Riemannian Integration

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Abstract: Riemann's Theory of Integration appeared in Riemann's paper "On the Representation of a Function by a Trigonometric Series".

There, Riemann

1. presents his Oscillation Conditions for integrability,
2. establishes the integrability of his example of a series of functions with infinitely many discontinuities in any interval of real numbers, and
3. obtains a sequence of necessary and sufficient conditions for the integration of a singular function.

These fundamental results constitute the Riemannian Integration Theory that we present here

In a forthcoming article, we show that Infinitesimal Calculus allows integration of functions that are not Riemann Integrable.

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Introduction.

In [Dan1] we have shown that Lebesgue's requirement of absolute integrability takes a toll on the range of validity of his Integration Theory.

It is well known that Riemann's Integral may exist when Lebesgue's does not. For instance, $\frac{1}{t}\sin\frac{1}{t}$ is Riemann integrable, but not Lebesgue integrable, over $(0,1]$.

It is well known that The Fundamental Theorem of Calculus in Lebesgue's Theory is far more restrictive than in Riemann's Theory. For instance, $F(x) = x^2 \sin\frac{\pi}{x^2}$ satisfies the Fundamental Theorem of the Integral Calculus on $0 \le x \le 1$ in Riemann's Theory, but violates it in Lebesgue Integration.

It is well known that the limit function of a sequence of integrable functions may be Riemann Integrable, but not Lebesgue Integrable. For instance, the limit function of

$$f_n(x) = \chi_{[0,1)}(x) - \frac{1}{2}\chi_{[2,3)}(x) + \frac{1}{3}\chi_{[3,2)}(x) + \ldots + \frac{(-1)^{n-1}}{n}\chi_{[n-1,n)}(x)$$

is Riemann integrable, but not Lebesgue integrable.

These facts, and other evidence lead to the conclusion that, in spite of statements to the opposite, Riemann's Integral is more general than Lebesgue's.

Lebesgue integration is based on the premise that any sequence in the domain of a bounded function is of measure zero. According to Lebesgue theory, any sequence has no length ,and the function values on it add nothing to the integral.

But we have shown [Dan2] that the real numbers in $[0,1]$ can be sequenced, and it is well known that the interval $[0,1]$ has non-zero length. Therefore, a Countable set need not have measure zero. This invalidates a fundamental notion of Lebesgue Theory.

Furthermore, we have shown [Dan3] that the Rational Numbers in $[0,1]$ are a Non-Measurable set.

Thus, the Dirichlet Function [Dan1], is a Non-Measurable limit of measurable functions, and has no Lebesgue Integral, while Riemann's Function [Dan1] is Riemann integrable over its Non-Measurable set of discontinuities.

These facts disprove the main results of Lebesgue Theory: We have established in [Dan1] that

1. L^1 has Cauchy sequences of Lebesgue Measurable Integrable functions with Lebesgue Non-Measurable, Non-Integrable Limits. Hence L^1 is an Incomplete, Normed Linear space, and its Completion to a Banach Space has Non-Measurable, Non-Integrable functions

2. Lebesgue Dominant Convergence is invalid. Only Arzela Bounded Convergence for Riemann's Integral holds.
3. Lebesgue Monotone Bounded Convergence is invalid.
4. Fatou's Lemma is invalid
5. Beppo-Levi term by term integration of a series of Lebesgue integrable functions- is invalid
6. Fubini and Tonelli Theorems for iterated Lebesgue integration are invalid.

Consequently, to date, Riemannian Integration is the only Integration Theory that we have.

Riemann's Theory of Integration appears in Riemann's paper on the representation of a function by a Trigonometric Series.

In four results-packed pages, Riemann presents his Oscillation Conditions for integrability, establishes the integrability of his example of a series of functions with infinitely many discontinuities in any interval of real numbers, and obtains a sequence of necessary and sufficient conditions for the integration of a singular function.

These fundamental results constitute the Riemannian Integration Theory that we present here.

In [Dan6], we show that Infinitesimal Calculus allows integration of functions that may not be Riemann Integrable.

1.

Riemann's Sums, Integral, and Oscillation Sums

1.1 Riemann Sums

Riemann wrote

Take an increasing sequence between a, *and* b

$$a < x_1 < x_2 <,...,< x_{n-1} < b,$$

and denote

$$\delta(1) = x_1 - a,$$

$$\delta(2) = x_2 - x_1,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\delta(n) = b - x_{n-1},$$

$$\delta_n = \max\left\{\delta(1),...\delta(n)\right\}.$$

Let

$$0 < \rho_1 < 1; \ 0 < \rho_2 < 1; \ldots\ldots\ldots\ldots\ldots\ldots, 0 < \rho_n < 1,$$

and denote

$$\boxed{S(\delta_n) = \delta(1)f(a + \rho_1\delta(1)) + \delta(2)f(x_1 + \rho_2\delta(2)).... + \delta(n)f(x_{n-1} + \rho_n\delta(n))}.$$

1.2 Riemann Integral

Riemann wrote

What is the meaning of $\int_{x=a}^{x=b} f(x)dx$ *?*

If for any choice of $x_1, x_2, ..., x_{n-1}$, *and for any choice of* $\rho_1, ..., \rho_n$, *so that* $\delta_n \downarrow 0$, *we have* $S(\delta_n) \to A$,

then A *is called the integral of* $f(x)$ *from* a *to* b.

1.3 The Oscillation of a bounded $f(x)$ over $a < x < b$ is

$$\omega(f)\Big|_a^b = \sup_{a \le t \le b} f(t) - \inf_{a \le s \le b} f(s)$$

1.4 Riemann's 1st Oscillation Sum

Riemann wrote

Suppose that $f(x)$ *is bounded, and integrable from* a *to* b,

and denote

$$\omega\Big|_a^{x_1} = \sup_{a \le t \le x_1} f(t) - \inf_{a \le s \le x_1} f(s),$$

$$\omega\Big|_{x_1}^{x_2} = \sup_{x_1 < t < x_2} f(t) - \inf_{x_1 < s < x_2} f(s),$$

....,

$$\omega\Big|_{x_{n-1}}^{b} = \sup_{x_{n-1} < t < b} f(t) - \inf_{x_{n-1} < s < b} f(s),$$

$$\omega_n = \max\left\{\omega\Big|_a^{x_1}, \dots \omega\Big|_{x_{n-1}}^{b}\right\},$$

$$\delta_n = \max\left\{\delta(1), \dots \delta(n)\right\},$$

and

$$\boxed{\Delta(\delta_n) = \sup\left\{\delta(1)\,\omega\Big|_a^{x_1} + \delta(2)\,\omega\Big|_{x_1}^{x_2} + \dots + \delta(n)\,\omega\Big|_{x_{n-1}}^{b}\right\}},$$

were the sup is taken over all the partitions of $[a,b]$.

1.4 Riemann's 1st Oscillation Condition for Integrability

For any sequence of partitions $x_1, x_2, \dots, x_{n-1}$ *with* $\delta_n \downarrow 0$,

$$\Delta(\delta_n) \downarrow 0$$

1.5 Riemann's 2nd Oscillation Sum

Fix $\varepsilon > 0$,

$$\boxed{s_n(\varepsilon) = \text{ the total length of the intervals in which } \omega_n \geq \varepsilon}$$

1.6 Riemann's 2nd Oscillation Condition for Integrability

For any sequence of partitions $x_1, x_2, \dots, x_{n-1}$ *with* $\delta_n \downarrow 0$,

$$s_n(\varepsilon) \downarrow 0$$

2.

Riemann's Integrability

2.1 Riemann's Integrability Conditions

For a bounded $f(x)$, the following are equivalent

I $f(x)$ is integrable from a to b

II For any sequence of partitions $x_1, x_2, ..., x_{n-1}$ with $\delta_n \downarrow 0$,

and for any $\rho_1, ..., \rho_n$, **Riemann Sums** converge to a limit

III For any sequence of partitions $x_1, x_2, ..., x_{n-1}$ with $\delta_n \downarrow 0$,

Riemann 1st Oscillation Sums $\Delta(\delta_n)$ decrease to 0.

IV For any $\varepsilon > 0$,

and for any sequence of partitions $x_1, x_2, ..., x_{n-1}$ with $\delta_n \downarrow 0$,

Riemann 2nd Oscillation Sums $s_n(\varepsilon)$ decrease to 0.

V For any $\varepsilon > 0$,

and for any sequence of partitions $x_1, x_2, ..., x_{n-1}$ with

infinitesimal $\delta = \langle \delta_n \rangle$, $s(\varepsilon) = \langle s_n(\varepsilon) \rangle$ is infinitesimal.

Lebesgue's Integration is based on misinterpretation of statement V, as well as miscomprehension of infinities.

In particular,

Infinitesimals and sets of measure zero are unrelated.

Infinitesimals are convergent to zero sequences, while sequences of measure zero must be divergent.

Proof:

$(II \Rightarrow III)$ Note that

$$\delta(1)\,\omega\Big|_{a}^{x_1} + \delta(2)\,\omega\Big|_{x_1}^{x_2} + ... + \delta(n)\,\omega\Big|_{x_{n-1}}^{b} =$$

$$= \left(\delta(1)\sup_{a\le t\le x_1} f(t) + ... + \delta(n)\sup_{x_{n-1}<t<b} f(t)\right) - \left(\delta(1)\inf_{a\le s\le x_1} f(s) + ... + \delta(n)\inf_{x_{n-1}<s<b} f(s)\right),$$

and take any sequence of partitions $x_1, x_2, ..., x_{n-1}$ with $\delta_n \downarrow 0$.

If $f(x)$ is the integrable, the right hand side converges to the

Upper Integral – Lower Integral,

which equals to zero.

Therefore, the Oscillation sums on the left hand side converge to zero.$\square$

$(II \Leftarrow III)$, if the oscillation sums converge to zero, the right hand side converges to zero.

Since the oscillations sums terms are non-negative, their convergence is absolute, and independent of the order. Therefore, the right hand side can be rearranged so that it converges to

Upper Integral – Lower Integral.

Since this is zero, the Upper, and Lower integrals are equal, and $f(x)$ is the integrable.□

$(II + III \Rightarrow IV)$

Riemann wrote

Fix $\varepsilon > 0$, *and let*

$s_n(\varepsilon) =$ *the total length of the intervals in which* $\omega_n \geq \varepsilon$.

Then,

$$\varepsilon s_n(\varepsilon) \leq \delta(1)\omega\Big|_a^{x_1} + \delta(2)\omega\Big|_{x_1}^{x_2} + ... + \delta(n)\omega\Big|_{x_{n-1}}^{b} \leq \Delta(\delta_n).$$

Hence,

$$s_n(\varepsilon) \leq \frac{\Delta(\delta_n)}{\varepsilon}.$$

Thus,

$$\delta_n \downarrow 0 \Rightarrow s_n(\varepsilon) \downarrow 0.\square$$

$(IV \Leftrightarrow V)$

By [Dan4], the monotonic decreasing to zero sequence $\langle \delta_n \rangle$ represents an infinitesimal δ, and the monotonic decreasing to zero sequence $\langle s_n(\varepsilon) \rangle$ represents an infinitesimal $s(\varepsilon)$.

Riemann wrote,

Therefore, for a fixed $\varepsilon > 0$, *we can choose infinitesimally small* δ, *so that* $\Delta(\delta)$, *and* $s(\varepsilon)$, *are infinitesimally small.*

Consequently,

If $f(x)$ *is finite and integrable from* a *to* b, *then for any* $\varepsilon > 0$*, there is a suitable infinitesimal* δ *that makes* $s(\varepsilon)$ *infinitesimal.*□

$(IV \Rightarrow I)$

Riemann wrote,

If $f(x)$ *is finite,*

and if for $\varepsilon > 0$, $\delta_n \downarrow 0$ *implies* $s_n(\varepsilon) \to 0$,

then for any $x_1, x_2, ..., x_{n-1}$ *with* $\delta_n \downarrow 0$*, and for any* $\rho_1, ..., \rho_n$,

S_n *has a limit.*

Indeed, since $f(x)$ *is finite,* $\omega(f)\Big|_a^b \le M$,

and the contribution to $\delta_1\,\omega\Big|_a^{x_1} + \delta_2\,\omega\Big|_{x_1}^{x_2} + ... + \delta_n\,\omega\Big|_{x_{n-1}}^{b}$

from the length-$s_n(\varepsilon)$ *subintervals is less than*

$$s_n(\varepsilon)\underbrace{\omega(f)\Big|_a^b}_{\le M} \le s_n(\varepsilon)M.$$

The rest of the intervals contribute to

$\delta_1\,\omega\Big|_a^{x_1} + \delta_2\,\omega\Big|_{x_1}^{x_2} + ... + \delta_n\,\omega\Big|_{x_{n-1}}^{b}$ *less than* $\varepsilon(b-a)$.

Taking any $x_1, x_2, ..., x_{n-1}$ *with arbitrarily small* δ*, and taking arbitrarily small* $\varepsilon > 0$*, we obtain arbitrarily small* $s_n(\varepsilon)$*, and arbitrarily small* $\delta_1\,\omega\Big|_a^{x_1} + \delta_2\,\omega\Big|_{x_1}^{x_2} + ... + \delta_n\,\omega\Big|_{x_{n-1}}^{b}$,

so that for any $\rho_1, ..., \rho_n$, S_n *has a limit.*□

2.3 Riemann's Non-Integrability Conditions

For a bounded $f(x)$, the following are equivalent

I $f(x)$ is non-integrable from a to b

II There is a sequence of partitions $x_1, x_2, ..., x_{n-1}$ with $\delta_n \downarrow 0$, so that **Riemann Sums** do not converge to a limit.

III There is a sequence of partitions $x_1, x_2, ..., x_{n-1}$ with $\delta_n \downarrow 0$, so that **Riemann Oscillation Sums** do not converge to 0.

IV There is $\varepsilon_0 > 0$, so that for any sequence of partitions $x_1, x_2, ..., x_{n-1}$ with $\delta_n \downarrow 0$, the sequence $s_n(\varepsilon_0)$, is not decreasing to zero.

V There is $\varepsilon_0 > 0$, so that for any sequence of partitions $x_1, x_2, ..., x_{n-1}$ with infinitesimal $\delta = \langle \delta_n \rangle$, $s(\varepsilon_0) = \langle s_n(\varepsilon_0) \rangle$ is not infinitesimal.

3.

Infinitely many Discontinuities in any interval

Riemann wrote

We consider functions that have infinitely many discontinuities between any two numbers, however close.

Since such functions have never been considered before, we start with an example

Denote

$$(x) = \begin{cases} \text{ the excess of } x \text{ over the closest integer} \\ 0, \text{ if } x = \text{midpoint between two integers} \end{cases}.$$

Define the series

$$f(x) = \frac{(x)}{1^2} + \frac{(2x)}{2^2} + \frac{(3x)}{3^2} + ...$$

This series converges for each x.

Denote

$$n = \text{ integer},$$

$$p = \text{ odd integer},$$

where p *and* n *are relatively prime.*

$f(x)$ *is continuous at* $\frac{p}{2n} + 0$, *and at* $\frac{p}{2n} - 0$.

$$If\ \ x \downarrow \frac{p}{2n} + 0, \quad then \quad f(x) \to f(\tfrac{p}{2n} + 0),$$

and

$$if\ \ x \uparrow \frac{p}{2n} - 0, \quad then \quad f(x) \to f(\tfrac{p}{2n} - 0).$$

We have

$$f(\tfrac{p}{2n} + 0) = f(\tfrac{p}{2n}) - \frac{1}{2n^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + ...\right) = f(\tfrac{p}{2n}) - \frac{\pi^2}{16n^2},$$

$$f(\tfrac{p}{2n} - 0) = f(\tfrac{p}{2n}) + \frac{1}{2n^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + ...\right) = f(\tfrac{p}{2n}) + \frac{\pi^2}{16n^2}.$$

For all $x \neq \dfrac{p}{2n}$,

$$f(x+0) = f(x), \ \ and \ \ f(x-0) = f(x).$$

At any $x = \dfrac{p}{2n}$, $f(x)$ *is discontinuous.*

Thus, $f(x)$ *has infinitely many discontinuities between any two interval endpoints, however close.*

But $f(x)$ *has only finitely many jumps larger than* σ.

$f(x)$ *is integrable over any interval because* $f(x)$ *is finite, and*

1. $f(x+0)$, *and* $f(x-0)$ *exist for each* x,
2. $f(x)$ *has only finitely many jumps* $\geq \sigma$.

1), and 2) imply that δ *can be taken so small, that* $s(\sigma)$ *is arbitrarily small, and in the rest of the subintervals,* $\omega < \sigma$.

Functions that -unlike the above $f(x)$*- have only finitely many maxima and minima, always satisfy 1), and 2), and are integrable wherever they are finite.*

4.
Riemann's (x) Function

Riemann defined

4.1 The Riemann (x) Function

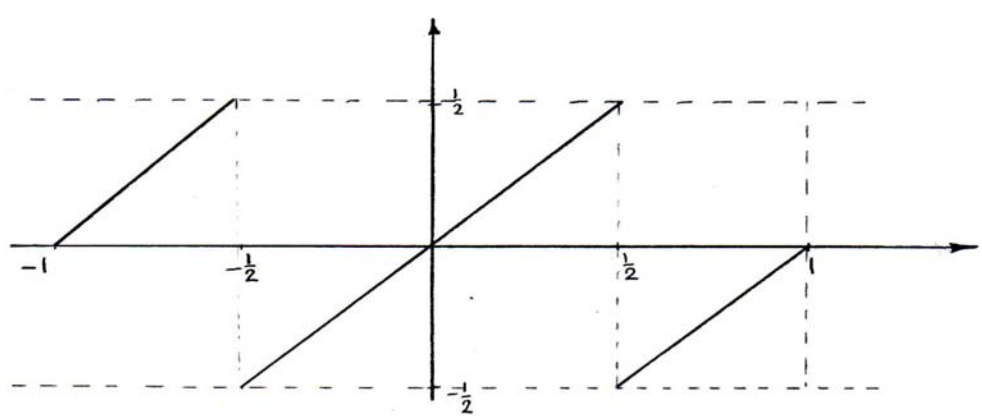
$$(x) = \begin{cases} x - nearest\ \text{int}eger,\ if\ \ x \neq n + \frac{1}{2} \\ 0,\ \text{if}\ x = n + \frac{1}{2} \end{cases}$$

4.2 (x) **is periodic with period** 1**,**

is continuous in $..., -\frac{3}{2} < x < -\frac{1}{2},\ -\frac{1}{2} < x < \frac{1}{2},$

$\frac{1}{2} < x < \frac{3}{2},..$

is discontinuous at $.....-\dfrac{3}{2},\ -\dfrac{1}{2},\ \dfrac{1}{2},\ \dfrac{3}{2},.....$



and has the Fourier series

$$(x) = \frac{1}{\pi}\left(\frac{\sin 2\pi x}{1} - \frac{\sin 2\cdot 2\pi x}{2} + \frac{\sin 3\cdot 2\pi x}{3} - ...\right)$$

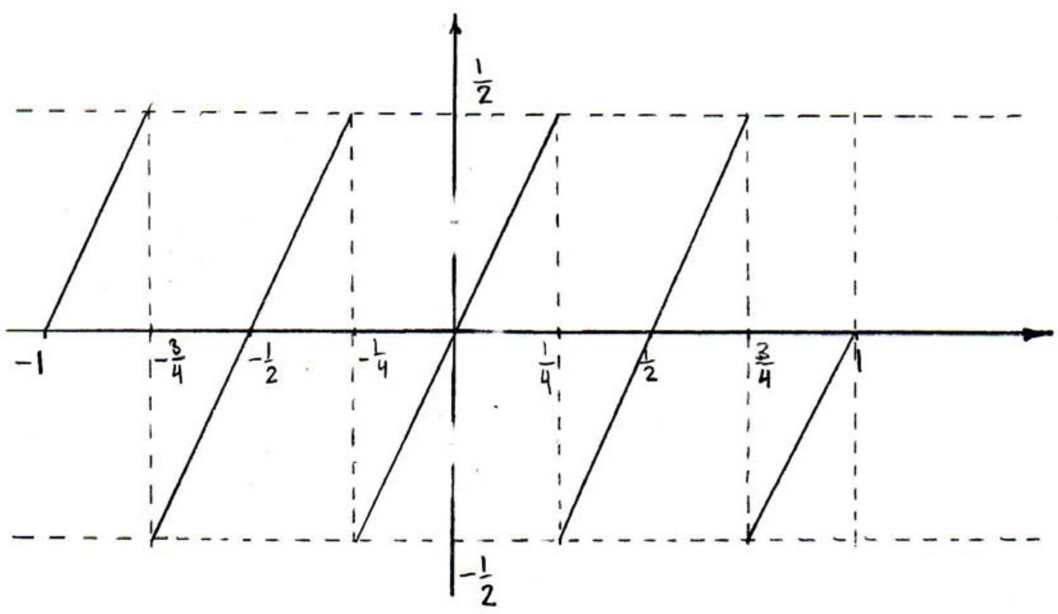
4.3 $(2x)$ **is periodic with period** $\frac{1}{2}$**,**

is continuous in $..,-\frac{3}{4} < x < -\frac{1}{4},\ -\frac{1}{4} < x < \frac{1}{4},$

$\frac{1}{4} < x < \frac{3}{4},....$

is discontinuous at $.....-\frac{3}{4},\ -\frac{1}{4},\ \frac{1}{4},\ \frac{3}{4},.....$

has the graph



and has the Fourier series

$$(2x) = \frac{1}{\pi}\left(\frac{\sin 4\pi x}{1} - \frac{\sin 2\cdot 4\pi x}{2} + \frac{\sin 3\cdot 4\pi x}{3} - ...\right)$$

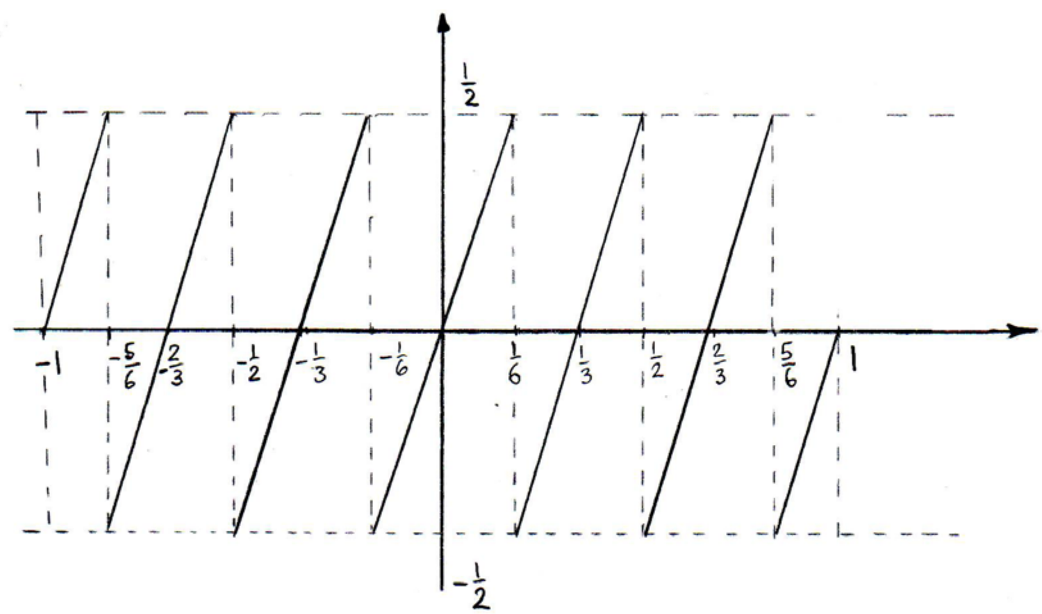
4.4 $(3x)$ **is periodic with period** $\frac{1}{3}$**,**

is continuous in $..., -\frac{3}{6} < x < -\frac{1}{6}, \quad -\frac{1}{6} < x < \frac{1}{6},$

$\frac{1}{6} < x < \frac{3}{6}, ...$

is discontinuous at $.....-\frac{3}{6}, -\frac{1}{6}, \frac{1}{6}, \frac{3}{6}, ...$

has the graph



and has the Fourier series

$$(3x) = \frac{1}{\pi}\left(\frac{\sin 6\pi x}{1} - \frac{\sin 2 \cdot 6\pi x}{2} + \frac{\sin 3 \cdot 6\pi x}{3} - ...\right)$$

....

5.

Riemann's (x) Series

5.1 Riemann's (x) Series is

$$f(x) = \frac{(x)}{1^2} + \frac{(2x)}{2^2} + \frac{(3x)}{3^2} + \ldots$$

5.2 $$\left| f(x) \right| \le \frac{\pi^2}{8}$$

Proof: $\dfrac{\left|(x)\right|}{1^2} + \dfrac{\left|(2x)\right|}{2^2} + \dfrac{\left|(3x)\right|}{3^2} + \ldots \le \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots = \dfrac{\pi^2}{8}$

Therefore,

5.3 Riemann's (x) Series converges absolutely, and uniformly in $-\infty < x < \infty$

5.4 Riemann's (x) Series has the Fourier Series

$$f(x) = \frac{1}{\pi}\left(\frac{\sin 2\pi x}{1} - \frac{\sin 2 \cdot 2\pi x}{2} + \frac{\sin 3 \cdot 2\pi x}{3} - \frac{\sin 4 \cdot 2\pi x}{4} + \frac{\sin 5 \cdot 2\pi x}{5} - \ldots\right)$$

$$+ \frac{1}{2^2}\frac{1}{\pi}\left(\frac{\sin 4\pi x}{1} - \frac{\sin 2 \cdot 4\pi x}{2} + \frac{\sin 3 \cdot 4\pi x}{3} - \frac{\sin 4 \cdot 4\pi x}{4} + \frac{\sin 5 \cdot 4\pi x}{5} \ldots\right)$$

$$+\frac{1}{3^2}\frac{1}{\pi}\left(\frac{\sin 6\pi x}{1}-\frac{\sin 2\cdot 6\pi x}{2}+\frac{\sin 3\cdot 6\pi x}{3}-\frac{\sin 4\cdot 6\pi x}{4}+\frac{\sin 5\cdot 6\pi x}{5}...\right)$$

$$+\frac{1}{4^2}\frac{1}{\pi}\left(\frac{\sin \pi 8x}{1}-\frac{\sin 2\cdot 8\pi x}{2}+\frac{\sin 3\cdot 8\pi x}{3}-\frac{\sin 4\cdot 8\pi x}{4}+\frac{\sin 5\cdot 8\pi x}{5}-...\right)$$

$$+\frac{1}{5^2}\frac{1}{\pi}\left(\frac{\sin 10\pi x}{1}-\frac{\sin 2\cdot 10\pi x}{2}+\frac{\sin 3\cdot 10\pi x}{3}-\frac{\sin 4\cdot 10\pi x}{4}+\frac{\sin 5\cdot 10\pi x}{5}-...\right)$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$=\frac{1}{\pi}\sin 2\pi x$$

$$-\frac{1}{\pi}\left(\frac{1}{2}-\frac{1}{2^2}\right)\sin 4\pi x$$

$$+\frac{1}{\pi}\left(\frac{1}{3}+\frac{1}{3^2}\right)\sin 6\pi x$$

$$-\frac{1}{\pi}\left(\frac{1}{2^2}+\frac{1}{2^3}-\frac{1}{2^4}\right)\sin 8\pi x$$

$$+\frac{1}{\pi}\left(\frac{1}{5}+\frac{1}{5^2}\right)\sin 10\pi x$$

$$-\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

6.

Riemann's (x) Series Discontinuities

6.1 Riemann's series is discontinuous at

$$...,-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2},\ldots$$

$$...,-\frac{7}{4},-\frac{5}{4},-\frac{3}{4},-\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{5}{4},\frac{7}{4}\ldots$$

$$...,-\frac{9}{6},-\frac{7}{6},-\frac{5}{6},-\frac{3}{6},-\frac{1}{6},\frac{1}{6},\frac{3}{6},\frac{5}{6},\frac{7}{6},\frac{9}{6},\ldots$$

$$\ldots,-\frac{13}{8},-\frac{11}{8},-\frac{9}{8},-\frac{7}{8},-\frac{5}{8},-\frac{3}{8},-\frac{1}{8},-\frac{5}{8},-\frac{3}{8},-\frac{1}{8},\frac{1}{8},\frac{3}{8},\frac{5}{8},\frac{7}{8},\frac{9}{8},\frac{11}{8},\frac{13}{8},\ldots$$

$$\ldots-\frac{15}{10},-\frac{13}{10},-\frac{11}{10},-\frac{9}{10},-\frac{7}{10},-\frac{5}{10},-\frac{3}{10},-\frac{1}{10},\frac{1}{10},\frac{3}{10},\frac{5}{10},\frac{7}{10},\frac{9}{10},\frac{11}{10},\frac{13}{10},\frac{15}{10},\ldots$$

....

6.2 $$f(\tfrac{1}{2}-0)=\frac{\pi^2}{16}$$

Proof: Since

$$\frac{1}{2}=\frac{3}{2\times 3}=\frac{5}{2\times 5}=\frac{7}{2\times 7}=\ldots,$$

$f(\frac{1}{2}-0)$ is generated by $(x),(3x),(5x),(7x),\ldots$ at $x=\frac{1}{2}-0$, and it is the sum

$$\underbrace{(x)\Big|_{x=\frac{1}{2}-0}}_{=\frac{1}{2}} + \frac{1}{3^2}\underbrace{(3x)\Big|_{x=\frac{1}{2}-0}}_{=\frac{1}{2}} + \frac{1}{5^2}\underbrace{(5x)\Big|_{x=\frac{1}{2}-0}}_{=\frac{1}{2}} + \frac{1}{7^2}\underbrace{(7x)\Big|_{x=\frac{1}{2}-0}}_{=\frac{1}{2}} + \ldots.$$

$$= \frac{\pi^2}{8} \times \frac{1}{2}. \square$$

Similarly,

6.3 $$f(\tfrac{1}{2}+0) = -\frac{\pi^2}{16}$$

Therefore, The jump discontinuity at $x = \dfrac{1}{2}$ is

6.4 $$\Big| f(\tfrac{1}{2}+0) - f(\tfrac{1}{2}-0) \Big| = \frac{\pi^2}{8}$$

Similarly,

6.5 $$f(\tfrac{2k+1}{2n}-0) = \frac{\pi^2}{16n^2}$$

6.6 $$f(\tfrac{2k+1}{2n}+0) = -\frac{\pi^2}{16n^2}$$

and the jump discontinuity at $x = \dfrac{2k+1}{2n}$ is

6.7 $$\Big| f(\tfrac{2k+1}{2n}+0) - f(\tfrac{2k+1}{2n}-0) \Big| = \frac{\pi^2}{8n^2}$$

7.

Riemann (x) Series Integrability

By **5.3**

7.1 Riemann (x) Series is bounded, and continuous off its discontinuity points

7.2 Riemann's Series is Riemann-Integrable.

Proof:

Given arbitrarily small $\varepsilon > 0$, there are only finitely many natural numbers n's so that $n^2 \leq \frac{1}{\varepsilon}$.

In $-\frac{1}{2} < x < \frac{1}{2}$, there are only finitely many, say N, numbers $\frac{2k+1}{2n}$ so that by **6.7**

$$\left| f(\tfrac{2k+1}{2n} - 0) - f(\tfrac{2k+1}{2n} + 0) \right| = \frac{\pi^2}{8n^2}.$$

Therefore,

$$\frac{\pi^2}{8}\varepsilon \leq \left| f(\tfrac{2k+1}{2n} - 0) - f(\tfrac{2k+1}{2n} + 0) \right| \leq \frac{\pi^2}{8}.$$

These points occupy at most N subintervals, $I_1, \dots I_N$, each of length $\delta < \frac{\varepsilon}{N}$.

The contribution to the Riemann Oscillation Sums from these subintervals is

$$\left|\max_{t \in I_1} f(t) - \min_{s \in I_1} f(s)\right|\delta + ... + \left|\max_{t \in I_N} f(t) - \min_{s \in I_N} f(s)\right|\delta$$

$$\leq \underbrace{\left|f(\tfrac{2k_1+1}{2n_1}+0) - f(\tfrac{2k_1+1}{2n_1}-0)\right|}_{\leq \frac{\pi^2}{8}}\delta + ... + \underbrace{\left|f(\tfrac{2k_N+1}{2n_N}+0) - f(\tfrac{2k_N+1}{2n_N}-0)\right|}_{\leq \frac{\pi^2}{8}}\delta$$

$$\leq \frac{\pi^2}{8}N\delta < \frac{\pi^2}{8}\varepsilon$$

These N subintervals, are separated by l subintervals each of length $\delta < \frac{\varepsilon}{N}$.

Over these subintervals, the oscillations are

$$\omega^{(1)} < \frac{\pi^2}{8}\varepsilon,$$

$$\dots\dots\dots\dots\dots\dots,$$

$$\omega^{(l)} < \frac{\pi^2}{8}\varepsilon,$$

and the contribution to the Riemann Oscillation sum is

$$\underbrace{\omega^{(1)}}_{<\frac{\pi^2}{8}\varepsilon}\delta + ... + \underbrace{\omega^{(l)}}_{<\frac{\pi^2}{8}\varepsilon}\delta < \frac{\pi^2}{8}\varepsilon\underbrace{l\delta}_{<1} < \frac{\pi^2}{8}\varepsilon.$$

Therefore, the Riemann Oscillation Sums are bounded by $2\frac{\pi^2}{8}\varepsilon$.

If $\delta \downarrow 0$ then, $\varepsilon \downarrow 0$, and the Riemann Oscillation Sums vanish.

Therefore, by the Riemann Oscillation Sums Condition for Integrability, the Riemann Series $f(x)$ is Riemann Integrable.$\square$

8.

Necessary Conditions for Integrability of a Singular $f(x)$

Riemann wrote

8.1 *If $f(x)$ is integrable.*

$f(x) \uparrow \infty$, *as* $x \downarrow 0$,

$f(x)x$ *has only finitely many maxima, and minima in any* $-\varepsilon < x < \varepsilon$

then $f(x)x \to 0$, *as* $x \downarrow 0$.

Because if

$$f(x)x \geq c > 0,$$

then

$$\int_{t=x}^{t=a} f(t)dt \geq \int_{t=x}^{t=a} \frac{c}{t}dt = c\left(\log\tfrac{1}{x} - \log\tfrac{1}{a}\right) \uparrow \infty, \text{ as } x \downarrow 0. \square$$

Similarly to **8.1**,

8.2 If $f(x)$ is integrable.

$f(x) \uparrow \infty$, as $x \downarrow 0$,

$f(x)x\log\frac{1}{x}$ has finitely many maxima, and minima in any $-\varepsilon < x < \varepsilon$

then $f(x)x\log\frac{1}{x}\to 0$, as $x\downarrow 0$.

Proof: If

$$f(x)x\log\tfrac{1}{x}\geq c>0,$$

we have

$$\int_{t=x}^{t=a} f(t)dt \geq \int_{t=x}^{t=a}\frac{c}{t\log\frac{1}{t}}dt,$$

$$= -c\log\log\tfrac{1}{t}\Big|_{t=x}^{t=a}$$

$$= c\Big(\log\log\tfrac{1}{x}-\log\log\tfrac{1}{a}\Big)\uparrow\infty,\quad \text{as } x\downarrow 0.\square$$

Similarly,

8.3 *If $f(x)$ is integrable.*

$f(x)\uparrow\infty$, *as* $x\downarrow 0$,

$f(x)x\Big(\log\frac{1}{x}\Big)\Big(\log\log\frac{1}{x}\Big)$ *has only finitely many maxima,*

and minima in any $-\varepsilon<x<\varepsilon$

then $f(x)x\Big(\log\frac{1}{x}\Big)\Big(\log\log\frac{1}{x}\Big)\to 0$, *as* $x\downarrow 0$.

Proof:

If

$$f(x)x\Big(\log\tfrac{1}{x}\Big)\Big(\log\log\tfrac{1}{x}\Big)\geq c>0,$$

we have

$$\int_{t=x}^{t=a} f(t)dt \geq \int_{t=x}^{t=a}\frac{c}{t\Big(\log\frac{1}{t}\Big)\Big(\log\log\frac{1}{t}\Big)}dt,$$

$$= -c \log\log\log\tfrac{1}{t}\Big|_{t=x}^{t=a}$$

$$= c\Big(\log\log\log\tfrac{1}{x} - \log\log\log\tfrac{1}{a}\Big) \uparrow \infty\,,\ \text{as } x \downarrow 0\,.\square$$

Similarly,

8.4 *If* $f(x)$ *is integrable.*

$f(x) \uparrow \infty$, *as* $x \downarrow 0$,

$f(x)x\Big(\log\tfrac{1}{x}\Big)\Big(\log\log\tfrac{1}{x}\Big)\Big(\log\log\log\tfrac{1}{x}\Big)$ *has only finitely many maxima, and minima in any* $-\varepsilon < x < \varepsilon$

then $f(x)x\Big(\log\tfrac{1}{x}\Big)\Big(\log\log\tfrac{1}{x}\Big)\Big(\log\log\log\tfrac{1}{x}\Big) \to 0$, *as* $x \downarrow 0$.

Proof:

If

$$f(x)x\Big(\log\tfrac{1}{x}\Big)\Big(\log\log\tfrac{1}{x}\Big)\Big(\log\log\log\tfrac{1}{x}\Big) \geq c > 0\,,$$

we have

$$\int_{t=x}^{t=a} f(t)dt \geq \int_{t=x}^{t=a} \frac{c}{t\Big(\log\frac{1}{t}\Big)\Big(\log\log\frac{1}{t}\Big)\Big(\log\log\log\frac{1}{t}\Big)}dt\,,$$

$$= -c\log\log\log\log\tfrac{1}{t}\Big|_{t=x}^{t=a}$$

$$= c\Big(\log\log\log\log\tfrac{1}{x} - \log\log\log\log\tfrac{1}{a}\Big) \uparrow \infty\,, \text{as } x \downarrow 0\,.\square$$

In general,

8.5 *If* $f(x)$ *is integrable.*

$f(x) \uparrow \infty$, ***as*** $x \downarrow 0$,

$f(x)x\left(\log\frac{1}{x}\right)\left(\log\log\frac{1}{x}\right)...\left(\log...\log\frac{1}{x}\right)$ ***has only finitely many maxima, and minima in any*** $-\varepsilon < x < \varepsilon$

then $f(x)x\left(\log\frac{1}{x}\right)\left(\log\log\frac{1}{x}\right)...\left(\log...\log\frac{1}{x}\right) \to 0$, ***as*** $x \downarrow 0$.

9.

Sufficient Conditions for Integrability of Singular $f(x)$

Riemann wrote

9.1 *If* $f(x) \uparrow \infty$, *as* $x \downarrow 0$,

$\alpha < 1$

$f(x)x^{\alpha} \to 0$, *as* $x \to 0$

then $\int\limits_{t=x}^{t=a} f(t)dt$ *converges as* $x \to 0$

Because

$$f(x)x^{\alpha} = f(x)(1-\alpha)\frac{dx}{d(x^{1-\alpha})}.\square$$

We supply the proof

Proof: Rewriting Riemann's formula above, we have

$$f(t)dt = \frac{1}{1-\alpha} f(t)t^{\alpha} d(t^{1-\alpha})$$

Therefore,

$$\int\limits_{t=x}^{t=a} f(t)dt = \frac{1}{1-\alpha}\int\limits_{t=x}^{t=a} f(t)t^{\alpha} d(t^{1-\alpha}).$$

As $x \to 0$, we get $f(x)x^{\alpha} \to 0$. Therefore, $f(x)x^{\alpha}$ is bounded in $-\varepsilon \le x \le \varepsilon$. and the integral there converges.

Since it is implicitly assumed that $\int_{t=\varepsilon}^{t=a} f(t)dt$ converges, we conclude that $\int_{t=\varepsilon}^{t=a} f(t)dt$ converges for $\varepsilon \downarrow 0$.$\square$

Similarly,

9.2 If $f(x)$ is integrable on $\varepsilon \le x \le a$ for small ε

$f(x) \uparrow \infty$, as $x \downarrow 0$,

$\alpha > 1$

$f(x)x\left(\log\frac{1}{x}\right)^{\alpha} \to 0$, as $x \to 0$

then $\int_{t=x}^{t=a} f(t)dt$ converges as $x \to 0$

Proof: we have

$$f(t)dt = -\frac{1}{1-\alpha} f(t)t(\log\tfrac{1}{t})^{\alpha}\, d(\log\tfrac{1}{t})^{1-\alpha}$$

Therefore,

$$\int_{t=x}^{t=a} f(t)dt = -\frac{1}{1-\alpha}\int_{t=x}^{t=a} f(t)t(\log\tfrac{1}{t})^{\alpha}\, d(\log\tfrac{1}{t})^{1-\alpha}.$$

As $x \to 0$, we get $f(x)x\left(\log\frac{1}{x}\right)^{\alpha} \to 0$. Therefore, $f(x)x\left(\log\frac{1}{x}\right)^{\alpha}$ is bounded in $-\varepsilon \le x \le \varepsilon$. and the integral there converges.

Since $\int\limits_{t=\varepsilon}^{t=a} f(t)dt$ converges, we conclude that $\int\limits_{t=\varepsilon}^{t=a} f(t)dt$ converges for $\varepsilon \downarrow 0$.□

Similarly,

9.3 If $f(x)$ is integrable on $\varepsilon \le x \le a$ for small ε

$f(x) \uparrow \infty$, as $x \downarrow 0$,

$\alpha > 1$

$f(x)x\log\frac{1}{x}\left(\log\log\frac{1}{x}\right)^{\alpha} \to 0$, as $x \to 0$

then $\int\limits_{t=x}^{t=a} f(t)dt$ converges as $x \to 0$

Proof: we have

$$f(t)dt = -\frac{1}{1-\alpha} f(t)t\log\tfrac{1}{t}(\log\log\tfrac{1}{t})^{\alpha} d(\log\log\tfrac{1}{t})^{1-\alpha}$$

Therefore,

$$\int\limits_{t=x}^{t=a} f(t)dt = -\frac{1}{1-\alpha}\int\limits_{t=x}^{t=a} f(t)t\log\tfrac{1}{t}(\log\log\tfrac{1}{t})^{\alpha} d(\log\log\tfrac{1}{t})^{1-\alpha}.$$

As $x \to 0$, we get $f(x)x\log\frac{1}{x}\left(\log\log\frac{1}{x}\right)^{\alpha} \to 0$. Therefore, $f(x)x\log\frac{1}{x}\left(\log\log\frac{1}{x}\right)^{\alpha}$ is bounded in $-\varepsilon \le x \le \varepsilon$. and the integral there converges.

Since $\int_{t=\varepsilon}^{t=a} f(t)dt$ converges, then $\int_{t=\varepsilon}^{t=a} f(t)dt$ converges for $\varepsilon \downarrow 0$. $\square$

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