Lebesgue Integration

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Abstract:

We show that

1. the Riemann integral may exist when the Lebesgue integral does not
2. the Fundamental Theorem of Calculus may hold for Riemannian Integration but not for Lebesgue integration of the same function
3. The same limit function of Integrable functions may be Riemann-Integrable but not Lebesgue integrable.

Therefore, contrary to Common belief, Riemann Integration is more general than Lebesgue Integration.

Furthermore, we show that a convergent sequence of measurable functions may have a non-measurable limit function.

This cast a doubt on the validity of the main results of Lebesgue theory.

Consequently, Riemannian Integration is not only a superior Integration Theory. It is the only Integration Theory that we have.

Keywords: Lebesgue, Riemann, Integration, Measureable, Measure, Integrability, Completion, Banach spaces, L^p spaces, Dominant Convergence, Monotone Bounded Convergence, Fatou Lemma, Fubini, Tonelli, Beppo-Levi, Dirichlet function, Riemann Function,

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Contents

Introduction.

A continuous function is Cauchy integrable, if and only if the Sums that approximate the area under the function graph

$$f(t_1)(x_1 - x_0) + f(t_2)(x_2 - x_1) + ... + f(t_n)(x_n - x_{n-1}),$$

converge to a limit, as the size of the subintervals of the partition $a = x_0 < x_1 < ... < x_n = b$ approaches zero.

In the Cauchy Sums, $t_1, t_2, ..., t_n$ are the endpoints of the partition subintervals.

Riemann allowed a bounded function that may have infinitely many discontinuities, and let $t_1, t_2, ..., t_n$ be arbitrarily chosen points in each subinterval.

Riemann required that the oscillation sums

$$\left|\max_{x_0 \le t \le x_1} f(t) - \min_{x_0 \le s \le x_1} f(s)\right|(x_1 - x_0) + ... + \left|\max_{x_{n-1} \le t \le x_n} f(t) - \min_{x_{n-1} \le s \le x_n} f(s)\right|(x_n - x_{n-1})$$

will vanish as the size of the subintervals of the partition $a = x_0 < x_1 < ... < x_n = b$ approaches zero.

Lebesgue allowed a measurable function, and sequences of discontinuities. But restricted his integral by requiring the measurable function to be absolutely integrable.

We shall recall that similarly to series convergence, there are functions that are Riemann Integrable, but not absolutely integrable.

Lebesgue theory is based on the belief that any countable set has measure zero, so that the integral on a countable set is zero.

We recall that sequencing and measurability are unrelated. Some countable sets have non-zero measure, and some countable sets have no measure at all.

Consequently, Riemann presented two Functions that satisfy Riemann's Oscillation sums condition, and are Riemann Integrable, but not Lebesgue integrable.

Furthermore, the Dirichlet function, that is not Riemann Integrable, is not Lebesgue integrable either.

This suggests that Riemann's integral generalizes Lebesgue's.

While Riemann-integrability is preserved for the limit function under uniform convergence, Lebesgue theory claims that measurability is always preserved for the limit function.

We disprove Lebesgue's theory claim, with a convergent sequence of measurable functions, that does not converge to a measurable function.

That is, we show that the Dirichlet Function is a Non-Measurable limit of measurable functions, and has no Lebesgue Integral.

On the other hand, Riemann's Function is Riemann integrable over its Non-Measurable set of discontinuities.

These facts disprove the main results of Lebesgue Theory: In particular,

1. L^1 has Cauchy sequences of Lebesgue Measurable Integrable functions with Lebesgue Non-Measurable, Non-Integrable Limits. Hence L^1 is an Incomplete, Normed Linear space, and its Completion to a Banach Space has Non-Measurable, Non-Integrable functions
2. the Lebesgue function spaces L^p, $1 \le p \le \infty$, have Cauchy sequences of Riemann integrable functions with a limit function that is not measurable, and not Lebesgue integrable, and the L^p spaces are incomplete.
3. Lebesgue Dominant Convergence Theorem is invalid. Only the Bounded Convergence of Arzela for Riemann-Integrable functions holds.
4. Lebesgue Monotone Bounded Convergence is invalid.
5. Fatou's Lemma is invalid
6. Beppo-Levi term by term integration of a series of Lebesgue integrable functions- is invalid
7. Fubini and Tonelli Theorems for iterated Lebesgue integration are invalid.

Consequently, Riemannian Integration is not just a superior Theory. It is the only Integration Theory that we have.

1.

Riemann Integral may exist when Lebesgue's does not

Many series that converge conditionally, do not converge absolutely. The alternating series

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+...$$

converges to $\log 2$, and the alternating series

$$1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+...$$

converges to $\tfrac{1}{4}\pi$.

But the absolute values series diverge.

Similarly, the requirement of absolute integrability imposed on the Lebesgue integral, eliminates functions that are Riemann Integrable.

For instance $\dfrac{\sin x}{x}$, over the interval $0 \le x < \infty$, is Riemann Integrable, but not absolutely integrable.

Therefore, similarly to the series,

1.1 *Riemann's conditional integrability allows integrability that is not allowed by Lebesgue's absolute integrability.*

We establish this by either 1.2, or 1.3.

1.2 $\frac{\sin x}{x}$ *is Riemann integrable but not Lebesgue integrable over* $[0,\infty)$

Proof: It is well known that $\int\limits_{x=0}^{x=\infty} \frac{\sin x}{x} dx = \frac{\pi}{4}$.

Therefore, $\frac{\sin x}{x}$ is Riemann Integrable over $[0,\infty)$.□

But $\frac{\sin x}{x}$ is not absolutely integrable over$[0,\infty)$. Indeed,

$$\int\limits_{x=0}^{x=\infty} \left|\frac{\sin x}{x}\right| dx = \int\limits_{x=0}^{x=\pi} \frac{\sin x}{x} dx + \int\limits_{x=\pi}^{x=2\pi} \frac{|\sin x|}{x} dx + \int\limits_{x=2\pi}^{x=3\pi} \frac{\sin x}{x} dx + \int\limits_{x=3\pi}^{x=4\pi} \frac{\sin x}{x} dx + ...$$

$$\geq \frac{1}{\pi}\int\limits_{x=0}^{x=\pi} \sin x dx + \frac{1}{2\pi}\int\limits_{x=\pi}^{x=2\pi} |\sin x| dx + \frac{1}{3\pi}\int\limits_{x=2\pi}^{x=3\pi} \sin x dx + \frac{1}{4\pi}\int\limits_{x=3\pi}^{x=4\pi} |\sin x| dx + ...$$

$$= \frac{1}{\pi}\int\limits_{x=0}^{x=\pi} \sin x dx + \frac{1}{2\pi}\int\limits_{x=\pi}^{x=2\pi} -\sin x dx + \frac{1}{3\pi}\int\limits_{x=2\pi}^{x=3\pi} \sin x dx + \frac{1}{4\pi}\int\limits_{x=3\pi}^{x=4\pi} -\sin x dx + ...$$

$$= \frac{1}{\pi}\underbrace{\cos x\Big|_{\pi}^{0}}_{2} + \frac{1}{2\pi}\underbrace{\cos x\Big|_{\pi}^{2\pi}}_{2} + \frac{1}{3\pi}\underbrace{\cos x\Big|_{3\pi}^{2\pi}}_{2} + \frac{1}{4\pi}\underbrace{\cos x\Big|_{3\pi}^{4\pi}}_{2} + ...$$

$$= \frac{2}{\pi}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...\right) = \infty.\square$$

1.3 $\frac{1}{t}\sin\frac{1}{t}$ *is Riemann integrable, but not Lebesgue integrable, over* $(0,1]$.

Proof:

$$\int_{x=0}^{x=1} \frac{1}{t}\sin\frac{1}{t}dt = \lim_{n\to\infty}\int_{x=\frac{1}{n}}^{x=1} \frac{1}{t}\sin\frac{1}{t}dt$$

The transformation $x = \frac{1}{t}$ yields

$$= \lim_{n\to\infty}\int_{x=1}^{x=n} \frac{\sin x}{x}dx$$

$$= \int_{x=1}^{x=\infty} \frac{\sin x}{x}dx$$

$$= \int_{x=0}^{x=\infty} \frac{\sin x}{x}dx - \int_{x=0}^{x=1} \frac{\sin x}{x}dx$$

Since both Riemann Integrals $\int_{x=0}^{x=\infty} \frac{\sin x}{x}dx$, and $\int_{x=0}^{x=1} \frac{\sin x}{x}dx$ exist,

$\int_{x=0}^{x=1} \frac{1}{t}\sin\frac{1}{t}dt$ exists, and $\frac{1}{t}\sin\frac{1}{t}$ is Riemann Integrable over $(0,1]$.$\square$

But $\frac{1}{t}\sin\frac{1}{t}$ is not absolutely integrable over $(0,1]$. We have

$$\int_{x=0}^{x=1} \frac{1}{t}\left|\sin\frac{1}{t}\right|dt = \lim_{n\to\infty}\int_{x=\frac{1}{n}}^{x=1} \frac{1}{t}\left|\sin\frac{1}{t}\right|dt$$

The transformation $x = \frac{1}{t}$ yields

$$= \lim_{n\to\infty}\int_{x=1}^{x=n} \frac{|\sin x|}{x}dx$$

Similarly to the proof of 1.2,

$$\geq \frac{2}{\pi}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...\right)=\infty.$$

Therefore, $\frac{1}{t}\sin\frac{1}{t}$ is not Lebesgue Integrable over $(0,1]$.□

Therefore,

1.4 *Riemann Integral may exist when Lebesgue's does not*

Proof: 1.2, or 1.3.□

2.

The Fundamental Theorem of Calculus may hold for Riemannian but not Lebesgue-Integration of the same function

Lebesgue's requirement of absolute integrability takes a toll on the Fundamental Theorem of Calculus in Lebesgue Theory of integration.

The Fundamental Theorem of the Calculus states the conditions under which the integral of the derivative $F'(x)$ produces the function $F(x)$.

For Riemannian Integration we have

2.1 The Fundamental Theorem of Calculus

$$F(x) \in C[a,b], \textit{ and } F'(x) \in C(a,b) \Rightarrow \int_{t=a}^{t=b} F'(t)dt = F(b) - F(a)$$

The integration may exclude the interval endpoints, and be from $a + 0$, to $b - 0$.

2.2 $F(x) = x^2 \sin\frac{\pi}{x^2}$ *satisfies the Fundamental Theorem of Calculus in Riemannian Integration over* $0 \le x \le 1$

Namely,

$$\int_{x=0+}^{x=1} \frac{d}{dx}\left(x^2 \sin\frac{\pi}{x^2}\right)dx = \left(x^2 \sin\frac{\pi}{x^2}\right)_{x=0+}^{x=1}$$

Proof: Since $F'(x) = 2x\sin\frac{\pi}{x^2} - \frac{2\pi}{x}\cos\frac{\pi}{x^2}$ is discontinuous at $x = 0$, we consider the integral on $0 < x \le 1$. Then,

$$\int_{x=0+}^{x=1} \frac{d}{dx}\left(x^2 \sin\frac{\pi}{x^2}\right)dx = \lim_{0<\varepsilon\downarrow 0}\int_{x=\varepsilon}^{x=1} \frac{d}{dx}\left(x^2 \sin\frac{\pi}{x^2}\right)dx\,.$$

Now,

$F(x) = x^2 \sin\frac{\pi}{x^2}$ is continuous on $0 \le x \le 1$.

And

$F'(x) = 2x\sin\frac{\pi}{x^2} - \frac{2\pi}{x}\cos\frac{\pi}{x^2}$ is continuous on $0 < x < 1$.

Therefore, by the Fundamental Theorem of Calculus,

$$= \lim_{0<\varepsilon\downarrow 0}\left(x^2 \sin\frac{\pi}{x^2}\right)_{x=\varepsilon}^{x=1}$$

$$= \left(x^2 \sin\frac{\pi}{x^2}\right)_{x=0+}^{x=1} = 0.\square$$

Since the usefulness of the integral Calculus requires that integration yields the correct answer, Lebesgue Theory needed the same result to hold for its measurable, absolutely integrable functions.

It ended up requiring the function to have absolutely integrable derivative $F'(x)$ on $[a,b]$.

We have [Rudin, p.249],

2.3 The Fundamental Theorem of Calculus in Lebesgue Integration

If $F(x)$ *is differentiable on every point of* $[a,b]$ *,*

$F'(x)$ *is absolutely integrable on* $[a,b]$*,*

Then, $\int_{t=a}^{t=b} F'(t)dt = F(b) - F(a)$.

Without Differentiability at every point of $[a,b]$, 2.3 may fail.

For instance, $F(x) = \begin{cases} 0, & x \in [0,1] \\ 3, & x \in (1,2] \end{cases}$ has $F'(x) = 0$, for all $x \neq 1$,

and $\int_{x=0}^{x=2} F'(x)dx = 0 < 3 = F(2) - F(0)$.

These conditions are not satisfied by $F(x) = x^2 \sin \frac{\pi}{x^2}$.

That is,

2.4 *$F(x) = x^2 \sin\frac{\pi}{x^2}$ Violates the Fundamental Theorem of Calculus in Lebesgue integration over* $0 < x \leq 1$

Proof:

We'll see that $F'(x) = 2x\sin\frac{\pi}{x^2} - \frac{2\pi}{x}\cos\frac{\pi}{x^2}$ is not absolutely integrable over $0 < x \leq 1$, because $\frac{1}{x}\cos\frac{\pi}{x^2}$ is not.

$$\int_{0+}^{1}\frac{1}{x}\left|\cos\frac{\pi}{x^2}\right|dx \geq \int_{\frac{1}{\sqrt{1+\frac{1}{3}}}}^{\frac{1}{\sqrt{1-\frac{1}{3}}}}\frac{1}{x}\left|\cos\frac{\pi}{x^2}\right|dx + \int_{\frac{1}{\sqrt{2+\frac{1}{3}}}}^{\frac{1}{\sqrt{2-\frac{1}{3}}}}\frac{1}{x}\left|\cos\frac{\pi}{x^2}\right|dx + \int_{\frac{1}{\sqrt{3+\frac{1}{3}}}}^{\frac{1}{\sqrt{3-\frac{1}{3}}}}\frac{1}{x}\left|\cos\frac{\pi}{x^2}\right|dx + ...$$

In the first interval of integration, $\frac{1}{\sqrt{1+\frac{1}{3}}} < x < \frac{1}{\sqrt{1-\frac{1}{3}}}$,

$$\frac{2}{3}\pi \leq \frac{\pi}{x^2} \leq \frac{4}{3}\pi, \text{ and } \left|\cos\frac{\pi}{x^2}\right| \geq \frac{1}{2}.$$

In the second interval,

$$\frac{5}{3}\pi \leq \frac{\pi}{x^2} \leq \frac{7}{3}\pi, \text{ and } \left|\cos\frac{\pi}{x^2}\right| \geq \frac{1}{2}.$$

In the third interval,

$$\frac{8}{3}\pi \leq \frac{\pi}{x^2} \leq \frac{9}{3}\pi, \text{ and } \left|\cos\frac{\pi}{x^2}\right| \geq \frac{1}{2}.$$

....

Therefore, the integral is

$$\geq \frac{1}{2}\int_{\frac{1}{\sqrt{1+\frac{1}{3}}}}^{\frac{1}{\sqrt{1-\frac{1}{3}}}} \frac{1}{x}dx + \frac{1}{2}\int_{\frac{1}{\sqrt{2+\frac{1}{3}}}}^{\frac{1}{\sqrt{2-\frac{1}{3}}}} \frac{1}{x}dx + \frac{1}{2}\int_{\frac{1}{\sqrt{3+\frac{1}{3}}}}^{\frac{1}{\sqrt{3-\frac{1}{3}}}} \frac{1}{x}dx + \ldots$$

$$= \frac{1}{4}\log\frac{1+\frac{1}{3}}{1-\frac{1}{3}} + \frac{1}{4}\log\frac{2+\frac{1}{3}}{2-\frac{1}{3}} + \frac{1}{4}\log\frac{3+\frac{1}{3}}{3-\frac{1}{3}} + \ldots$$

$$= \frac{1}{4}\int_{1-\frac{1}{3}}^{1+\frac{1}{3}} \frac{1}{x}dx + \frac{1}{4}\int_{2-\frac{1}{3}}^{2+\frac{1}{3}} \frac{1}{x}dx + \frac{1}{4}\int_{3-\frac{1}{3}}^{3+\frac{1}{3}} \frac{1}{x}dx + \ldots$$

$$\geq \frac{1}{4}\frac{1}{1+\frac{1}{3}}\frac{2}{3} + \frac{1}{4}\frac{1}{2+\frac{1}{3}}\frac{2}{3} + \frac{1}{4}\frac{1}{3+\frac{1}{3}}\frac{2}{3} + \ldots$$

$$= \frac{1}{6}\left(\frac{1}{1+\frac{1}{3}} + \frac{1}{2+\frac{1}{3}} + \frac{1}{3+\frac{1}{3}} + \ldots\right)$$

$$\geq \frac{1}{6}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\right) = \infty . \square$$

Therefore,

2.5 *The Fundamental Theorem of Calculus may hold for Riemannian Integration, but not for Lebesgue Integration of the same function*

Proof: 2.2, and 2.4.□

3.

The same limit function of Integrable functions may be Riemann-Integrable but not Lebesgue-integrable

Lebesgue requirement of absolute integrability may render the Riemann Integrable limit function of a sequence of Riemann Integrable functions, Lebesgue non-integrable.

This further suggests that Riemannian Integration generalizes Lebesgue Integration.

Denote by $\chi_{[a,b]}(x)$ the function that equals 1 on $[a,b]$, and 0 out of $[a,b]$. Then we have,

3.1 *The limit of the Riemann integrable functions*

$$f_n(x) = \chi_{[0,1)}(x) - \frac{1}{2}\chi_{[2,3)}(x) + \frac{1}{3}\chi_{[3,2)}(x) + \ldots + \frac{(-1)^{n-1}}{n}\chi_{[n-1,n)}(x)$$

is Riemann Integrable

Proof:

Each $f_n(x)$ has finitely many jump discontinuities, and is Integrable. Its Riemann integral is

$$\int_{x=0}^{x=\infty} f_n(x)dx = \int_{x=0}^{x=\infty} \chi_{[0,1)}(x)dx - \frac{1}{2}\int_{x=0}^{x=\infty} \chi_{[2,3)}(x)dx + ... + \frac{(-1)^{n-1}}{n}\int_{x=0}^{x=\infty} \chi_{[n-1,n)}(x)dx$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - ... + \frac{(-1)^{n-1}}{n}.$$

The limit function of $f_n(x)$ is the alternating series

$$f(x) = \chi_{[0,1)} - \frac{1}{2}\chi_{[2,3)} + \frac{1}{3}\chi_{[3,2)} - \frac{1}{4}\chi_{[3,4)} + ...$$

Its Riemann Integral is

$$\int_0^\infty f(x)dx = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... = \ln 2\,.\square$$

3.2 *The limit of the Lebesgue integrable functions*

$$f_n(x) = \chi_{[0,1)}(x) - \frac{1}{2}\chi_{[2,3)}(x) + \frac{1}{3}\chi_{[3,2)}(x) + ... + \frac{(-1)^{n-1}}{n}\chi_{[n-1,n)}(x)$$

is not Lebesgue Integrable

Proof: Each $f_n(x)$ is absolutely integrable. The absolute value integral is

$$\int_{x=0}^{x=\infty} \left|f_n(x)\right| dx = \int_{x=0}^{x=\infty} \chi_{[0,1)}(x)dx + \frac{1}{2}\int_{x=0}^{x=\infty} \chi_{[2,3)}(x)dx + ... + \frac{1}{n}\int_{x=0}^{x=\infty} \chi_{[n-1,n)}(x)dx$$

$$= 1 + \frac{1}{2} + \frac{1}{3} - ... + \frac{1}{n}$$

and Its Lebesgue integral is

$$\int_{x=0}^{x=\infty} f_n(x)dx = \int_{x=0}^{x=\infty} \chi_{[0,1)}(x)dx - \frac{1}{2}\int_{x=0}^{x=\infty} \chi_{[2,3)}(x)dx + ... + \frac{(-1)^{n-1}}{n}\int_{x=0}^{x=\infty} \chi_{[n-1,n)}(x)dx$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - ... + \frac{(-1)^{n-1}}{n}.$$

The limit function of $f_n(x)$ is the alternating series

$$f(x) = \chi_{[0,1)} - \frac{1}{2}\chi_{[2,3)} + \frac{1}{3}\chi_{[3,2)} - \frac{1}{4}\chi_{[3,4)} + ...$$

Hence,

$$\left|f(x)\right| = \chi_{[0,1)} + \frac{1}{2}\chi_{[2,3)} + \frac{1}{3}\chi_{[3,2)} + \frac{1}{4}\chi_{[3,4)} + ...$$

Its Integral is

$$\int_0^\infty \left|f(x)\right|dx = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... = \infty.$$

Thus, $f(x)$ is not Lebesgue integrable.□

Therefore,

3.3 *The same limit function of integrable functions may be Riemann-Integrable, but not Lebesgue-Integrable*

Proof: 3.1, and 3.2.□

4.

A Countable Set Need Not Have Measure Zero

Lebesgue integration is based on the notion of "almost everywhere", on the premise that any sequence in the domain of a bounded function, has no contribution to the integral, and may be tossed away when we evaluate the integral.

We have shown constructively, in [Dan1], and in [Dan2] that the real numbers in $[0,1]$ can be sequenced, although the interval $[0,1]$ has non-zero length.

The sequencing proof in [Dan1] uses the Cantor Set, and is a little longer.

We outline here the sequencing proof of [Dan2].

4.1 The Sequencing of the real Numbers in $[0,1]$

We list the real numbers in $[0,1]$, in consecutive rows, using their binary representation.

The 1^{st} row has the 2^1 sequences representing 0, and $\frac{1}{2}$,

$$(0,0,0,\dots,0,0,0\dots) \leftrightarrow 0,$$

$$(1,0,0,\ldots,0,0,0\ldots) \leftrightarrow \tfrac{1}{2}$$

The 2^{nd} row has the 2^2 sequences

$$(0,0,0,\ldots,0,\ldots) \leftrightarrow 0,$$

$$(0,1,0,\ldots,0\ldots) \leftrightarrow \tfrac{1}{2^2},$$

$$(1,0,0,\ldots,0,\ldots) \leftrightarrow \tfrac{2}{2^2}$$

$$(1,1,0,\ldots,0,\ldots) \leftrightarrow \tfrac{3}{2^2}$$

The 3^{rd} row has the 2^3 sequences

$$(0,0,0,\ldots,0,\ldots) \leftrightarrow 0,$$

$$(0,0,1,0,\ldots) \leftrightarrow \tfrac{1}{2^3},$$

$$(0,1,0,0,\ldots) \leftrightarrow \tfrac{2}{2^3},$$

$$(0,1,1,0,\ldots) \leftrightarrow \tfrac{3}{2^3},$$

$$(1,0,0,0,\ldots) \leftrightarrow \tfrac{4}{2^3},$$

$$(1,0,1,0,\ldots) \leftrightarrow \tfrac{5}{2^3},$$

$$(1,1,0,0,\ldots) \leftrightarrow \tfrac{6}{2^3},$$

$$(1,1,1,0,\ldots) \leftrightarrow \tfrac{7}{2^3}.$$

The n^{th} row lists the 2^n sequences that start with

$$(0,0,0,0,\ldots) \leftrightarrow 0,$$

and end with

$$(1,1,1,1,\ldots 1,0,\ldots) \leftrightarrow \tfrac{2^n-1}{2^n}.$$

0 1

00 01 10 11

000 001 010 011 100 101 110 111

... ...

The $Card\mathbb{N}$ row has $2^{Card\mathbb{N}}$ sequences that represent all the real numbers in $[0,1]$.

This listing enumerates all the real numbers in $[0,1]$.$\square$

Equivalently, we have the equality

$$Card\mathbb{N} = Card\mathbb{R} = 2^{Card\mathbb{N}}.$$

In [Dan2], we established this equality in another three technical, non-constructive proofs. The first proof, uses a result of Tarski. The second uses the properties of cardinal, and ordinal numbers. The third uses the cardinality property of ordinals.

So far, we established the sequencing of the real numbers in five different proofs.

The Midpoints Set that is used in [Dan2] to well-order the real numbers in $[0,1]$, can supply a sixth constructive proof.

Therefore,

4.2 *A countable set need not have measure zero*

Proof: By 4.1, the reals in $[0,1]$ are countable, with measure 1.$\square$

5.

The Rationals in $[0,1]$ are Non-Measurable

In [Dan3] we have shown that the set of rational numbers does not have measure zero. In fact, by Lebesgue own criteria, the set of rational numbers in $[0,1]$ is not measurable.

We shall outline here the main arguments of [Dan3] that establish that the rationals are non-measurable

5.1 Lebesgue's Procedure

Lebesgue sequenced the rationals

$$\{r_1, r_2, r_3, ...\}$$

and covered them by the intervals

$$(r_1 - \tfrac{1}{4}\varepsilon, r_1 + \tfrac{1}{4}\varepsilon),$$

$$(r_2 - \tfrac{1}{8}\varepsilon, r_2 + \tfrac{1}{8}\varepsilon),$$

$$(r_3 - \tfrac{1}{16}\varepsilon, r_3 + \tfrac{1}{16}\varepsilon)$$

$$\dots\dots\dots\dots\dots\dots$$

of lengths

$$\frac{1}{2}\varepsilon, \frac{1}{2^2}\varepsilon, \frac{1}{2^3}\varepsilon, ...$$

Then,

$$m(E) \leq \frac{1}{2}\varepsilon + \frac{1}{2^2}\varepsilon + \frac{1}{2^3}\varepsilon + ... = \varepsilon .$$

Taking the infimum on $\varepsilon > 0$, Lebesgue effectively set ε to zero, and concluded that $m(E) = 0$.

However,

5.2 *The Rationals in* $[0,1]$ *cannot be squeezed into an interval of length* ε

Proof: There are no rational-only intervals, or irrationals-only intervals. In any interval with irrational endpoints, there are infinitely many rational numbers, and in any interval with rational endpoints, there are infinitely many irrational numbers.

The sequencing of the rationals does not alter their dense distribution in the irrationals. We can sequence the rationals, but we cannot squeeze them into any subinterval of $[0,1]$. Not even into a subinterval of size $1-\delta$, for any $\delta > 0$. Similarly, the irrationals are dense in the rationals.

The cardinality of the rationals and irrationals is irrelevant to the density of each set in the other, and to the inability to separate the two sets.

Recall Lebesgue's cover of the rationals in $[0,1]$

$$(r_1 - \tfrac{1}{4}\varepsilon, r_1 + \tfrac{1}{4}\varepsilon),$$

$$(r_2 - \tfrac{1}{8}\varepsilon, r_2 + \tfrac{1}{8}\varepsilon),$$

$$(r_3 - \tfrac{1}{16}\varepsilon, r_3 + \tfrac{1}{16}\varepsilon)$$

$$\dots\dots\dots\dots\dots\dots$$

with length tailored to be $< \varepsilon$.

Its complement in $[0,1]$ is a union of intervals with length $> 1 - \varepsilon$.

And according to Lebesgue, there are no rational numbers in those non-degenerate intervals…

Can there be a non-degenerate interval void of rational numbers?

Lebesgue's claim to be able to keep rationals out of infinitely many intervals in $[0,1]$ is not credible.

There is no open cover of the rationals in $[0,1]$ of length $\varepsilon < 1$ that contains all the rational numbers in $[0,1]$.

The minimal open cover of all the rational numbers in $[0,1]$ is $(0,1)$, that has length 1.

Thus, the Lebesgue procedure to extend the definition of measure to the rationals in $[0,1]$ is based on an impossibility, and is invalid.$\square$

Moreover, by Lebesgue's own criteria for measurability,

5.3 *The Rational Numbers in* $[0,1]$ *are Non-Measurable Set*

Proof: Lebesgue's procedure ignores his own characterization of a measurable set. We quote Lebesgue from [Hawking, p.1051]

> *"A set* E *is measurable if and only if*
> *for as small as we wish* $\varepsilon > 0$*,* E *has a cover by* $\alpha(\varepsilon)$ *open intervals, and* E^c *has a cover by* $\beta(\varepsilon)$ *open intervals so that the sum of the lengths of the intervals of intersection of the covers is* $< \varepsilon$*"*

Lebesgue's definition is applicable to the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...$$

Then, the complement of the sequence in $[0,1]$,

$$(\frac{1}{2}, 1) \cup (\frac{1}{3}, \frac{1}{2}) \cup (\frac{1}{4}, \frac{1}{3}) \cup ....$$

has the length

$$(1 - \tfrac{1}{2}) + (\tfrac{1}{2} - \tfrac{1}{3}) + (\tfrac{1}{3} - \tfrac{1}{4}) + ... = 1.$$

Therefore, $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}$ is measurable, and its measure is

$$m[0,1] - m\left[(\frac{1}{2}, 1) \bigcup (\frac{1}{3}, \frac{1}{2}) \bigcup (\frac{1}{4}, \frac{1}{3}) \cup ....\right] = 0.$$

Similarly, the Cantor set [Carothers], constructed so that its complement

$$(\frac{1}{3},\frac{2}{3})\cup(\frac{1}{3^2},\frac{2}{3^2})\cup(\frac{7}{3^2},\frac{8}{3^2})\cup....$$

has length 1, is measurable, and its measure is

$$m[0,1]-m\left[(\frac{1}{3},\frac{2}{3})\cup(\frac{1}{3^2},\frac{2}{3^2})\cup(\frac{7}{3^2},\frac{8}{3^2})\cup....\right]=0\,.$$

In both cases, the complement is the union of disjoint open intervals. Then, the open covers may be refined so that their common intersection shrinks and is $<\varepsilon$.

But rational numbers cannot be separated from each other by open intervals of irrational numbers.

The density of the rationals in $[0,1]$ guarantees their presence in any interval in the open cover of the irrationals in $[0,1]$.

Therefore, the smallest ε equals 1, and there are no refined open covers, so that the sum of the lengths of the intervals that belong to the intersection of the covers is <1.

That is, by Lebesgue's characterization, both the rationals and the irrationals in $[0,1]$ are non-measurable.□

Thus,

5.4 *The concept of length in* $[0,1]$ *applies only to a union of disjoint open intervals. or a set of points separated by intervals.*

6.

Dirichlet Function

Dirichlet Function is the Characteristic function of the rational numbers in $[0,1]$.

We define it for any real number, $-\infty < x < \infty$, by

6.1 $$\chi_{[0,1]\cap Q}(x) = \begin{cases} 1, & x = rational\ in\ [0,1] \\ 0, & x \neq rational\ in\ [0,1] \end{cases}$$

6.2 *Dirichlet Function is Non-Measurable*

Proof: To be a measurable function, any measurable set in the range has to inverse-map to a measurable set in the domain.

Here,

$$\chi^{-1}_{[0,1]\cap Q}\{1\} = Q \cap [0,1] = rational\ numbers\ in\ [0,1].$$

Since by 5.3, the rational numbers in $[0,1]$ are non-measurable, the Dirichlet function is a Non-measurable function.□

Hence,

6.3 *Dirichlet Function is Lebesgue Non-Integrable Function*

Let the sequence of the rationals in $[0,1]$ be

$$r_1, r_2, r_3, \ldots\ldots,$$

Then,

6.4 *Dirichlet Function is the limit of the Bounded Monotonic Increasing sequence of Lebesgue Measurable, and Integrable Functions*

$$\chi_{\{r_1\}}(x),$$

$$\chi_{\{r_1, r_2\}}(x),$$

$$\chi_{\{r_1, r_2, r_3\}}(x),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

6.5 *Dirichlet Function is the* $\liminf$ *of the Bounded Monotonic Increasing sequence of Lebesgue Measurable, and Integrable Functions*

$$\chi_{\{r_1\}}(x),$$

$$\chi_{\{r_1, r_2\}}(x),$$

$$\chi_{\{r_1, r_2, r_3\}}(x),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Proof: $\lim f_n(x) = \liminf f_n(x) = \limsup f_n(x)$,

and apply 6.4.□

6.6 *Dirichlet Function is the limit of the Bounded Monotonic Increasing sequence of Lebesgue Measurable, and Integrable Functions*

$$\chi_{\{r_1\}}(x),$$

$$\chi_{\{r_1\}}(x) + \chi_{\{r_2\}}(x),$$

$$\chi_{\{r_1\}}(x) + \chi_{\{r_2\}}(x) + \chi_{\{r_3\}}(x),$$

....

Proof: $\chi_{\{r_1,r_2\}}(x) = \chi_{\{r_1\}}(x) + \chi_{\{r_2\}}(x)$,

and apply 6.4.□

7.

Riemann's Function

Riemann's Function is defined for, $0 \le x \le 1$, [Olmsted, p.142] by

7.1 $$R(x) = \begin{cases} \frac{1}{q}, \ x = rational \ \frac{p}{q} \ in \ [0,1] \\ 0, \ x \neq rational \ in \ [0,1] \end{cases}$$

7.2 *Riemann's Function is Non-Measurable*

Proof: By 7.1, $R^{-1}(0,1] = Q \cap [0,1] = rational \ numbers \ in \ [0,1]$,

and by 5.3, the rational numbers in $[0,1]$ are non-measurable.□

Hence,

7.3 *Riemann's Function is Not Lebesgue-Integrable*

Let the sequence of the rationals in $[0,1]$ be

$$\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}, ...,$$

Then,

7.4 *Riemann's Function is the limit of the Bounded Monotonic Increasing sequence of Measurable, and Integrable Functions*

$$R_1(x) = \begin{cases} \dfrac{1}{q_1}, & x = \dfrac{p_1}{q_1} \ in \ [0,1] \\ 0, & x \neq rational \ in \ [0,1] \end{cases}$$

$$R_2(x) = \begin{cases} \dfrac{1}{q_1}, & x = \dfrac{p_1}{q_1} \ in \ [0,1] \\ \dfrac{1}{q_2}, & x = \dfrac{p_2}{q_2} \ in \ [0,1] \\ 0, & x \neq rational \ in \ [0,1] \end{cases}$$

$$R_3(x) = \begin{cases} \dfrac{1}{q_1}, & x = \dfrac{p_1}{q_1} \ in \ [0,1] \\ \dfrac{1}{q_2}, & x = \dfrac{p_2}{q_2} \ in \ [0,1] \\ \dfrac{1}{q_3}, & x = \dfrac{p_3}{q_3} \ in \ [0,1] \\ 0, & x \neq rational \ in \ [0,1] \end{cases}$$

……………………………………………………………………………

7.5 *Riemann's Function is Discontinuous at the Non-Measurable Rational Numbers*

Proof: For any rational $\frac{p}{q}$, we want to find a sequence $x_n \to \frac{p}{q}$,

so that

the $R(x_n)$ do not converge to $R(\frac{p}{q})$

Indeed, any rational $\frac{p}{q}$ is the limit of a sequence of irrationals $\alpha_n \to \frac{p}{q}$. And $R(\alpha_n) = 0 \to 0 \neq \frac{1}{q} = R(\frac{p}{q})$.

7.6 *Riemann's Function is Continuous at the Non-Measurable Irrational Numbers*

Proof: For any irrational α, we want to show that given arbitrarily small $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ so that for $\alpha - \delta < x < \alpha + \delta$, we have $\left|\underbrace{R(x)}_{\geq 0} - \underbrace{R(\alpha)}_{=0}\right| < \varepsilon$.

While $\frac{1}{\varepsilon}$ may be large, there are only finitely many natural numbers q's so that $q \leq \frac{1}{\varepsilon}$.

Since $p < q$, there are only finitely many rational numbers $\frac{p}{q}$ so that $R(\frac{p}{q}) = \frac{1}{q} \geq \varepsilon$.

Thus, there is an interval $\alpha - \delta < x < \alpha + \delta$, that contains none of those finitely many rationals. For x's in that interval, whether they are rationals, or irrationals, we have $R(x) < \varepsilon$.$\square$

7.7 *Riemann's Function is Riemann-Integrable over a Non-Measurable set of Discontinuities*

Proof: Given arbitrarily small $\varepsilon > 0$, there are only finitely many natural numbers q's so that $q \leq \frac{1}{\varepsilon}$.

Since $p < q$, there are only finitely many, say N, rational numbers $\frac{p}{q}$ so that $R(\frac{p}{q}) = \frac{1}{q} \geq \varepsilon$.

These points occupy at most N subintervals, $I_1,...I_N$, each of length $\delta < \frac{\varepsilon}{N}$.

The contribution to the Riemann Sum from these sub-intervals is

$$\leq \underbrace{R(\frac{p_1}{q_1})}_{\leq 1}\delta + ...\underbrace{R(\frac{p_N}{q_N})}_{\leq 1}\delta \leq N\delta < \varepsilon$$

These N subintervals are separated by l subintervals each of length $\delta < \frac{\varepsilon}{N}$.

At arbitrarily chosen points $\xi_1,...\xi_l$ in each of the subintervals, we have

$$R(\xi_1) < \varepsilon,$$

$$\text{.................},$$

$$R(\xi_l) < \varepsilon,$$

and the contribution to the Riemann Sum is

$$\underbrace{R(\xi_1)}_{<\varepsilon}\delta + ... + \underbrace{R(\xi_l)}_{<\varepsilon}\delta < \varepsilon \underbrace{l\delta}_{<1} < \varepsilon .$$

Consequently, the Riemann Sums are bounded by 2ε.

If $\delta \downarrow 0$ then, $\varepsilon \downarrow 0$, and the Riemann Sums converge to zero.

Therefore, $\int\limits_{x=0}^{x=1} R(x)dx = 0$. $\square$

8.

A Convergent Sequence of Lebesgue Measurable, Integrable Functions may have Non-measurable, Non-integrable limit Function

Lebesgue theory claims that every limit of measurable functions is measurable.

This enables every limit of Lebesgue integrable functions to be Lebesgue integrable, and permits us to do away with the uniform convergence that is required for the limit function of Riemann integrable functions to be Riemann integrable.

We shall disprove Lebesgue's theory claim with convergent sequences of Lebesgue measurable, and integrable functions which limit function is Lebesgue Non-measurable, and Non-integrable

8.1 *The Lebesgue Non-Measurable, Non-integrable* <u>*Dirichlet Function*</u> *is the limit of Lebesgue measurable, integrable functions*

Proof: By 6.2, 6.3, and 6.4.□

8.2 *The Lebesgue Non-Measurable, Non-integrable* <u>*Riemann Function*</u> *is the limit of Lebesgue measurable, integrable functions*

Proof: By 7.2, 7.3, and 7.4.□

Therefore,

8.3 *A Convergent Sequence of Lebesgue Measurable and Integrable Functions may have a Lebesgue Non-Measurable, Non-integrable limit Function*

Proof: By 8.1, or 8.2.□

9.

L^1, is an incomplete Normed Linear Space

For L^1 to be a complete (Banach) space, every Cauchy sequence of Lebesgue Integrable functions, should converge to a Lebesgue integrable function.

But the Lebesgue Non-integrable Dirichlet function is the limit of a Cauchy sequence of Lebesgue-integrable functions.

And the Lebesgue Non-Integrable, Riemann function is the limit of Cauchy sequence of Lebesgue-integrable functions.

That is,

9.1 *L^1 has Cauchy sequences of Lebesgue-integrable functions with Lebesgue Non-integrable Limits*

Therefore,

9.2 *L^1 is an Incomplete, Normed Linear space*

Similarly, we conclude that

9.3 *L^p for $1 \le p \le \infty$, is an Incomplete, Normed Linear space.*

10.

The Completion of L^1 to a Banach Space has Lebesgue Non-integrable functions

The completion of a function space is based on the completion of the rational numbers into the real line. Then, the irrational numbers are defined as the limits of Cauchy sequences of rationals.

The completion space for the real line is an infinite dimensional space, where the irrationals are represented by non-constant Cauchy sequences of rationals.

Similarly, the completion of L^1, contains Lebesgue Non-measurable, Non-integrable functions. These are the limits of Cauchy sequences of Lebesgue measurable, integrable functions.

To expect the completion of L^1 to contain only integrable functions, is analogous to expecting the real line to contain only rational numbers.

The existence of the irrationals, used to be incomprehensible, just as the existence of non-integrable functions was.

Perhaps, accepting the irrationals was helped by the fact that not all numbers have to be fractions.

Similarly, not all functions have to be integrable.

The completion of L^1, includes the Lebesgue Non-integrable functions that are limits of Cauchy sequences of Lebesgue-integrable functions.

10.1 *The Completion of* L^1 *to a Banach Space has Lebesgue Non-Integrable functions*

Proof: The Riemann function, and the Dirichlet function are Lebesgue non-integrable limits of Cauchy sequences of Lebesgue measurable and integrable functions.□

11.

Dominant Convergence

It is well-known that uniform convergence of Riemann-integrable functions is required to guarantee the Riemann integrability of the Limit function on $[a,b]$, and the convergence of the integrals.

Namely,

11.1 Convergence of Riemann-integrable Functions

If $f_n(x)$ *are Riemann-integrable,*

$f_n(x) \xrightarrow{u} f(x)$ *on* $[a,b]$*,*

Then, $f(x)$ *is Riemann-integrable,*

$$\int_{x=a}^{x=b} f_n(x)dx \to \int_{x=a}^{x=b} f(x)dx.$$

Arzela replaced the uniform convergence with integrability of $f(x)$, and boundedness of the $f_n(x)$ [Olmsted, p. 152].

11.2 Arzela Bounded Convergence (1885)

If $f_n(x)$*, and* $f(x)$ *are Riemann Integrable on* $[a,b]$*,*

$f_n(x) \to f(x)$ *on* $[a,b]$*,*

$\left| f_n(x) \right| \le M$ ***on*** $[a,b]$,

Then, $\int\limits_{x=a}^{x=b} f_n(x)dx \to \int\limits_{x=a}^{x=b} f(x)dx$.

Relaxing Arzela conditions, led Lebesgue to his failed Dominated Convergence.

11.3 Lebesgue Dominant Convergence

If $f_n(x)$***, and*** $g(x)$ ***are Lebesgue Measurable, and Integrable on*** $[a,b]$,

$f_n(x) \to f(x)$ ***on*** $[a,b]$,

$\left| f_n(x) \right| \le g(x)$ ***on*** $[a,b]$,

Then, $f(x)$ ***is Lebesgue Measurable, and Integrable on*** $[a,b]$, ***and***

$$\int\limits_{x=a}^{x=b} f_n(x)dx \to \int\limits_{x=a}^{x=b} f(x)dx.$$

11.4 *Lebesgue Dominant Convergence fails*

Proof: Let the sequence of the rationals in $[0,1]$ be $r_1, r_2, r_3, ...$

By 6.4, and 6.6, the Non-Measurable, Non-Integrable Dirichlet function is the limit of Lebesgue Measurable, and Integrable $f_n(x)$ so that

$$f_1(x) = \chi_{\{r_1\}}(x),$$

$$f_2(x) = \chi_{\{r_1\}}(x) + \chi_{\{r_2\}}(x),$$

$$f_3(x) = \chi_{\{r_1\}}(x) + \chi_{\{r_2\}}(x) + \chi_{\{r_3\}}(x),$$

....

And

$$\left| f_n(x) \right| \leq 1 \text{ on } [0,1].$$

Since $g(x) = 1$ is Lebesgue measurable and integrable on $[0,1]$, the conditions of Lebesgue dominant convergence are satisfied.

However the conclusion that $f(x)$ must be measurable, and Lebesgue integrable is invalid.□

To save Lebesgue Dominant Convergence, $f(x)$ has to be required to be integrable in 11.3.

We will have

11.5 Lebesgue Dominant Convergence modified

If $f_n(x)$, $f(x)$, *and* $g(x)$ *are Lebesgue Measurable, and Integrable*

on $[a,b]$,

$f_n(x) \to f(x)$ *on* $[a,b]$,

$\left| f_n(x) \right| \leq g(x)$ *on* $[a,b]$,

Then, $\int\limits_{x=a}^{x=b} f_n(x)dx \to \int\limits_{x=a}^{x=b} f(x)dx$.

But it could be easier to find a bound M, than to find a bounding Lebesgue measurable, and integrable $g(x)$.

That is,

11.6 *The modified Lebesgue Dominant Convergence may not improve on Arzela's Bounded Convergence*

12.

Lebesgue Monotone Bounded Convergence

It is well-known that uniform convergence of Monotonic sequence of functions guarantees the Riemann integrability of the Limit function on $[a,b]$, and the convergence of the integrals.

Namely,

12.1 Riemann Monotone Uniform Convergence

If $f_n(x)$ *are Riemann-Integrable*

$f_n(x)$ *is monotonic* $\uparrow$ *to* $f(x)$ *on* $[a,b]$,

$f_n(x) \xrightarrow{u} f(x)$ *on* $[a,b]$,

Then, $f(x)$ *is Riemann-Integrable,*

$$\int_{x=a}^{x=b} f_n(x)dx \to \int_{x=a}^{x=b} f(x)dx .$$

Lebesgue Theory replaces the uniform convergence of the $f_n(x)$ with Lebesgue integrability, and boundedness of the $\int_{x=a}^{x=b} \left|f_n(x)\right|dx$.

This leads to Lebesgue's failed Monotone Bounded Convergence.

12.2 Lebesgue Monotone Bounded Convergence

If $f_n(x)$*, are Lebesgue Integrable on* $[a,b]$*,*

$f_n(x)$ *is monotonic* $\uparrow$ *to* $f(x)$ *on* $[a,b]$*,*

$$\int_{x=a}^{x=b} |f_n(x)|dx \leq M \text{ on } [a,b],$$

Then, $f(x)$ *is Lebesgue Integrable,*

$$\int_{x=a}^{x=b} f_n(x)dx \to \int_{x=a}^{x=b} f(x)dx.$$

12.3 *Lebesgue's Monotone Bounded Convergence fails*

Proof: By 6.4, and 6.6 the Non-Measurable, Non-Integrable Dirichlet function is the limit of the bounded monotonic increasing sequence of Lebesgue Measurable, and Integrable function $f_n(x)$ so that

$$f_1(x) = \chi_{\{r_1\}}(x),$$

$$f_2(x) = \chi_{\{r_1\}}(x) + \chi_{\{r_2\}}(x),$$

$$f_3(x) = \chi_{\{r_1\}}(x) + \chi_{\{r_2\}}(x) + \chi_{\{r_3\}}(x),$$

....

And

$$|f_n(x)| \leq 1 \text{ on } [0,1].$$

Thus, Lebesgue bounded Monotone convergence conditions are satisfied. However, the conclusion that $f(x)$ must be Lebesgue integrable is invalid.□

12.4 Lebesgue Monotone Bounded Convergence Modified

If $f_n(x)$*, and* $f(x)$ *are Lebesgue Integrable on* $[a,b]$*,*

$f_n(x)$ *is monotonic* $\uparrow$ *to* $f(x)$ *on* $[a,b]$*,*

$$\int_{x=a}^{x=b} |f_n(x)|dx \le M \text{ on } [a,b],$$

Then, $\int_{x=a}^{x=b} f_n(x)dx \to \int_{x=a}^{x=b} f(x)dx$.

But it could be easier to show uniform convergence, than Lebesgue integrability of $f(x)$.

That is,

12.5 *The modified Lebesgue Monotone Bounded Convergence may not improve on Riemann Monotone Uniform Convergence*

13.

Fatou's Lemma

Lebesgue Theory includes

13.1 Fatou's Lemma

If $f_n(x) \geq 0$*, are Lebesgue Integrable on* $[a,b]$*,*

$$\liminf \int_{x=a}^{x=b} f_n(x)dx < \infty$$

Then, $\liminf f_n(x)$ *is Lebesgue Integrable on* $[a,b]$*,*

$$\int_{x=a}^{x=b} \liminf f_n(x)dx \leq \liminf \int_{x=a}^{x=b} f(x)dx .$$

13.2 *Fatou's Lemma fails*

Proof: Let the sequence of the rationals in $[0,1]$ be $r_1, r_2, r_3, \ldots\ldots$.

By 6.5, and 6.6, the Non-Measurable, Non-Integrable Dirichlet function is $\liminf f_n(x)$ of the bounded, monotonic increasing, Lebesgue Measurable, and Integrable $f_n(x)$ so that

$$f_1(x) = \chi_{\{r_1\}}(x),$$

$$f_2(x) = \chi_{\{r_1\}}(x) + \chi_{\{r_2\}}(x),$$

$$f_3(x) = \chi_{\{r_1\}}(x) + \chi_{\{r_2\}}(x) + \chi_{\{r_3\}}(x),$$

....

Since for any $n = 1, 2, 3...$

$$\int_{x=0}^{x=1} f_n(x)dx = 0,$$

we have

$$\liminf \int_{x=0}^{x=1} f_n(x)dx = 0 < \infty .$$

That is, the conditions of the Fatou Lemma are satisfied, but the limit function $\liminf f_n(x)$ is Lebesgue non-integrable, the Lebesgue integral $\int_{x=0}^{x=1} \liminf f_n(x)dx$ does not exist. and the conclusions in 14.1 are invalid.□

13.3 Fatou's Lemma modified

If $f_n(x) \geq 0$*, and* $\liminf f_n(x)$ *are Lebesgue Integrable on* $[a, b]$*,*

$$\liminf \int_{x=a}^{x=b} f_n(x)dx < \infty$$

Then, $\int_{x=a}^{x=b} \liminf f_n(x)dx \leq \liminf \int_{x=a}^{x=b} f_n(x)dx .$

The usefulness of this modified Fatou Lemma is unknown to us.

14.

Term by Term Series Integration

It is well-known that uniform convergence of a series of Riemann Integrable functions $\sum_{n=1}^{\infty} f_n(x)$ is required to guarantee the Riemann integrability of the series on $[a,b]$, and the term by term integration of the series.

Namely,

14.1 Riemann Term by Term Series Integration

If $f_n(x)$ *are Riemann Integrable,*

$\sum_{n=1}^{\infty} f_n(x)$ *converges uniformly on* $[a,b]$,

Then, $\sum_{n=1}^{\infty} f_n(x)$ *is Riemann Integrable,*

$$\int_{x=a}^{x=b} \sum_{n=1}^{\infty} f_n(x)dx = \sum_{n=1}^{\infty} \int_{x=a}^{x=b} f_n(x)dx .$$

Beppo-Levi replaced the uniform convergence of the $\sum_{n=1}^{\infty} f_n(x)$

with convergence of $\sum_{n=1}^{\infty} \int_{x=a}^{x=b} |f_n(x)| dx$, and used the failed Lebesgue Dominant Convergence to obtain his Theorem for series integration

14.2 Beppo-Levi Term by Term Series Integration

If $f_n(x)$ *are Lebesgue Integrable on* $[a, b]$,

$$\sum_{n=1}^{\infty} \int_{x=a}^{x=b} |f_n(x)| dx < \infty$$

Then, $\sum_{n=1}^{\infty} f_n(x)$ *is Lebesgue Integrable on* $[a, b]$,

$$\int_{x=a}^{x=b} \sum_{n=1}^{\infty} f_n(x) dx = \sum_{n=1}^{\infty} \int_{x=a}^{x=b} f_n(x) dx .$$

14.3 *Beppo-Levi Term by Term Series Integration fails*

Proof: Let the sequence of the rationals in $[0,1]$ be $r_1, r_2, r_3, \ldots\ldots$.

By 6.6, the Non-Measurable, Non-Integrable Dirichlet function is $\sum_{n=1}^{\infty} f_n(x)$, where $f_n(x) = \chi_{\{r_n\}}(x)$ are Lebesgue-Integrable on $[0,1]$,

And $$\sum_{n=1}^{\infty} \int_{x=0}^{x=1} |f_n(x)| dx = \sum_{n=1}^{\infty} \int_{x=0}^{x=1} \chi_{\{r_n\}}(x) dx = \sum_{n=1}^{\infty} 0 = 0 < \infty .$$

Thus, $f_n(x) = \chi_{\{r_n\}}(x)$ satisfy the condition of Beppo-Levi Term by Term Series Integration.

However, the Dirichlet function, which is $\sum_{n=1}^{\infty} f_n(x)$, is not Lebesgue integrable, and the conclusion in 15.2 is invalid.□

15.

Iterated Lebesgue Integration

Continuity is needed to change order of Riemann Integration

15.1 Iterated Riemann Integration

If $f(x,y)$ *is continuous on* $[a,b]\times[c,d]$

Then, $$\int_{y=c}^{y=d}\left(\int_{x=a}^{x=b} f(x,y)dx\right)dy = \int_{x=a}^{x=b}\left(\int_{y=c}^{y=d} f(x,y)dy\right)dx$$

The function $f(x,y) = \dfrac{x^2 - y^2}{(x^2 + y^2)^2}$ is discontinuous at (0,0), and

$$\int_{y=0}^{y=1}\int_{x=0}^{x=1} f(x,y)dxdy = \int_{y=0}^{y=1}\int_{x=0}^{x=1} \frac{x^2 - y^2}{(x^2 + y^2)^2} dxdy = \int_{y=0}^{y=1} \frac{-1}{1 + y^2} dy = -\frac{\pi}{4}.$$

$$\int_{x=0}^{x=1}\int_{y=0}^{y=1} f(x,y)dydx = \int_{x=0}^{x=1}\int_{y=0}^{y=1} \frac{x^2 - y^2}{(x^2 + y^2)^2} dydx = \int_{x=0}^{x=1} \frac{1}{1 + x^2} dx = \frac{\pi}{4}.$$

Fubini Theorem replaces the continuity that allows iterated integration of the Riemann Integral, with absolute Lebesgue Integrability:

15.2 Fubini Theorem

If $$\iint\limits_{[a,b]\times[c,d]} \left|f(x,y)\right| dxdy < \infty$$

Then,

$$\textit{for almost all } x \in [a,b], \quad \int\limits_{y=c}^{y=d} \left|f(x,y)\right| dy < \infty$$

$$\textit{for almost all } y \in [c,d], \quad \int\limits_{x=a}^{x=b} \left|f(x,y)\right| dx < \infty,$$

$$\int\limits_{x=a}^{x=b} \left| \int\limits_{y=c}^{y=d} f(x,y)dy \right| dx < \infty,$$

$$\int\limits_{y=c}^{y=d} \left| \int\limits_{x=a}^{x=b} f(x,y)dx \right| dy < \infty,$$

and

$$\iint\limits_{[a,b]\times[c,d]} f(x,y)dxdy = \int\limits_{y=c}^{y=d} \left(\int\limits_{x=a}^{x=b} f(x,y)dx \right) dy = \int\limits_{x=a}^{x=b} \left(\int\limits_{y=c}^{y=d} f(x,y)dy \right) dx.$$

15.3 *Fubini Theorem fails*

Proof: The proof of Fubini's Theorem [Royden, p.269] makes extensive use of the claim that the limit of measurable functions is measurable. By 8, this claim is false, and 15.2 is invalid.□

Similarly,

15.4 *Tonelli's Theorem for iterated Lebesgue integration fails*.

16.

Riemann Integral generalizes Lebesgue's

Contrary to common belief that Lebesgue integral generalizes Riemann's, we know of no example of a function that is Lebesgue integrable, and not Riemann Integrable.

On the other hand, sections 1-4 present examples of

Functions that are Riemann Integrable, and not Lebesgue integrable. (section 1)

A function that satisfies the Fundamental Theorem of Calculus for Riemann's Integral but not for Lebesgue's Integral (section 2)

A limit function of integrable functions that is Riemann-Integrable, but not Lebesgue-Integrable (section 3)

Riemann's Function that is Riemann-Integrable over a Non-Measurable set of Discontinuities.

This suggests that

16.1 *Riemann Integral generalizes Lebesgue's*

In particular, the fact that Riemann's Function is Riemann-Integrable over a Non-Measurable set of Discontinuities, suggests that

16.2 *Measurability is a stricter criteria for integrability than continuity*

The fact that the Lebesgue integral cannot be defined over the non-measurable rationals, while the Riemann Integral may be defined over the rationals, suggests that

16.3 *The Lebesgue-integrable functions are a subset of the Riemann-integrable functions*

The irrelevance of measurability for the Riemann Integral indicates that

16.4 *Lebesgue Measurability fails to extend integrability.*

17.

The Space of Riemann Integrable functions

17.1 *The Riemann Integrable functions constitute a linear space*

Define a semi-norm of a Riemann-Integrable function on $[a,b]$ by

17.2
$$\|f\| \equiv \left| \int_{x=a}^{x=b} f(x)dx \right|$$

$\| \ \|$ is a semi-norm, because $\|f\| = 0$ does not imply $f = 0$.

17.3 *The Riemann Integrable Functions so that* $\|f\| < \infty$ *constitute a semi-normed space*

Since there are Cauchy sequences of Riemann Integrable Functions that converge to functions that are Riemann-integrable but not Lebesgue-Integrable, the space of Riemann-Integrable Functions contains the set of Lebesgue integrable functions

That is,

17.4 *The semi-normed linear space of Riemann Integrable functions contains* L^1

Since there are Cauchy sequences of Riemann integrable functions that have Riemann Non-Integrable limit, such as the Dirichlet function, the space of Riemann Integrable functions is not complete.

17.5 *The semi-normed linear space of Riemann-Integrable functions is incomplete*

18.

Cantor Function

Cantor's Function is defined by an inductive process that follows the construction of the Cantor set on the interval $0 \le x \le 1$.

In the first step of the construction of the Cantor set, we delete the interval $\frac{1}{3} < x < \frac{2}{3}$.

$$\text{On } \frac{1}{3} \le x \le \frac{2}{3}, \quad \text{Cantor's function is defined to be } \frac{1}{2}.$$

In the second step, we delete the intervals,

$$\frac{1}{9} < x < \frac{2}{9},$$

and

$$\frac{7}{9} < x < \frac{8}{9}.$$

$$\text{On } \frac{1}{9} \le x \le \frac{2}{9}, \quad \text{Cantor's function is defined to be } \frac{1}{4},$$

$$\text{On } \frac{7}{9} \le x \le \frac{8}{9}, \quad \text{Cantor's function is defined to be } \frac{3}{4}.$$

In the third step, we delete the intervals,

$$\frac{1}{27} < x < \frac{2}{27},$$

$$\frac{7}{27} < x < \frac{8}{27},$$

$$\frac{19}{27} < x < \frac{20}{27},$$

$$\frac{25}{27} < x < \frac{26}{27}.$$

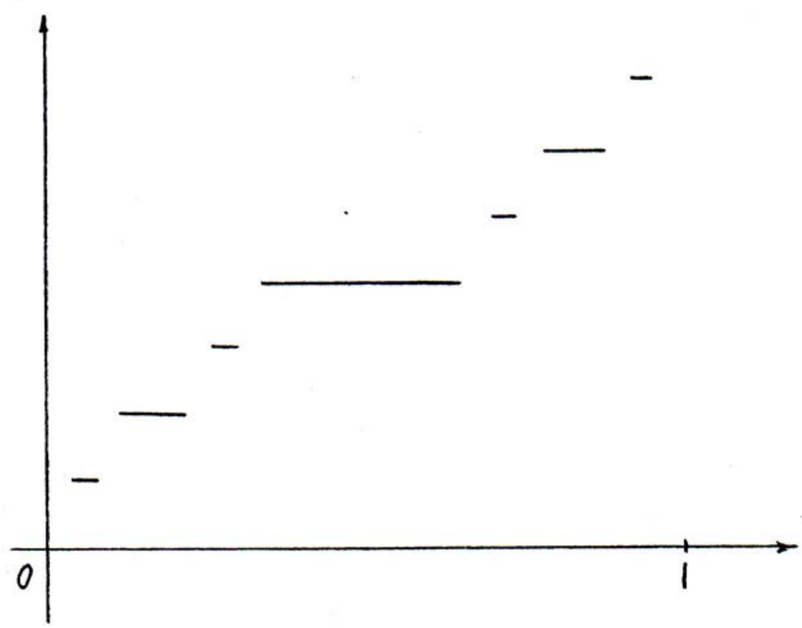
On $\frac{1}{27} \le x \le \frac{2}{27}$, Cantor's function is defined to be $\frac{1}{8}$,

On $\frac{7}{27} \le x \le \frac{8}{27}$, Cantor's function is defined to be $\frac{3}{8}$,

On $\frac{19}{27} \le x \le \frac{20}{27}$, Cantor's function is defined to be $\frac{5}{8}$,

On $\frac{25}{27} \le x \le \frac{26}{27}$, Cantor's function is defined to be $\frac{7}{8}$

After three steps, the graph is



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It is well known that

18.1 *Cantor's Function is Continuous, Thus, Riemann-Integrable on* $[0,1]$*, and is Differentiable with* $F'(x) = 0$ *on the Complement of the Cantor Set in* $[0,1]$

The Cantor Set, which is a set of points separated by open intervals, and the complement of the Cantor set in $[0,1]$, which is a collection of open disjoint intervals, will serve us in the next section as a model for a measurable set.

The Cantor Function will serve us as model for measurable function.

We aim to clarify the meaning of measurable sets, and measurable functions.

19.

The Meaning of a Measurable Set

We have seen that

both the rationals, and the irrationals are not measurable sets because they are dense in the real numbers.

To be measurable the points have to be separated by intervals, that is be discrete, or occupy intervals continuously.

A set of discrete points has measure zero, because the separating intervals have the same measure as the whole interval.

A union of disjoint intervals has measure which is the sum of the lengths of its member intervals.

A set of discrete points need not be constructed as a monotonic sequence. The Cantor Set that is constructed as a non-monotonic sequence, has measure zero.

19.1 *A Measurable Set is either*

a Union of Disjoint Intervals, Namely, an Open Set, *Then, its measure is the sum of the intervals' lengths*

or, a set of discrete points,

Then, its measure is zero

or, a union of both

Then, its measure is the sum of the intervals' lengths

20.

The Meaning of a Measurable Function

To be a measurable function, $f(x)$ has to satisfy

$$f^{-1}(measurable\ set) = measurable\ set$$

That is, the inverse-map of a union of discrete points and disjoint intervals, is another union of discrete points and disjoint intervals.

By Littlewood's characterization, a *measurable function is continuous almost everywhere.*

Indeed, to Lebesgue integrate, over a domain with no singularities, we may ignore the discrete points in the domain.

Then, if the function is measurable, ignoring the domain's discrete points, the source of a measurable image will be disjoint open intervals. That is, an open set.

If every measurable set in the image is an open set,

Then, the measurable function satisfies

$$f^{-1}(open\ set) = open\ set.$$

Then, a Measurable Function is a Continuous Function.

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