Infinitesimals

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Abstract Logicians never constructed Leibnitz infinitesimals and proved their being on a line. They only postulated it.

We construct infinitesimals, and investigate their properties.

Each real number α can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant hyper-real.

Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of positive infinitesimal hyper-reals.

The positive infinitesimals are smaller than any positive real number, yet strictly greater than zero.

Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the positive infinite hyper-reals.

The positive infinite hyper-reals are greater than any real number, yet strictly smaller than infinity.

The negative infinitesimals are greater than any negative real number, yet strictly smaller than zero.

The negative infinite hyper-reals are smaller than any real number, yet strictly greater than $-\infty$.

The sum of a real number with an infinitesimal is a non-constant hyper-real.

The Hyper-reals are the totality of constant hyper-reals, a family of positive and negative infinitesimals, their associated infinite hyper-reals, and non-constant hyper-reals.

The hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.

That line includes the real numbers separated by the non-constant hyper-reals. Each real number is the center of an interval of hyper-reals, that includes no other real number.

In particular, zero is separated from any positive real by the positive infinitesimals, and from any negative real by the negative infinitesimals, $-dx$.

Zero is not an infinitesimal, because zero is not strictly greater than zero, or strictly smaller than zero.

We do not add infinity to the Hyper-real line.

The infinitesimals, and the infinite hyper-reals are semi-groups with respect to addition. Neither set includes zero.

The Hyper-real line is embedded in $\mathbb{R}^{\infty}$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the hyper-real line onto the real line.

In particular, there are no points on the real line that can be assigned uniquely to

the infinitesimal hyper-reals,

or to

the infinite hyper-reals,

or to

the non-constant hyper-reals.

Thus, the real line is inadequate for Infinitesimal Calculus.

Consequently, whenever only infinitesimals or infinite hyper-reals support the derivation of a result, the Calculus of Limits on the real line, fails to deliver that result.

No neighbourhood of a hyper-real is homeomorphic to an $\mathbb{R}^{n}$ ball. Therefore, the hyper-real line is not a manifold.

The hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

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Contents

Introduction

Leibnitz wrote the derivative as a quotient of two differentials,

$$f'(x) = \frac{df}{dx}.$$

The differential df is the difference between two extremely close values of f, attained on two extremely close values of x. The differential dx is the difference between those values of x.

Since $f'(x)$ may vanish, $df = f'(x)dx$ can vanish. But dx cannot vanish because division by zero is undefined.

The differential dx is a positive infinitesimal. it is smaller than any positive real number, yet it is greater than zero.

That characterization was met with understandable scepticism.

How can there be positive numbers that are smaller than any positive number and yet are greater than zero?

The concept of a limit seems to answer that question.

$f'(x_0) = \frac{df}{dx}(x_0)$ means that if for any sequence $x_n \to x_0$, so that $x_n \neq x_0$, the sequence of the quotient differences

$$\frac{f(x_n) - f(x_0)}{x_n - x_0}$$

converges to a number A, then $f'(x_0) = A$.

In other words, the positive differential dx may be interpreted as a shorthand for all the sequences

$$a_n = x_n - x_0, \text{ so that } a_n > 0, \text{ and } a_n \to 0.$$

However, the real numbers have the Archimedean property. It means that if $a > 0$, and $b > 0$, are real numbers so that

$$a < b,$$

then there is some natural number N_0, so that

$$N_0 a > b.$$

This property does not hold for the two sequences

$$\left\langle \frac{1}{n} \right\rangle, \text{ and } \left\langle \frac{1}{n^2} \right\rangle$$

because there is no N_0 so that $N_0 \dfrac{1}{n^2}$ is strictly greater than $\dfrac{1}{n}$, for all $n = 1, 2, 3, \ldots$

Consequently, the Archimedean real number system does not contain infinitesimals, and the addition of infinitesimals to the real numbers, extends the real numbers into a Non-Archimedean number system.

Thus, to establish the existence of infinitesimals, we have to construct a system of the real numbers, that includes the Non-Archimedean infinitesimals. A Hyper-Reals Number system that will contain infinitesimal hyper-reals.

To obtain infinitesimals, Schmieden, and Laugwitz [Laugwitz], added in 1958, an infinity to the rational numbers.

"*...adjoined to the field of rational numbers*
an infinitely large natural number Ω*...*"

Clearly, this must be the first infinite ordinal

$$\omega,$$

that is equal to the first infinite Cardinal

$$Card\mathbb{N}.$$

According to [Laugwitz],

> *To adjoin* Ω *was meant in a more general sense than in algebra but defined by the following* <u>*Postulate:*</u>
>
> *Whatever is true for all finite large natural numbers will, by definition, remain true for* Ω.
>
> *More precisely, if* $A(n)$ *is a formula that is true for all sufficiently large finite natural* n*, then* $A(\Omega)$ *is*

true. (Leibnitz Principle)

Since $n > 10^{100}$ *for all large* n *it follows that* $\Omega > 10^{100}$*, etc.*

Thus Ω *is indeed infinitely large.*

Since n^n*, and* 2^{n^2} *are natural numbers, and since* $n^n < 2^{n^2}$ *for all large* n*, the inequality*

$$\Omega^{\Omega} < 2^{\Omega^2}$$

is true, and both sides are infinitely large natural numbers.

The Schmieden and Laugwitz Postulate is wrong.

What is true for natural numbers, need not hold for infinities. For instance, for any natural number n

$$n < n + n$$

But it is well known that

$$Card\mathbb{N} = Card\mathbb{N} + Card\mathbb{N}$$

Similarly, while $n^n < 2^{n^2}$, we have $\Omega^{\Omega} = 2^{\Omega^2}$.

Indeed, it is well known that

$$\Omega^{\Omega} = (Card\mathbb{N})^{Card\mathbb{N}} = 2^{Card\mathbb{N}} = 2^{(Card\mathbb{N})^2} = 2^{\Omega^2}.$$

Consequently, Schmieden and Laugwitz approach to the infinitesimals is founded on ignorance of infinities, and has to be rejected.

In 1961, Robinson produced, in the spirit of the Schmieden and Laugwitz, the

Transfer Axiom.

It says [Keisler, p.908],

> *Every real statement that holds for all real numbers holds for all hyper-real numbers.*

How can a generalization be restricted the same as the special case?

The Transfer Axiom is a guess that guarantees that conjecture from the special to the general, must be true, contradictory to logical deduction.

To ensure the validity of his theory, Robinson stated his

Extension Axiom

that claims amongst other things [Keisler, p. 906]

> *There is a positive hyper-real number that is smaller than any positive real number.*

In other words, Robinson's theory of Non Standard Analysis produces infinitesimals, by postulating them, and determines their properties by adding the right axioms.

This wrong answer to Leibnitz problem of infinitesimals demonstrates the divide between Robinson's Logic, and Mathematical Analysis.

Here, we observe that the solution to the infinitesimals problem was overlooked when the real numbers were constructed from Cauchy sequences.

That is, when we represent the Real Line as the infinite dimensional space of all the Cauchy sequences of rational numbers, we can construct families of infinitesimal hyper-reals.

In the following, we investigate the structure and properties of such families of infinitesimal hyper-reals, and construct the associated families of their reciprocals, the infinite hyper-reals.

1.

The Constant Hyper-reals

The construction of real numbers from the rationals is based on the fact that for any real number α there is a Cauchy sequence of rational numbers r_n that converges to α. Indeed, there are infinitely many such sequences

$$r_n \to \alpha,$$

and each real number α represents the equivalence class of the Cauchy sequences

$$< r_1, r_2, r_3, ... >$$

that converge to α.

Every real number α can be represented by the constant $\mathbb{R}^\infty$ vector $\begin{pmatrix} \alpha \\ \alpha \\ \alpha \\ \dots \end{pmatrix}$.

We identify the constant hyper-real with the <u>real number</u> α.

The $\mathbb{R}^\infty$ zero vector is identified with the real number 0, and it represents any sequence with components

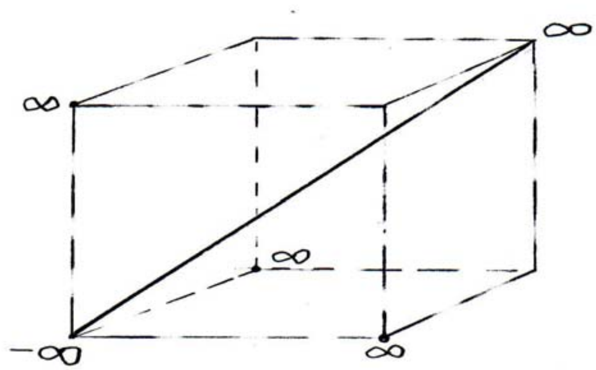
$$a_n = 0, \text{ for all } n > n_0.$$

The constant hyper-reals are on the diagonal of $\mathbb{R}^\infty$.

1.1 *Let* $\mathbb{R}$ *be the real line,*

and let $\mathbb{R}^\infty$ *be the infinite dimensional cube of side* $\mathbb{R}$.

Then, *<u>the constant hyper-reals are on the diagonal of the cube</u>*



A <u>non-constant hyper-real</u> will be defined as

<u>constant hyper-real + infinitesimal hyper-real</u>

We aim to show that each constant hyper-real is the center of an interval of hyper-reals, that includes no other constant hyper-real.

That is, we aim to show that the real numbers are separated from each other by intervals of non-constant hyper-reals.

To that end, we need to construct infinitesimal hyper-reals so that they, and the hyper-reals that are based on them, are totally, and linearly ordered.

2.

Infinitesimal Hyper-reals

To construct strictly positive infinitesimals, we restrict the Cauchy sequences that represent α to be monotonic.

Then, the convergence $r_n \downarrow \alpha$, can be written

$$\alpha + (r_n - \alpha) \downarrow \alpha,$$

where

$$r_n - \alpha \downarrow 0,$$

$$r_1 - \alpha > r_2 - \alpha > r_3 - \alpha > ... > r_n - \alpha > ... > 0.$$

The convergence $r_n \uparrow \alpha$, can be written

$$r_n + (\alpha - r_n) \downarrow \alpha,$$

where

$$\alpha - r_n \downarrow 0,$$

$$\alpha - r_1 > \alpha - r_2 > \alpha - r_3 > ... > \alpha - r_n > ... > 0.$$

To become positive infinitesimal hyper-reals, such monotonically decreasing-to-zero sequences need to be totally and linearly ordered like the real numbers.

We define the order between two positive monotonically decreasing-to-zero sequences ι=*iota*, and o=*omicron*, by

2.1 $\iota < o$ *means that for all* $n > n_0$, $\iota_n < o_n$

$\iota = o$ *means that for all* $n > n_0$, $\iota_n = o_n$

We require that

For every two infinitesimal hyper-reals ι, and o,

either $\iota < o$, or $\iota = o$ or $\iota > o$.

This requirement mandates that the definition of the infinitesimal hyper-reals involves at least two, and in fact, an infinite Family of Infinitesimals.

2.2 *A set of positive monotonically decreasing-to-zero sequences constitutes a Family of positive Infinitesimal Hyper-reals in* $\mathbb{R}^\infty$ *iff*

1. *the components of each infinitesimal hyper-real* ι *satisfy* $\iota_n \downarrow 0$,

$$\iota_1 > \iota_2 > \iota_3 > ... > \iota_n > ... > 0.$$

2. *for every two infinitesimal hyper-reals* ι*, and* o*,*

either $\iota < o$*, or* $\iota = o$ *or* $\iota > o$*.*

Such totally ordered family can be generated by one positive infinitesimal ι:

2.3 *Let* ι *be a positive monotonically decreasing to zero sequence. Then, linear combinations of the powers of* ι

$$\iota, \iota^2, \iota^3, ..., \iota^n, ...,$$

with positive scalar coefficients are a family of positive infinitesimals based on ι.

In the following we'll use the family based on $\iota = \left\langle \frac{1}{n} \right\rangle$:

That is,

2.4 *The linear combinations of* $\left\langle \frac{1}{n} \right\rangle, \left\langle \frac{1}{n^2} \right\rangle \left\langle \frac{1}{n^3} \right\rangle, ...$

with positive coefficients are a family of positive infinitesimals.

Alternatives are,

2.5 *Linear combinations of* $\left\langle \frac{1}{n^a} \right\rangle, \left\langle \frac{1}{n^{2a}} \right\rangle \left\langle \frac{1}{n^{3a}} \right\rangle, ..., a > 1$

with positive coefficients are a family of positive Infinitesimals.

2.6 *Linear combinations of* $\left\langle \frac{1}{2^n} \right\rangle, \left\langle \frac{1}{2^{2n}} \right\rangle, \left\langle \frac{1}{2^{3n}} \right\rangle, \ldots\ldots$ *with positive coefficients are a family of positive Infinitesimals.*

2.7 *Linear combinations of* $\left\langle \frac{1}{3^n} \right\rangle, \left\langle \frac{1}{3^{2n}} \right\rangle, \left\langle \frac{1}{3^{3n}} \right\rangle, \ldots\ldots$ *with positive coefficients are a family of positive Infinitesimals.*

2.8 *Linear combinations of* $\left\langle \frac{1}{4^n} \right\rangle, \left\langle \frac{1}{4^{2n}} \right\rangle, \left\langle \frac{1}{4^{3n}} \right\rangle, \ldots\ldots$ *with positive coefficients are a family of positive Infinitesimals.*

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

2.9 *Linear combinations of* $\left\langle \frac{1}{b^n} \right\rangle, \left\langle \frac{1}{b^{2n}} \right\rangle, \left\langle \frac{1}{b^{3n}} \right\rangle, \ldots\ldots, b > 1,$ *with positive coefficients are a family of positive Infinitesimals.*

2.10 *The infinitesimal hyper-reals are Non-Archimedean.*

Proof:

$\iota > \iota^2$. But there is no N_0, and no n_0 so that $N_0\iota_n^2$ is strictly greater than ι_n, for all $n > n_0$.□

2.11 $$dx > 0.$$

Proof: For any $n = 1,2,3...$, $\iota_n > 0$.□

Therefore,

2.12 $$-dx < 0.$$

That is, the convergence $r_n \uparrow \alpha$, that produces the infinitesimal dx, in a family of positive infinitesimals associated with the hyper-real line, can be written

$$\alpha + (r_n - \alpha) \uparrow \alpha.$$

Then,

$$r_n - \alpha \uparrow 0,$$

$$r_1 - \alpha < r_2 - \alpha < r_3 - \alpha < ... < r_n - \alpha < ... < 0,$$

produces the negative infinitesimal $-dx$.

2.13 *The hyper-reals include the family of negative infinitesimals. But, since zero is not an infinitesimal,*
The infinitesimals are only a *semi-group* *with respect to addition*.

2.14 *The positive Infinitesimals are Strictly Smaller than any positive real.*

Proof: Consider the positive infinitesimal ι so that

$$\iota_n \downarrow 0,$$

and

$$\iota_1 > \iota_2 > \iota_3 > ... > \iota_n > ... > 0.$$

If $\varepsilon > 0$ is any small real number, then there is a number $N(\varepsilon)$, so that for all $n > N(\varepsilon)$,

$$\varepsilon > \iota_n.$$

Thus,

$$\varepsilon = \langle \varepsilon, \varepsilon, \varepsilon, ... \rangle > \langle \iota_1, \iota_2, \iota_3, ... \rangle = \iota. \square$$

Therefore,

2.15 *The negative Infinitesimals are Strictly greater than any negative real.*

Consequently, zero is shielded from any reals by infinitesimals:

2.16 *Zero is Separated from Positive Reals by positive Infinitesimals, and from Negative Reals by negative infinitesimals.*

2.17 *For any positive infinitesimal* ι,

$$\iota > \iota^2 > \iota^3 > ... > \iota^n > ...0$$

and

$$-\iota < -\iota^2 < -\iota^3 < ... < -\iota^n < ... < 0$$

3.

The Infinitesimal line

3.1 *For each positive infinitesimal*

$$\iota = \left\langle \iota_1, \iota_2, \iota_3, ... \right\rangle,$$

and for each $t > 0$*,*

$$t\iota = \left\langle t\iota_1, t\iota_2, t\iota_3, ... \right\rangle$$

is a positive infinitesimal hyper-real,

and

$$-t\iota = \left\langle -t\iota_1, -t\iota_2, -t\iota_3, ... \right\rangle$$

is a negative infinitesimal hyper-real.

3.2 $\mathbb{R}_+\iota \equiv \left\{ t\iota : t > 0 \right\} = \left\{ \left\langle t\iota_1, t\iota_2, t\iota_3, ... \right\rangle : t > 0 \right\}$

$\mathbb{R}_-\iota \equiv \left\{ -t\iota : t > 0 \right\} = \left\{ t\iota : t < 0 \right\}$

are lines of infinitesimals.

3.3 $\mathbb{R}_+\iota$, and $\mathbb{R}_-\iota$ *are Totally ordered*.

Proof:

If $t_1\iota$, and $t_2\iota$ are two infinitesimals in $\mathbb{R}_+\iota$,

Then, either $t_2 > t_1 > 0 \Rightarrow t_2\iota > t_1\iota$

or $t_2 = t_1 \Rightarrow t_2\iota = t_1\iota$

or $0 < t_2 < t_1 \Rightarrow t_2\iota < t_1\iota$.□

4.

Infinitesimals Neighborhood

The total order of the infinitesimal hyper-reals says that they are on a line.

But in an infinite dimensional cube of side $\mathbb{R}_{+}\iota$, each infinitesimal line through the constant hyper-real zero, has a different direction:

For instance, the infinitesimal hyper-real

$$\left\langle 1, \frac{1}{2}, \frac{1}{3}, \ldots \right\rangle$$

has l^2 length

$$\sqrt{1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots} = \frac{\pi}{\sqrt{6}},$$

and its direction cosines are

$$\left\langle \frac{\sqrt{6}}{\pi}, \frac{\sqrt{6}}{2\pi}, \frac{\sqrt{6}}{3\pi}, \ldots \right\rangle.$$

The infinitesimal hyper-real

$$\left\langle 1, \frac{1}{2^2}, \frac{1}{3^2}, \ldots \right\rangle$$

has l^2 length

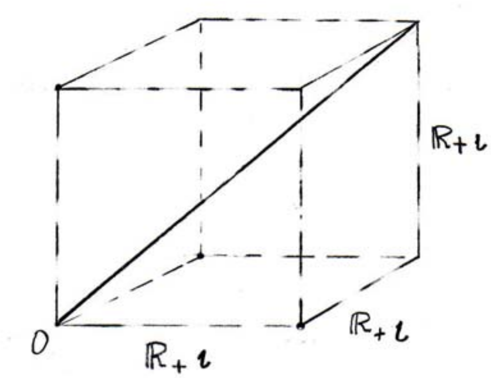
$$\sqrt{1+\frac{1}{2^4}+\frac{1}{3^4}+...}=\frac{\pi^2}{\sqrt{90}},$$

and its direction cosines are

$$\left\langle \frac{\sqrt{90}}{\pi^2},\frac{\sqrt{90}}{4\pi^2},\frac{\sqrt{90}}{9\pi^2},...\right\rangle.$$

Therefore, these two infinitesimals point in different directions from the zero at the vertex of the cube, and we are inclined to describe the infinitesimal hyper-real as a cloud of points in the neighborhood of zero.

4.1 *Let* $\mathbb{R}_+\iota$ *be the infinitesimal hyper-reals interval, and let* $(\mathbb{R}_+\iota)^\infty$ *be the infinite dimensional cube of side* $\mathbb{R}_+\iota$*. Then, the infinitesimal hyper-reals in the cube point in different directions from the constant hyper-real zero .*



5.

The Non-Constant Hyper-reals

We define the order between $\alpha + \iota$, and $\beta + o$, by

5.1 $\alpha + \iota < \beta + o$,

iff $\alpha < \beta$,

or if $\alpha = \beta$, and $\iota < o$

$\alpha + \iota = \beta + o$,

iff $\alpha = \beta$ and $\iota = o$

5.2 For any infinitesimal ι, *each hyper-real in*

$$\alpha + \mathbb{R}_{+}\iota = \left\{ \alpha + t\iota : t > 0 \right\},$$

or in

$$\alpha + \mathbb{R}_{\_}\iota$$

is a non-constant hyper-real

5.3
$$\alpha + \iota > \alpha + \iota^2 > \alpha + \iota^3 > ..... > \alpha$$

$$\alpha - \iota < \alpha - \iota^2 < \alpha - \iota^3 < .... < \alpha$$

Since $\mathbb{R}_{+}\iota$ and are Totally Ordered,

5.4 $\alpha + \mathbb{R}_+\iota$ *and* $\alpha + \mathbb{R}_-\iota$ *are Totally Ordered*

Proof:

If $\alpha + t_1\iota$, and $\alpha + t_2\iota$ are two non-constant hyper-reals in $\alpha + \mathbb{R}_+\iota$,

Then, either $t_2\iota > t_1\iota \Rightarrow \alpha + t_2\iota > \alpha + t_1\iota$

or $t_2\iota = t_1\iota \Rightarrow \alpha + t_2\iota = \alpha + t_1\iota$

or $t_2\iota < t_1\iota \Rightarrow \alpha + t_2\iota < \alpha + t_1\iota$.$\square$

6.

The Infinite Hyper-Reals

6.1 $$\frac{1}{\iota} = \left\langle \frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3} ..... \right\rangle$$

is the infinite hyper-real reciprocal to $\iota = \left\langle \iota_1, \iota_2, \iota_3, ... \right\rangle$

6.2 *If the family of infinitesimals is* **2.4***, the associated family of infinite hyper-reals are the divergent sequences*

$$< n >, < n^2 >, < n^3 > ...$$

We define the order between infinite hyper-reals by

6.3 $\frac{1}{\iota} < \frac{1}{o}$ means that for all $n > n_0$, $o_n < \iota_n$

$\frac{1}{\iota} = \frac{1}{o}$ means that for all $n > n_0$, $\iota_n = o_n$

6.4 *The infinite hyper-reals are Non-Archimedean*

Proof: Since the infinitesimals are Non-Archimedean, if

$k < K$, then

$$\iota_n{}^K < \iota_n{}^k,$$

and there is no N_0 so that $N_0 \iota_n^K$ is strictly greater than ι_n^k,

for all $n > n_0$.

Therefore,

$$\left(\frac{1}{\iota_n}\right)^k < \left(\frac{1}{\iota_n}\right)^K,$$

and there is no N_0 so that $N_0 \dfrac{1}{\iota_n^k}$ is strictly greater than $\dfrac{1}{\iota_n^K}$,

for all $n > n_0$.□

6.5 *The infinite hyper-reals are totally ordered*

Proof: Since the infinitesimals are totally ordered,

then, either $\iota > o \Rightarrow \dfrac{1}{o} > \dfrac{1}{\iota}$,

or $\iota = o \Rightarrow \dfrac{1}{o} = \dfrac{1}{\iota}$,

or $\iota < o \Rightarrow \dfrac{1}{o} < \dfrac{1}{\iota}$.□

6.6 *Infinite hyper-reals are greater than any real number*

Proof: For any large number N, there is $n_0(N)$, so that for all $n > n_0$, $\iota_n < \frac{1}{N}$.

That is, $N < \frac{1}{\iota_n}$.

Hence, $\frac{1}{\iota} = \left\langle \frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ... \right\rangle > \left\langle N, N, N, ... \right\rangle . \square$

6.7 *Any infinite hyper-real is Strictly Smaller than* ∞.

Proof: Since for any $n = 1, 2, 3...$,

$$\iota_n > 0,$$

we have

$$\frac{1}{\iota_n} < \infty .$$

That is,

$$\frac{1}{\iota} = \left\langle \frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3} ..... \right\rangle < \left\langle \infty, \infty, \infty, ... \right\rangle . \square$$

Similarly,

6.8 *Any negative infinite hyper-real is Strictly Greater than* $-\infty$.

This distinction between infinite hyper-reals and Infinity is crucial to the development of Infinitesimal Calculus, as is the distinction between infinitesimals and Zero.

Infinitesimal Calculus is based on these two distinctions:

For any infinitesimal, dx, we have

$$dx > 0,$$

and

$$\frac{1}{dx} < \infty.$$

In [Dan], we established the equality of all positive infinities:

We proved that

the number of the Natural Numbers, $Card\mathbb{N}$,

equals

the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$,

and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty.$$

However, none of the infinite hyper-reals equals the constant hyper-real infinity

$$\langle \infty, \infty, \infty, ... \rangle,$$

and none may be identified with it.

6.9 *For any infinitesimal* ι*,*

$$\frac{1}{\iota} < \frac{1}{\iota^2} < .... < \frac{1}{\iota^n} < ... < \infty,$$

$$-\frac{1}{\iota} > -\frac{1}{\iota^2} > .... > -\frac{1}{\iota^n} > ... > -\infty.$$

7.

The Hyper-real Line

The total order defined on the hyper-reals, aligns them on a line: the Hyper-real Line.

That line, includes the real numbers separated by the non-constant hyper-reals.

In particular,

zero is separated from any positive real by a family of infinitesimals, and from any negative real by these infinitesimals with negative signs, $-dx$.

Zero is not an infinitesimal, because zero is not strictly greater than zero.

The positive hyper-reals include

the infinite hyper-reals that are strictly smaller than Infinity.

The negative hyper-reals include

the infinite hyper-reals with negative signs, that are strictly greater than $-\infty$.

We do not add infinity to the hyper-real line, and

no infinite hyper-real equals infinity.

The infinitesimals, and the infinite hyper-reals are semi-groups with respect to addition. Neither set includes zero.

The hyper-real line is embedded in $\mathbb{R}^{\infty}$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the hyper-real onto the real line.

In particular,

there are no points on the real line that can be assigned uniquely to

the infinitesimal hyper-reals,

or to

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or to

the non-constant hyper-reals.

Thus, the real line is inadequate for Infinitesimal Calculus.

Consequently,

whenever only infinitesimals or infinite hyper-reals support the derivation of a result, the Calculus of Limits on the real line, fails to deliver that result.

No neighbourhood of a hyper-real is homeomorphic to an $\mathbb{R}^n$ ball. Therefore, the hyper-real line is not a manifold.

The hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

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