\frac{3}{4}}\sqrt{y - (2\sqrt{n} - na)}},$$

and the first integral transforms to

$$\int_{y=3\sqrt{n}-na}^{y=2\sqrt{n}-na} \sin y \left(-\frac{dy}{2n^{\frac{3}{4}}\sqrt{y - (2\sqrt{n} - na)}} \right) = n^{-\frac{3}{4}} \frac{1}{2} \int_{y=2\sqrt{n}-na}^{y=3\sqrt{n}-na} \frac{\sin y}{\sqrt{y - (2\sqrt{n} - na)}} dy .$$

In $[\frac{1}{\sqrt{n}}, \frac{2}{\sqrt{n}}]$, $\quad x > \frac{1}{\sqrt{n}}$,

$$dx = \frac{dy}{2n^{\frac{3}{4}}\sqrt{y - (2\sqrt{n} - na)}},$$

and the second integral transforms to

$$\int_{y=2\sqrt{n}-na}^{y=3\sqrt{n}-na} \sin y \left(\frac{dy}{2n^{\frac{3}{4}}\sqrt{y - (2\sqrt{n} - na)}} \right) = n^{-\frac{3}{4}} \frac{1}{2} \int_{y=2\sqrt{n}-na}^{y=3\sqrt{n}-na} \frac{\sin y}{\sqrt{y - (2\sqrt{n} - na)}} dy .$$

Therefore,

$$n^{1-\frac{\nu}{2}} \int_{x=0}^{x=\frac{2}{\sqrt{n}}} \sin y dx = n^{\frac{1}{4}-\frac{\nu}{2}} \int_{y=2\sqrt{n}-na}^{y=3\sqrt{n}-na} \frac{\sin y}{\sqrt{y - (2\sqrt{n} - na)}} dy .$$

Changing variable to

$$\xi = y - 2\sqrt{n} - na$$

$$= n^{\frac{1}{4}-\frac{\nu}{2}} \int\limits_{\xi=0}^{\xi=\sqrt{n}} \frac{\sin(\xi + 2\sqrt{n} - na)}{\sqrt{\xi}} d\xi$$

Since $\frac{1}{2} > \nu > 0$, we have, $\frac{1}{4} - \frac{\nu}{2} > 0$, and as $n \to \infty$,

$$n^{\frac{1}{4}-\frac{\nu}{2}} \to \infty.$$

But we don't know what is

$$\lim_{n\to\infty} \int\limits_{\xi=0}^{\xi=\sqrt{n}} \frac{\sin(\xi + 2\sqrt{n} - na)}{\sqrt{\xi}} d\xi.$$

Riemann wrote

"If

$$\int\limits_{\xi=0}^{\xi=\infty} \frac{\sin(\xi + \beta)}{\sqrt{\xi}} d\xi,$$

which equals

$$\sqrt{\pi}\sin(\beta + \tfrac{\pi}{4})$$

is not zero…"

demonstrating oblivion to the dependence of the integrand on n. Thus, leaving the proof hanging on an "if", incomplete, and his claim unproven.

6.

Fourier Series of Riemann's (x) Function

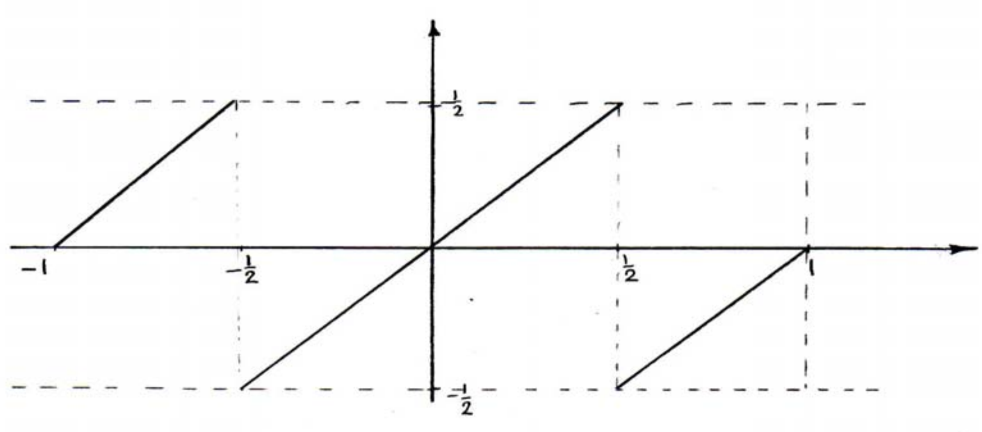
6.1 The Riemann (x) Function

$$(x) = \begin{cases} x - nearest \ \text{int}eger, \ if \ x \neq n + \frac{1}{2} \\ 0, \ \text{if } x = n + \frac{1}{2} \end{cases}$$

6.2 (x) *is periodic with period* 1,

is continuous in $..., -\frac{3}{2} < x < -\frac{1}{2}, -\frac{1}{2} < x < \frac{1}{2}, \frac{1}{2} < x < \frac{3}{2}, ..$

is discontinuous at..... $-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, ...$



and has the Fourier series

$$(x) = \frac{1}{\pi}\left(\frac{\sin 2\pi x}{1} - \frac{\sin 2 \cdot 2\pi x}{2} + \frac{\sin 3 \cdot 2\pi x}{3} - ... \right)$$

A detailed discussion of Riemann's (x) Function appears in [Dan1].

7.

Fourier Series of Riemann's $(2x)$ Function

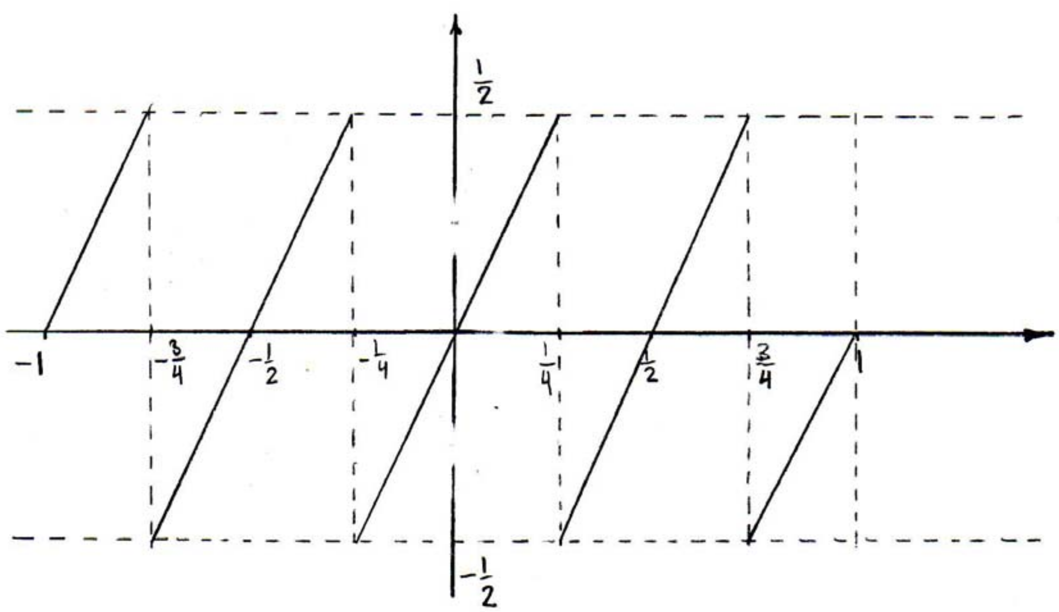
7.1 $(2x)$ *is periodic with period* $\frac{1}{2}$,

is continuous in $..,-\frac{3}{4} < x < -\frac{1}{4}$, $-\frac{1}{4} < x < \frac{1}{4}$,

$\frac{1}{4} < x < \frac{3}{4}$,....

is discontinuous at $.....-\frac{3}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}$,.....

has the graph



and has the Fourier series

$$(2x) = \frac{1}{\pi}\left(\frac{\sin 4\pi x}{1} - \frac{\sin 2\cdot 4\pi x}{2} + \frac{\sin 3\cdot 4\pi x}{3} - ...\right)$$

8.

Fourier Series of Riemann's $(3x)$ Function

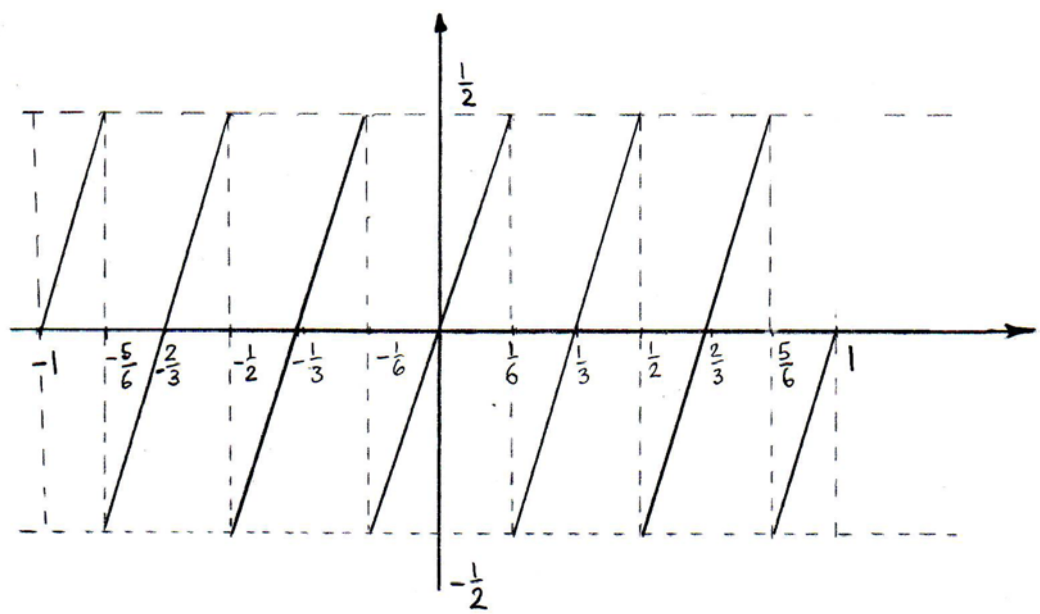
8.1 $(3x)$ *is periodic with period* $\frac{1}{3}$,

is continuous in $\ldots, -\frac{3}{6} < x < -\frac{1}{6}, \quad -\frac{1}{6} < x < \frac{1}{6},$

$$\frac{1}{6} < x < \frac{3}{6}, \ldots.$$

is discontinuous at $\ldots.. -\frac{3}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{3}{6}, \ldots..$

has the graph



and has the Fourier series

$$(3x) = \frac{1}{\pi}\left(\frac{\sin 6\pi x}{1} - \frac{\sin 2\cdot 6\pi x}{2} + \frac{\sin 3\cdot 6\pi x}{3} - ...\right)$$

9.

Fourier Series of Riemann's (x) Series

9.1 Riemann's (x) Series is

$$f(x) = \frac{(x)}{1} + \frac{(2x)}{2} + \frac{(3x)}{3} + ...$$

9.2 *Riemann (x) Series is unbounded in any interval*

Hence,

9.3 *Riemann (x) Series is nowhere integrable*

But

9.4 *Riemann (x) Series converges at each rational* $x = q$

And,

9.5 *At each rational* $x = q$*, Riemann's (q) Series has the Fourier Series*

$$f(q) = \frac{1}{\pi}\left(\frac{\sin 2\pi q}{1} - \frac{\sin 2\cdot 2\pi q}{2} + \frac{\sin 3\cdot 2\pi q}{3} - \frac{\sin 4\cdot 2\pi q}{4} + \frac{\sin 5\cdot 2\pi q}{5} - \frac{\sin 6\cdot 2\pi q}{6} + ...\right)$$

$$+\frac{1}{2}\frac{1}{\pi}\left(\frac{\sin 4\pi q}{1} - \frac{\sin 2\cdot 4\pi q}{2} + \frac{\sin 3\cdot 4\pi q}{3} - \frac{\sin 4\cdot 4\pi q}{4} + \frac{\sin 5\cdot 4\pi q}{5} - \frac{\sin 6\cdot 4\pi q}{6} + ...\right)$$

$$+\frac{1}{3}\frac{1}{\pi}\left(\frac{\sin 6\pi q}{1} - \frac{\sin 2\cdot 6\pi q}{2} + \frac{\sin 3\cdot 6\pi q}{3} - \frac{\sin 4\cdot 6\pi q}{4} + \frac{\sin 5\cdot 6\pi q}{5} - \frac{\sin 6\cdot 6\pi q}{6} + ...\right)$$

$$+\frac{1}{4}\frac{1}{\pi}\left(\frac{\sin 8\pi q}{1} - \frac{\sin 2\cdot 8\pi q}{2} + \frac{\sin 3\cdot 8\pi q}{3} - \frac{\sin 4\cdot 8\pi q}{4} + \frac{\sin 5\cdot 8\pi q}{5} - \frac{\sin 6\cdot 8\pi q}{6} + ...\right)$$

$$+\frac{1}{5}\frac{1}{\pi}\left(\frac{\sin 10\pi q}{1} - \frac{\sin 2\cdot 10\pi q}{2} + \frac{\sin 3\cdot 10\pi q}{3} - \frac{\sin 4\cdot 10\pi q}{4} + \frac{\sin 5\cdot 10\pi q}{5} - \frac{\sin 6\cdot 10\pi q}{6} + ...\right)$$

$$+\frac{1}{6}\frac{1}{\pi}\left(\frac{\sin 12\pi q}{1} - \frac{\sin 2\cdot 12\pi q}{2} + \frac{\sin 3\cdot 12\pi q}{3} - \frac{\sin 4\cdot 12\pi q}{4} + \frac{\sin 5\cdot 12\pi q}{5} - \frac{\sin 6\cdot 12\pi q}{6} + ...\right)$$

$$+\frac{1}{7}\frac{1}{\pi}\left(\frac{\sin 14\pi q}{1} - \frac{\sin 2\cdot 14\pi q}{2} + \frac{\sin 3\cdot 14\pi q}{3} - \frac{\sin 4\cdot 14\pi q}{4} + \frac{\sin 5\cdot 14\pi q}{5} - \frac{\sin 6\cdot 14\pi q}{6} + ...\right)$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \frac{1}{\pi}\sin 2\pi q$$

$$+\frac{2}{3\pi}\sin 6\pi q$$

$$-\frac{1}{4\pi}\sin 8\pi q$$

$$+\frac{2}{5\pi}\sin 10\pi q$$

$$+\frac{2}{7\pi}\sin 14\pi q$$

$$- \ldots\ldots\ldots\ldots\ldots$$

Riemann gave the formula

9.6 $f(q)$ ***has the Fourier Series***

$$\left\{\sum_{\theta=divisor\ \text{of}\ 1} -(-1)^{\theta}\right\}\frac{\sin(1\cdot 2\pi q)}{\pi}+\left\{\sum_{\theta=divisor\ \text{of}\ 2} -(-1)^{\theta}\right\}\frac{\sin(2\cdot 2\pi q)}{2\pi}+...$$

A detailed discussion of Riemann's (x) Series appears in [Dan1].

10.

Other Non Integrable Fourier Series

Riemann supplies other examples of Fourier series that converge at infinitely many points, but are nowhere integrable:

10.1 ***If*** *as* $n \to \infty$, $c_0 > c_1 > ... > c_n \downarrow 0$

$$c_0 + c_1 + ... + c_n \to \infty$$

$\dfrac{x}{2\pi}$ *is a rational* $\dfrac{p}{q}$ *in its lowest terms.*

Then,

$$c_0 + c_1 \cos x + c_2 \cos 2^2 x + c_3 \cos 3^2 x + ...$$

and

$$c_1 \sin x + c_2 \sin 2^2 x + c_3 \sin 3^2 x + ...$$

converge

$\Leftrightarrow$

$$c_0 + c_1 \cos x + c_2 \cos 2^2 x + ... + c_{q-1} \cos(q-1)^2 x = 0$$

and

$$c_1 \sin x + c_2 \sin 2^2 x + ... + c_{q-1} \sin(q-1)^2 x = 0$$

10.2 $$\frac{d^2}{dx^2}\operatorname{Im}\left\{\frac{(1-e^{ix})}{1^3}\log\frac{-\log(1-e^{ix})}{e^{ix}}+\frac{(1-e^{2ix})}{2^3}\log\frac{-\log(1-e^{2ix})}{e^{2ix}}+...\right\}$$

is a Trigonometric Series that converges infinitely often on any interval.

Its term by term integral

$$\frac{d}{dx}\operatorname{Im}\left\{\frac{(1-e^{ix})}{1^3}\log\frac{-\log(1-e^{ix})}{e^{ix}}+\frac{(1-e^{2ix})}{2^3}\log\frac{-\log(1-e^{2ix})}{e^{2ix}}+...\right\}$$

diverges infinitely often on any interval.

Its second term by term integral

$$\operatorname{Im}\left\{\frac{(1-e^{ix})}{1^3}\log\frac{-\log(1-e^{ix})}{e^{ix}}+\frac{(1-e^{2ix})}{2^3}\log\frac{-\log(1-e^{2ix})}{e^{2ix}}+...\right\}$$

is a Trigonometric Series

11.

Fourier Series with $A_n \not\to 0$

Riemann supplies an example of Fourier series that converges at infinitely many points, although its coefficients do not vanish as $n \to \infty$.

11.1 $$\sin(1!\pi x) + \sin(2!\pi x) + \sin(3!\pi x) + ...$$

converges at each rational x. *(Then it is a finite sum).*

and at infinitely many irrationals such as

$$\sin(1), \quad \cos(1), \quad \frac{2m}{e}, \quad (2m+1)e, \quad \frac{(2m+1)}{4}(e - \tfrac{1}{e}),$$ ***etc.***

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