Jordan Lemma Proof

H. Vic Dannon
vic0@comcast.net

May, 2011

Abstract The Jordan Lemma is needed in the evaluation of infinite real integrals. We observe that Jordan's proof, and Whittaker and Watson proof are both incomplete. Since all textbooks follow these proofs, we supply the complete proof.

Keywords: Complex Variable, Residue Theorem, Analytic Functions, Jordan Lemma,

2000 Mathematics Subject Classification 30D99; 30A10;

Introduction

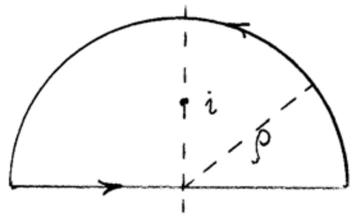
To evaluate the infinite integral

$$\int_{x=-\infty}^{x=\infty} \frac{\cos x}{x^2+1}dx$$

we consider the path integral

$$\int_{\gamma} \frac{e^{iz}}{z^2+1}dz$$

on the contour γ



with $\rho \to \infty$.

On the semi-circle,

$$\int_{\cap} \frac{e^{iz}}{z^2+1} dz \to 0, \text{ as } \rho \to \infty.$$

Therefore, by the Residue Theorem,

$$\int_{x=-\infty}^{x=\infty} \frac{e^{ix}}{x^2+1} dx = 2\pi i \mathrm{Res}\left\{\frac{e^{iz}}{(z+i)(z-i)}\right\}_{z=i}$$

$$= 2\pi i \lim_{z \to i}\left\{\frac{e^{iz}}{z+i}\right\}_{z=i}$$

$$= \pi e^{-1}.$$

It follows that this is the value of the desired integral.

The vanishing of the integral on the infinite semi-circle, holds under the conditions of a general result attributed by Whittaker and Watson, to Jordan.

If $f(z)$ is an analytic function in the upper-half complex plane, so that $f(z) \to 0$, uniformly on any upper semi-circle C_ρ with radius $\rho \to \infty$, centered at the origin, *Then, for* $m > 0$, $\int_{C_\rho} e^{imz} f(z)dz \to 0$, *as* $\rho \to \infty$.

We show here that the proof of the Lemma by Jordan, and the proof by Whittaker and Watson are incomplete.

Whittaker and Watson proof permeates many if not all textbooks on complex variables, and has to be corrected.

1. Jordan's Proof

In Jordan's proof, $m = 1$.

On the semi-circle, C_ρ

$$z = \rho e^{i\theta}, \ 0 \le \theta \le \pi .$$

We have

$$\left| \int_{C_\rho} e^{iz} f(z)dz \right| \le \int_{C_\rho} \left|e^{iz}\right| \left|f(z)\right| \left|dz\right|.$$

Substituting

$$\left|e^{iz}\right| = \left|e^{i\rho\cos\theta - \rho\sin\theta}\right| = e^{-\rho\sin\theta},$$

$$dz = \rho e^{i\theta} d\theta,$$

$$\left|dz\right| = \rho d\theta,$$

$$\left| \int_{C_\rho} e^{iz} f(z)dz \right| \le \max_{C_\rho} \left|f(z)\right| \rho \int_{\theta=0}^{\theta=\pi} e^{-\rho\sin\theta} d\theta .$$

Since

$$\max_{C_\rho}\left|f(z)\right| \to 0, \text{ as } \rho \to \infty,$$

we need to show that $\rho \int\limits_{\theta=0}^{\theta=\pi} e^{-\rho \sin\theta} d\theta$ is bounded.

Jordan writes with no further explanation

$$\rho \int\limits_{\theta=0}^{\theta=\pi} e^{-\rho \sin\theta} d\theta \leq 2\rho \int\limits_{\theta=0}^{\theta=\frac{\pi}{2}} e^{-\rho \sin\theta} d\theta, \qquad (1)$$

and proceeds to bound $\rho \int\limits_{\theta=0}^{\theta=\frac{\pi}{2}} e^{-\rho \sin\theta} d\theta$, by $\frac{\pi}{2}$.

Now, since

$$\int\limits_{\theta=0}^{\theta=\pi} e^{-\rho \sin\theta} d\theta = \int\limits_{\theta=0}^{\theta=\frac{\pi}{2}} e^{-\rho \sin\theta} d\theta + \int\limits_{\theta=\frac{\pi}{2}}^{\theta=\pi} e^{-\rho \sin\theta} d\theta,$$

the claim (1) amounts to

$$\int\limits_{\theta=\frac{\pi}{2}}^{\theta=\pi} e^{-\rho \sin\theta} d\theta \leq \int\limits_{\theta=0}^{\theta=\frac{\pi}{2}} e^{-\rho \sin\theta} d\theta.$$

Changing the integration variable in the first integral into

$$\phi = \theta - \tfrac{\pi}{2},$$

we have

$$e^{-\rho \sin\theta} = e^{-\rho \sin(\phi + \frac{\pi}{2})} = e^{-\rho \cos\phi},$$

and

$$\int_{\phi=0}^{\phi=\frac{\pi}{2}} e^{-\rho\cos\phi} d\phi \le \int_{\theta=0}^{\theta=\frac{\pi}{2}} e^{-\rho\sin\theta} d\theta.$$

That is,

$$\int_{\theta=0}^{\theta=\frac{\pi}{2}} e^{-\rho\cos\theta} d\theta \le \int_{\theta=0}^{\theta=\frac{\pi}{2}} e^{-\rho\sin\theta} d\theta.$$

Since this inequality is un-established, Jordan Proof is incomplete.

2. Whittaker and Watson Proof

Where Jordan saw an inequality, Whittaker and Watson saw an equality. They wrote with no further explanation

$$\rho\int_{\theta=0}^{\theta=\pi} e^{-m\rho\sin\theta} d\theta = 2\rho\int_{\theta=0}^{\theta=\frac{\pi}{2}} e^{-m\rho\sin\theta} d\theta, \qquad (2)$$

and proceeded to bound $\rho\int_{\theta=0}^{\theta=\frac{\pi}{2}} e^{-m\rho\sin\theta} d\theta$, by $\frac{\pi}{2m}$.

Now, since

$$\int_{\theta=0}^{\theta=\pi} e^{-m\rho\sin\theta} d\theta = \int_{\theta=0}^{\theta=\frac{\pi}{2}} e^{-m\rho\sin\theta} d\theta + \int_{\theta=\frac{\pi}{2}}^{\theta=\pi} e^{-m\rho\sin\theta} d\theta,$$

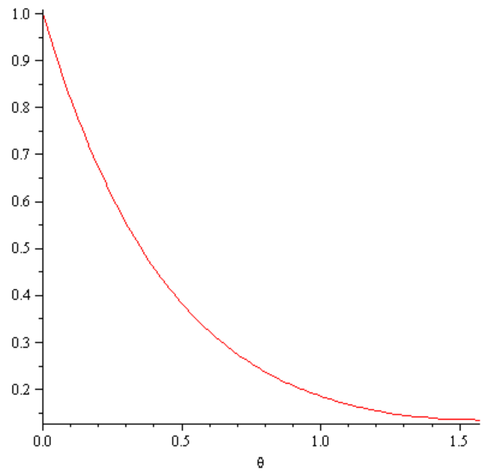
their claim (2) leads to

$$\int_{\theta=0}^{\theta=\frac{\pi}{2}} e^{-m\rho\cos\theta} d\theta = \int_{\theta=0}^{\theta=\frac{\pi}{2}} e^{-m\rho\sin\theta} d\theta.$$

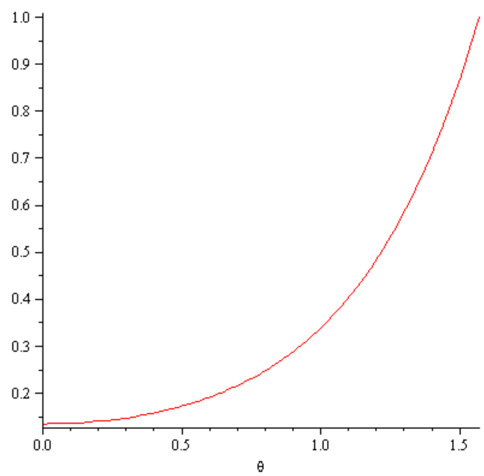
This is un-established, and Whittaker-Watson Proof is incomplete.

3. $\int_{\theta=0}^{\theta=\frac{\pi}{2}} e^{-m\rho\cos\theta}d\theta$, and $\int_{\theta=0}^{\theta=\frac{\pi}{2}} e^{-m\rho\sin\theta}d\theta$ by MAPLE

Using $plot\left(e^{-2\sin(\theta)}, \theta = 0 .. \frac{\pi}{2}\right)$ in MAPLE, we obtain



Using $plot\left(e^{-2\cos(\theta)}, \theta = 0 .. \frac{\pi}{2}\right)$ in MAPLE, we obtain



Thus, an equality of areas under the graphs may be due to symmetry of the graphs. But we cannot confirm it, because do not know how to evaluate the integrals.

In MAPLE, we confirm that

$$\int_{\theta=0}^{\theta=\frac{\pi}{2}} e^{-2\cos\theta}d\theta = \text{(1/2)*Pi*BesselI(0, 2)-(1/2)*Pi*StruveL(0, 2)}$$

and

$$\int_{\theta=0}^{\theta=\frac{\pi}{2}} e^{-2\cos\theta}d\theta = \text{(1/2)*Pi*BesselI(0, 2)-(1/2)*Pi*StruveL(0, 2)}$$

In the notations of Abramowitz, and Stegun

$$\text{BesselI(0, 2)=Modified Bessel Function } \mathrm{I}_0(2)$$

$$\text{StruveL(0, 2)=Struve Function } \mathrm{L}_0(2).$$

That is, both integrals equal

$$\frac{\pi}{2}\left(I_0(2) - L_0(2)\right).$$

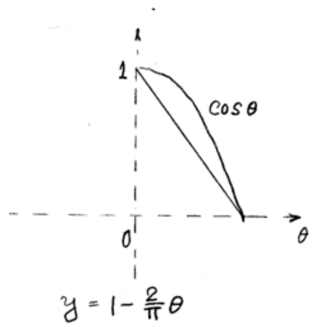
This confirms Whittaker's guess of equality.

Nevertheless, it seems easier to complete the Jordan Lemma Proof by bounding $\int_{\theta=0}^{\theta=\frac{\pi}{2}} e^{-m\rho\cos\theta}d\theta$, then by trying to prove the equality.

4. Completed Proof

To complete the Jordan Lemma Proof, we need to bound $\rho\int_{\theta=0}^{\theta=\frac{\pi}{2}} e^{-\rho\cos\theta}d\theta$, similarly to the bounding of $\rho\int_{\theta=0}^{\theta=\frac{\pi}{2}} e^{-\rho\sin\theta}d\theta$.

The cord between the endpoints over the interval $0 \le \theta \le \frac{\pi}{2}$



has the slope

$$\frac{\cos\frac{\pi}{2} - \cos 0}{\frac{\pi}{2} - 0} = -\frac{2}{\pi},$$

and its equation is

$$y - 1 = -\frac{2}{\pi}\theta .$$

Since $\cos\theta$ is concave down in $0 \le \theta \le \frac{\pi}{2}$, the cord between the endpoints is under the graph of $\cos\theta$. That is,

$$\cos\theta \ge y = 1 - \frac{2}{\pi}\theta .$$

Therefore,

$$-\cos\theta \le -1 + \frac{2}{\pi}\theta ,$$

and

$$\rho \int_{\theta=0}^{\theta=\frac{\pi}{2}} e^{m\rho(-\cos\theta)} d\theta \le \rho \int_{\theta=0}^{\theta=\frac{\pi}{2}} e^{-m\rho + m\rho\frac{2}{\pi}\theta} d\theta$$

$$= \frac{1}{e^{m\rho}} \rho \int_{\theta=0}^{\theta=\frac{\pi}{2}} e^{m\rho\frac{2}{\pi}\theta} d\theta$$

$$= \frac{1}{e^{m\rho}} \rho \frac{1}{m\rho \frac{2}{\pi}} e^{m\rho \frac{2}{\pi}\theta} \Big|_{\theta=0}^{\theta=\frac{\pi}{2}}$$

$$= \frac{1}{e^{m\rho}} \frac{1}{m} \frac{\pi}{2} \left(e^{m\rho \frac{2}{\pi}\frac{\pi}{2}} - 1 \right)$$

$$= \frac{\pi}{2m} \left(1 - \frac{1}{e^{m\rho}} \right)$$

$$\leq \frac{\pi}{2m}.$$

Similarly, [Whittaker and Watson], $\rho \int_{\theta=0}^{\theta=\frac{\pi}{2}} e^{-\rho\cos\theta} d\theta$ is bounded by

$\frac{\pi}{2m}$, and

$$\rho \int_{\theta=0}^{\theta=\pi} e^{-\rho\sin\theta} d\theta = \rho \int_{\theta=0}^{\theta=\frac{\pi}{2}} e^{-\rho\sin\theta} d\theta + \rho \int_{\theta=0}^{\theta=\frac{\pi}{2}} e^{-\rho\cos\theta} d\theta$$

is bounded by $\frac{\pi}{m}$.□

References

[Jordan] Jordan, M. C., *Cours D'ANALYSE de L'Ecole Polytechnique*, Tome Deuxieme, Calcul Integral, Gauthier-Villars, 1894, pp. 285-286

[Whittaker and Watson] Whittaker, E. T., and Watson G. N., *A Course of Modern Analysis*, Fourth Edition, Cambridge University press, 1927. p. 115.

[Abramowitz and Stegun] Abramowitz Milton, and Stegun, Irene, *Handbook of Mathematical Functions*, National Bureau of Standards Applied Mathematics Series, May 1958