Infinitesimal Calculus

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Abstract The controversy surrounding the infinitesimals, obstructed the development of the Infinitesimal Calculus.

Postulating infinitesimals, only revealed Logic's inapplicability to Mathematical Analysis, where claims must be proved, and unproven claims are ignored.

Recently we have shown that when the Real Line is represented as the infinite dimensional space of all the Cauchy sequences of rational numbers, the hyper-reals are spanned by the constant hyper-reals, a family of infinitesimal hyper-reals, and the associated family of infinite hyper-reals.

The infinitesimal hyper-reals are smaller than any real number, yet bigger than zero.

The reciprocals of the infinitesimal hyper-reals are the infinite hyper-reals. They are greater than any real number, yet strictly smaller than infinity.

A neighborhood of infinitesimals separates the zero hyper-real from the reals, and each real number is the center of an interval of hyper-reals, that includes no other real number.
The Hyper-reals are totally ordered, and are lined up on a line, the hyper-real line.
A hyper-real function is a mapping from the hyper-real line into the hyper-real line.
Infinitesimal Calculus is the Calculus of hyper-real functions.
Infinitesimal Calculus is far more precise, and effective than the ε,δ Calculus.
In particular, Infinitesimal Calculus enables us to differentiate over a discontinuity jump, as large as an infinite hyper-real, and integrate over such discontinuities

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Contents

Introduction

0.1 The origin of Infinitesimal Calculus

Leibnitz wrote the derivative as a quotient of two differentials,

$$f'(x) = \frac{df}{dx}.$$

The differential df is the difference between two extremely close values of f, attained on two extremely close values of x. The differential dx is the difference between those values of x. Since $f'(x)$ may vanish, $df = f'(x)dx$ can vanish. But dx cannot vanish because division by zero is undefined.

The differential dx is an infinitesimal. it is smaller than any real number, yet it is greater than zero.

That characterization was met with understandable scepticism.

How can there be numbers that are smaller than any number and yet are greater than zero?

Euler who used differentials, avoided this fundamental question, and did not develop the infinitesimal Calculus.

Calculus after Euler, avoided the use of infinitesimals. Even the theory of differential equations is inundated with ε, δ arguments, and the Infinitesimal Calculus remained undeveloped.

0.2 Postulating Infinitesimals

Attempts to prove the existence of infinitesimals by Schmieden, and Laugwitz [Laug] were flawed. They assumed that what is true for natural numbers must hold true for infinities. But while for any natural number n,

$$n < n + n,$$

it is well known that

$$Card\mathbb{N} = Card\mathbb{N} + Card\mathbb{N}$$

In 1961, Robinson assumed that infinitesimals exist, and with that assumption managed to prove that they exist.

Robinson first produced, in the spirit of the Schmieden and Laugwitz Postulate, the

Transfer Axiom.

It says [Keisler, p.908],

> *Every real statement that holds for all real numbers holds for all hyper-real numbers.*

The Transfer Axiom is a guess that guarantees that conjecture from the special to the general, must be true, contradictory to logical deduction.

To ensure the validity of his theory, Robinson stated his

Extension Axiom.

It claims amongst other things [Keisler, p. 906]

> *There is a positive hyper-real number that is smaller than any positive real number.*

In short, Robinson's theory of Non Standard Analysis establishes infinitesimals, by postulating them, and their properties.

The postulated infinitesimals were ignored in Mathematical Analysis, and were not followed by a mathematical development of Infinitesimal Calculus.

The Infinitesimal Calculus books by Hoskins, and Keisler restate the known ε, δ Calculus, while missing the precise, and powerful, tools and results of Infinitesimal Calculus.

0.3 Hyper-reals, and Infinitesimals

In [Dan 2] we observed that the solution to the infinitesimals problem was overlooked when the real numbers were constructed from Cauchy sequences.

When we represent the Real Line as the infinite dimensional space of all the Cauchy sequences of rational numbers, the hyper-reals are spanned by the constant hyper-reals, a family of infinitesimal hyper-reals, and the associated family of infinite hyper-reals.

The infinitesimal hyper-reals are smaller than any real number, yet bigger than zero.

The reciprocals of the infinitesimal hyper-reals are the infinite hyper-reals. They are greater than any real number, yet strictly smaller than infinity.

Each real number is the center of an interval of hyper-reals, that includes no other real number.

The hyper-reals are totally, and linearly ordered.

0.4 The ε Substitute, and the ε, δ Calculus

The ε, δ Calculus was created to replace the infinitesimals, that elude us on the real line because they are not there.

In the ε, δ Calculus, the limit process is detailed into the existence of an $\varepsilon > 0$, that is decreasing to zero, while remaining positive.

To steer clear from infinitesimals, a statement would be

“ For every $\varepsilon > 0$,...”,

while the meaning in plain words is

“For $\varepsilon > 0$, which is small beyond imagination…”

Indeed, only that kind of ε, will keep the quotient difference

$$\frac{f(x+\varepsilon)-f(x)}{\varepsilon}$$

away from division by zero.

But such ε is precisely an infinitesimal, and finding it on the real line, were it does not exist, amounts to postulating infinitesimals on the real line.

The ε, δ Calculus amounts to restatement of Euler-time infinitesimals, in an awkward legal language.

0.5 The Precision of Infinitesimal Calculus

We proceed to present the Infinitesimal Calculus of Functions that map the Hyper-reals into the Hyper-reals.

To check continuity, we focus on the oscillation of a Hyper-real function over an infinitesimal interval.

Differentiation in the ε, δ Calculus, is undefined over a jump discontinuity.

In infinitesimal Calculus, we can differentiate over jump discontinuities, that may be as large as infinite hyper-reals.

The derivative is sharply defined in terms of infinitesimals, compared with its definition in terms of limits.

Integration in the ε, δ Calculus is undefined over jump discontinuities, and for unbounded functions.

In infinitesimal Calculus, using infinite Riemann Sums, we can integrate over jump discontinuities, that may be as large as infinite hyper-reals.

Infinitesimal Calculus is far more effective than the ε, δ Calculus, because being based on almost zero numbers, it allows us to deal with their reciprocals, the almost infinite numbers. We have no use for infinity by itself, but to comprehend the effects of singularities, we have use for the almost infinite.

Infinitesimals are a precise tool compared to the vague limit concept, and the awkward ε, δ statements.

Infinitesimal Calculus is the Calculus of hyper-real functions.

The minimal domain and range, needed for the definition and analysis of a hyper-real function, is the hyper-real line.

1.

Hyper-real Line

Each real number α can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of infinitesimal hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the infinite hyper-reals.
4. The infinite hyper-reals are greater than any real number, yet strictly smaller than infinity.
5. The infinite hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a

non-constant hyper-real.

7. The Hyper-reals are the totality of constant hyper-reals, a family of infinitesimals, a family of infinitesimals with negative sign, a family of infinite hyper-reals, a family of infinite hyper-reals with negative sign, and non-constant hyper-reals.
8. The hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.
9. That line includes the real numbers separated by the non-constant hyper-reals. Each real number is the center of an interval of hyper-reals, that includes no other real number.
10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.
11. Zero is not an infinitesimal, because zero is not strictly greater than zero.
12. We do not add infinity to the hyper-real line.
13. The infinitesimals, the infinitesimals with negative signs, the infinite hyper-reals, and the infinite hyper-reals with negative signs are semi-groups with

respect to addition. Neither set includes zero.

14. The hyper-real line is embedded in $\mathbb{R}^{\infty}$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the hyper-real onto the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal hyper-reals, or to the infinite hyper-reals, or to the non-constant hyper-reals.

16. No neighbourhood of a hyper-real is homeomorphic to an $\mathbb{R}^{n}$ ball. Therefore, the hyper-real line is not a manifold.

17. The hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

2.

Hyper-real Function

2.1 Definition of a hyper-real function

$f(x)$ is a hyper-real function, iff it is from the hyper-reals into the hyper-reals.

This means that any number in the domain, or in the range of a hyper-real $f(x)$ is either one of the following

real

real + infinitesimal

real – infinitesimal

infinitesimal

infinitesimal with negative sign

infinite hyper-real

infinite hyper-real with negative sign

Clearly,

2.2 *Every function from the reals into the reals is a hyper-real function.*

2.3 $\dfrac{\sin(dx)}{dx}$ **has the constant hyper-real value** 1

Proof: $\sin(dx) = dx - \dfrac{(dx)^3}{3!} + \dfrac{(dx)^5}{5!} - ...$

$$\frac{\sin(dx)}{dx} = 1 - \frac{(dx)^2}{3!} + \frac{(dx)^4}{5!} - ...$$

2.4 $\cos(dx)$ **has the constant hyper-real value** 1

Proof: $\cos(dx) = 1 - \dfrac{(dx)^2}{2!} + \dfrac{(dx)^4}{4!} - ...$

2.5 e^{dx} **has the constant hyper-real value** 1

Proof: $e^{dx} = 1 + dx + \dfrac{(dx)^2}{2!} + \dfrac{(dx)^3}{3!} + \dfrac{(dx)^4}{4!} + ...$

2.6 $e^{\frac{1}{dx}}$ **has an infinite hyper-real value** $> \frac{1}{(dx)^k}$**, for any** k**.**

Proof: $e^{\frac{1}{dx}} = 1 + \frac{1}{dx} + \frac{1}{2!(dx)^2} + \frac{1}{3!(dx)^3} + ... > \frac{1}{(dx)^k}$, for any k.

2.7 $\left\langle e^n \right\rangle > \left\langle n^k \right\rangle$**, for any** k**.**

Proof: $e^n = 1 + n + \frac{1}{2!}n^2 + \frac{1}{3!}n^3 + ... > n^k$, for any k.

2.8 $\log(dx)$ **has an infinite hyper-real value** $-\left\langle \log n \right\rangle > -\frac{1}{dx}$

Proof: For the infinitesimal $dx = \left\langle \frac{1}{n} \right\rangle$,

$$\log(dx) = \log\left\langle \frac{1}{n} \right\rangle = -\left\langle \log n \right\rangle > -\left\langle n \right\rangle = -\frac{1}{dx}$$

3.

Oscillation

3.1 The oscillation of $f(x)$ over $[x_0 - \frac{dx}{2}, x_0 + \frac{dx}{2}]$

The <u>oscillation</u> of a hyper-real function $f(x)$, over the infinitesimal interval $[x_0 - \frac{dx}{2}, x_0 + \frac{dx}{2}]$ is

$$\sup_{[x_0 - \frac{dx}{2}, x_0 + \frac{dx}{2}]} f(t) - \inf_{[x_0 - \frac{dx}{2}, x_0 + \frac{dx}{2}]} f(s)$$

where the sup, and the inf are taken over the interval.

3.2 Continuity Definition

We require that the oscillation of the hyper-real function, over the infinitesimal interval $[x_0 - \frac{dx}{2}, x_0 + \frac{dx}{2}]$ will be infinitesimal. That is,

$f(x)$ is continuous at x_0 iff for any infinitesimal dx,

$$\sup_{[x_0 - \frac{dx}{2}, x_0 + \frac{dx}{2}]} f(t) - \inf_{[x_0 - \frac{dx}{2}, x_0 + \frac{dx}{2}]} f(s) = \text{infinitesimal}.$$

3.3 Continuity Example

$f(x) = x^2$ is continuous at $x = 1$, because for any dx,

$$\sup_{[1-\frac{dx}{2},1+\frac{dx}{2}]} f(t) - \inf_{[1-\frac{dx}{2},1+\frac{dx}{2}]} f(s) = (1+\tfrac{1}{2}dx)^2 - (1-\tfrac{1}{2}dx)^2$$

$$= 2dx\,.\square$$

3.4 Jump Discontinuity Example

The step function

$$f(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases}$$

is discontinuous at $x = 0$, because for any dx,

$$\sup_{[-\frac{dx}{2},\frac{dx}{2}]} f(t) - \inf_{[-\frac{dx}{2},\frac{dx}{2}]} f(s) = 1\,.\square$$

3.5 Comparison with ε,δ continuity

Continuity at $x = x_0$, based on the oscillation of the function $f(x)$ over a shrinking interval $(x_0 - \delta, x_0 + \delta)$ involves two limit processes:

We say that $f(x)$ is continuous at x_0

iff for any $\delta > 0$, with $\delta \downarrow 0$,

$$\sup_{[x_0-\delta,x_0+\delta]} f(t) - \inf_{[x_0-\delta,x_0+\delta]} f(s) \downarrow 0.$$

This statement requires imagining numbers that remain positive while they are shrinking to zero. In other words, it requires infinitesimals. But there are no infinitesimals on the real line, and it is difficult to imagine them on the real line.

Consequently, at some point, the endless process of approaching an unimaginable limit defeats the mind, and the limit concept remains foggy, imprecise, and ill-defined.

Moreover, to ensure the dependence of the two limit processes on each other, we need to restate this in ε, δ terms.

In these terms we say that

For any $\varepsilon > 0$, as small as it my be,

we can find small enough $\delta(\varepsilon) > 0$, so that

$$\sup_{[x_0-\delta, x_0+\delta]} f(t) - \inf_{[x_0-\delta, x_0+\delta]} f(s)) < \varepsilon .$$

The comprehension of the two entangled limit processes, is even more hopeless.

4.

$f(x) = \dfrac{1}{x}$, $x \neq 0$

In the Calculus of Limits, the function $f(x) = \dfrac{1}{x}$ is defined for all $x \neq 0$.

We avoid $x = 0$, because the oscillation of $f(x) = \dfrac{1}{x}$ over an interval that includes $x = 0$, is infinite.

Avoiding $x = 0$, still allows the statement

$$x \downarrow 0 \Rightarrow \frac{1}{x} \uparrow \infty,$$

which is imprecise due to the appearance of ∞ on the right.

It follows that the domain of $f(x) = \dfrac{1}{x}$ has to avoid an <u>interval</u> that includes $x = 0$.

Consequently, our conception of the function close to its singularity is imprecise.

4.1 *In the Calculus of limits,* $f(x) = \dfrac{1}{x}$ *, is precisely defined only out of an interval about* 0

In Infinitesimal Calculus, infinite hyper-reals give a precise value.

For instance, the oscillation of $\dfrac{1}{x}$ over $[\tfrac{1}{2}dx, \tfrac{3}{2}dx]$ is

$$\tfrac{4}{3}\frac{1}{dx} < \infty\,.$$

Therefore,

4.2 *In Infinitesimal Calculus, The Hyper-real function*

$f(x) = \dfrac{1}{x}$, is precisely defined for any $x \neq 0$.

5.

$\log x$, $x \neq 0$

In the Calculus of Limits, the function $f(x) = \log x$ is defined for all $x \neq 0$.

We avoid $x = 0$, because the oscillation of $f(x) = \log x$ over an interval that includes $x = 0$, is infinite.

However, for $\varepsilon > 0$,

$$-\frac{1}{2}\log\varepsilon = \frac{1-\varepsilon}{1+\varepsilon} + \frac{1}{3}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^3 + \frac{1}{5}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^5 + ...$$

To first order $\dfrac{1}{1+\varepsilon} \approx 1 - \varepsilon$, and we have,

$$-\frac{1}{2}\log\varepsilon \approx 1 - 2\varepsilon + \left(\frac{1}{3} - 2\varepsilon\right) + \left(\frac{1}{5} - 2\varepsilon\right) + ...$$

$$\log\varepsilon \approx -2 + 4\varepsilon - 2\left(\frac{1}{3} - 2\varepsilon\right) - 2\left(\frac{1}{5} - 2\varepsilon\right) + ...$$

Therefore,

$$\varepsilon \downarrow 0 \Rightarrow \log\varepsilon \downarrow -\infty,$$

which is imprecise due to the appearance of ∞ on the right.

It follows that the domain of $f(x) = \log x$ has to avoid an interval that includes $x = 0$.

5.1 *In the Calculus of limits,* $f(x) = \log x$ *, is precisely defined only out of an interval about* 0

In Infinitesimal Calculus, infinite hyper-reals give a precise value.

For instance, the oscillation of $\log x$ over $[\frac{1}{2}dx, \frac{3}{2}dx]$ is

$$\log 3.$$

Consequently,

5.2 *In Infinitesimal Calculus, The Hyper-real function* $f(x) = \log x$*, is defined for any* $x \neq 0$.

6.

Derivative

6.1 Left Derivative

Hyper-real $f(x)$ defined at x_0, has a Left Derivative at x_0, $\mathrm{D}f(x_0-)$, <u>iff</u> for any infinitesimal dx, the slope of the chord over $[x_0 - dx, x_0]$,

$$\frac{f(x_0) - f(x_0 - dx)}{dx},$$

equals a unique hyper-real number.

If the hyper-real is infinite, it is denoted $\mathrm{D}f(x_0-)$.

If the hyper-real is finite, its constant part is $\mathrm{D}f(x_0-)$.

6.2 Left Derivative of $|x|$ at $x = 0$

$$\frac{d}{dx}|x|\Big|_{x=0-} = -1$$

Proof: For any dx, $\dfrac{f(0) - f(-dx)}{dx} = \dfrac{0 - |-dx|}{dx} = -1$.□

6.3 Right Derivative

Hyper-real $f(x)$ defined at x_0, has a Right Derivative at x_0, $\mathrm{D}f(x_0+)$, iff for any infinitesimal dx, the slope of the chord over $[x_0, x_0 + dx]$,

$$\frac{f(x_0 + dx) - f(x_0)}{dx},$$

equals a unique hyper-real number.

If the hyper-real is infinite, it equals $\mathrm{D}f(x_0+)$.

If the hyper-real is finite, its constant part equals $\mathrm{D}f(x_0+)$.

6.4 Right Derivative of $|x|$ at $x = 0$

$$\frac{d}{dx}|x|\Big|_{x=0+} = 1$$

Proof: For any dx, $\dfrac{f(dx) - f(0)}{dx} = \dfrac{dx - 0}{dx} = 1.\square$

6.5 Derivative

A hyper-real function $f(x)$ defined at x_0, has a derivative at x_0, $\mathrm{D}f(x_0)$, iff both its left and right derivatives at x_0 exist, and are equal. Then,

$$\mathrm{D}f(x_0-) = \mathrm{D}f(x_0+)$$

is the derivative $\mathrm{D}f(x_0)$.

6.6 $|x|$ has no derivative at $x = 0$

Proof: The right and left derivatives at $x = 0$ are unequal.

6.7 Derivative of x^3 at $x = 1$, Example

For any dx,

$$\frac{(1)^3 - (1 - dx)^3}{dx} = 3 - 3dx + (dx)^2 \Rightarrow \mathrm{D}x^3\Big|_{x=1-} = 3.$$

$$\frac{(1 + dx)^3 - (1)^3}{dx} = 3 + 3dx + (dx)^2 \Rightarrow \mathrm{D}x^3\Big|_{x=1+} = 3.$$

Therefore, $\mathrm{D}x^3\Big|_{x=1} = 3.\square$

6.8 Fermat Theorem in Infinitesimal Calculus

If $f(x)$ *differentiable on* (a,b), *has maximum at* $c \in (a,b)$

Then $f'(c) = 0$.

Proof: for any infinitesimal dx,

$$f'(c) = f'(c-) = \frac{f(c) - f(c - dx)}{dx} \geq 0,$$

and

$$f\,'(c) = f\,'(c+) = \frac{f(c+dx) - f(c)}{dx} \le 0\,.$$

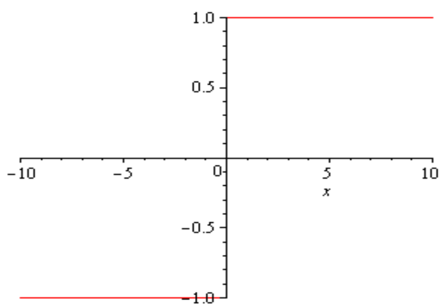
Therefore, $f\,'(c) = 0\,.\square$

7.

Derivative of the Sign Function is $\delta(x)$

7.1 Sign Function Definition

$$g(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$



7.2 Sign Function Derivative

$$\mathrm{D}g(0) = \frac{1}{dx}$$

$$\mathrm{D}g(x) = \frac{1}{dx}\chi\left[-\tfrac{dx}{2}, \tfrac{dx}{2}\right]$$

Proof:

At the jump over $[-dx, 0]$, from -1 to 0, for any dx,

$$\frac{g(0)-g(0-dx)}{dx}=\frac{0-(-1)}{dx}=\frac{1}{dx}$$

Therefore, the left derivative at $x=0$ is

$$\mathrm{D}g(0-)=\frac{1}{dx}.$$

At the jump over $[0, dx]$, from 0 to 1, for any dx,

$$\frac{g(0+dx)-g(0)}{dx}=\frac{1-0}{dx}=\frac{1}{dx}$$

Therefore, the right derivative at $x=0$ is

$$\mathrm{D}g(0+)=\frac{1}{dx}.$$

Since the right and left derivatives are equal, the derivative at $x=0$ is

$$\mathrm{D}g(0)=\frac{1}{dx}.\square$$

Therefore,

$$\text{p.v.}\mathrm{D}g(0)=\frac{1}{dx}.$$

That is, for any dx, the slope of the chord over $[-\frac{dx}{2},\frac{dx}{2}]$

$$\frac{g(\frac{dx}{2})-g(-\frac{dx}{2})}{dx}=\frac{1}{dx}$$

Thus, over $\left[-\frac{dx}{2},\frac{dx}{2}\right]$

$$\mathrm{D}g(x) = \frac{1}{dx}.$$

This means that

at $-\frac{dx}{2}$, $\mathrm{D}g(x)$ jumps from 0 to $\frac{1}{dx}$,

at $\frac{dx}{2}$, $\mathrm{D}g(x)$ drops from $\frac{1}{dx}$ to 0.

at $x \neq 0$, $\mathrm{D}g(x) = 0$.

Using

$$\chi\left[-\tfrac{dx}{2}, \tfrac{dx}{2}\right] = \begin{cases} 1, x \in \left[-\tfrac{dx}{2}, \tfrac{dx}{2}\right] \\ \\ 0, \text{otherwise} \end{cases}$$

we have

$$\mathrm{D}g(x) = \frac{1}{dx}\chi\left[-\tfrac{dx}{2}, \tfrac{dx}{2}\right].\square$$

7.3 Delta Function $\delta(x)$ Definition

The derivative of the sign function in Infinitesimal Calculus is the Hyper-real Function, Delta Function $\delta(x)$

$$\delta(x) = \frac{1}{dx}\chi\left[-\tfrac{dx}{2}, \tfrac{dx}{2}\right]$$

8.

Principal Value Derivative

Principal Value Derivative fails to be effective in the Calculus of Limits, and is not applied there.

In Infinitesimal Calculus, the Principal Value Derivative, can be effectively applied.

8.1 Definition of Principal Value Derivative in the Calculus of Limits

In the Calculus of Limits, we consider the quotient difference

$$\frac{f(x_0+\frac{\delta}{2})-f(x_0-\frac{\delta}{2})}{(x_0+\frac{\delta}{2})-(x_0-\frac{\delta}{2})}=\frac{f(x_0+\frac{\delta}{2})-f(x_0-\frac{\delta}{2})}{\delta}.$$

If it has a limit for $0<\delta\downarrow 0$, then that limit is the Principal Value Derivative of $f(x)$ at $x=x_0$, $\text{p.v.D}f(x_0)$

8.2 Principal Value Derivative is ineffective in the Calculus of Limits

$|x|$ *has no Principal Value Derivative at* $x=0$.

Proof:

$$\frac{f(x_0 + \frac{\delta}{2}) - f(x_0 - \frac{\delta}{2})}{\delta} = \frac{\left|0 + \frac{\delta}{2}\right| - \left|0 - \frac{\delta}{2}\right|}{\delta} = \frac{0}{\delta}.$$

Since the limits of the numerator, and the denominator exist, we have

$$\lim_{\delta \to 0} \frac{0}{\delta} = \frac{0}{\lim_{\delta \to 0} \delta}.$$

the Calculus of Limits mandates that as $\delta \downarrow 0$, we have

$$\lim_{\delta \to 0} \delta = 0.$$

Thus, we obtain $\dfrac{0}{0}$, the limit does not exist, and $\left|x\right|$ has no Principal Value Derivative at $x = 0$, in the Calculus of Limits.

We cannot argue that $\delta > 0$, and $\dfrac{0}{\delta} = 0$, because that would mean that δ an infinitesimal, and there are no infinitesimals on the real line.

So short of postulating that $\lim\limits_{\delta \to 0} \dfrac{0}{\delta}$ is zero, we must conclude that there is no limit.

The Limit concept is inadequate for singularities, and fails when applied to the Principal value Derivative.

In Infinitesimal Calculus the Principal Value Derivative is an effective concept.

8.3 Definition of Principal Value Derivative in Infinitesimal Calculus

Hyper-real $f(x)$ has a Principal Value Derivative at $x = x_0$ <u>iff</u> for any infinitesimal dx, the slope of the chord over $[x_0 - \frac{dx}{2}, x_0 + \frac{dx}{2}]$

$$\frac{f(x_0 + \frac{dx}{2}) - f(x_0 - \frac{dx}{2})}{dx}$$

equals a unique hyper-real number

If the hyper-real is infinite, it is denoted $\text{p.v.D}f(x_0)$.

If the hyper-real is finite, its constant part is $\text{p.v.D}f(x_0)$

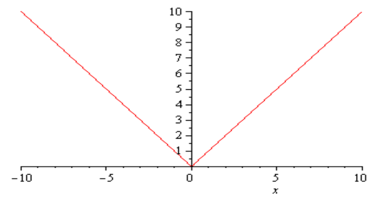
8.4 $\text{p.v.D}\left|x\right|_{x=0} = 0$

Proof: For any dx, $\dfrac{\left|0 + \frac{dx}{2}\right| - \left|0 - \frac{dx}{2}\right|}{dx} = \dfrac{0}{dx} = 0 . \square$

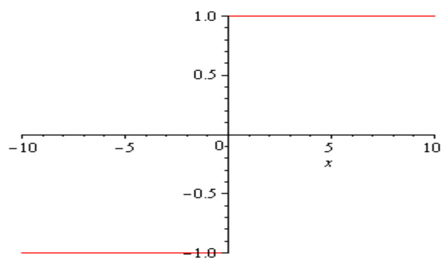
8.5 Principal Value Derivative of $\left|x\right|$ is the Sign

Function

If $f(x) = \left| x \right|$,



Then, $$\text{p.v.D}\left| x \right| = \begin{cases} -1, x < 0 \\ 0, x = 0 \\ 1, x > 0 \end{cases}$$



9.

Principal Value Derivative of the Step Function is $\delta(x)$

9.1 Step Function Definition

$$h(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases},$$

9.2 The Step Function has no derivative at $x = 0$

Proof: For any dx,

$$\frac{h(0) - h(-dx)}{dx} = \frac{0}{dx} = 0 \qquad \Rightarrow h'(0-) = 0.$$

$$\frac{h(dx) - h(0)}{dx} = \frac{1-0}{dx} = \frac{1}{dx} \qquad \Rightarrow h'(0+) = \frac{1}{dx}.$$

Therefore, the Step Function has no derivative at $x = 0$.□

But the Principal Value derivative of the step Function at $x = 0$ exists.

9.3 $$\text{p.v.D}h(0) = \frac{1}{dx}$$

$$\text{p.v.D}h(x) = \frac{1}{dx}\chi\left[-\tfrac{dx}{2}, \tfrac{dx}{2}\right] = \delta(x)$$

Proof:

$$\frac{h(\frac{dx}{2}) - h(-\frac{dx}{2})}{dx} = \frac{1-0}{dx} = \frac{1}{dx}.$$

That is

$$\text{p.v.D}h(0) = \frac{1}{dx}$$

and

$$\text{p.v.D}h(x) = \frac{1}{dx}\chi\left[-\tfrac{dx}{2}, \tfrac{dx}{2}\right] = \delta(x).\square$$

10.

Differentials and Infinitesimal Functions

10.1 Differential of the hyper-real function $f(x)$

The differential is the Hyper-real function, which is the product of the infinitesimal dx *by the derivative* $f'(x)$

$$df(x) = f'(x)dx.$$

10.2 *We use the same* dx *in the definition of all differentials.*

For instance, $d(x^2) = 2xdx$; $d(e^x) = e^x dx$;

Since $df(x)$ vanishes at the zeros of $f'(x)$, it follows that

10.3 $df(x)$ **is not an infinitesimal**

10.4 *We call the product of a hyper-real function by an infinitesimal an* **infinitesimal function**.

10.5 *Infinitesimal function may not be an infinitesimal*

11.

Impulse Function in the Calculus of Limits

In infinitesimal Calculus, an Impulse function is the hyper-real function

$$\delta(x) \equiv \frac{1}{dx}\chi\left[-\tfrac{dx}{2},\tfrac{dx}{2}\right],$$

that is the Principal value Derivative of the Step Function.

The origin of the Delta function is in the Calculus of Limits.

There, infinitesimals do not exist, and the Delta spike is defined as the limit of a sequence of narrowing pulses

11.1 Definition of an Impulse Function in the Calculus of Limits

In the Calculus of Limits, an Impulse Function is defined as the limit of a sequence that spikes at a point.

For instance, the limit of the Delta Sequence

$$\delta_n(x) = n\chi\left[-\tfrac{1}{2n}, \tfrac{1}{2n}\right] = \begin{cases} 0, x < -\frac{1}{2n} \\ n, x \in \left[-\frac{1}{2n}, \frac{1}{2n}\right] \\ 0, x > \frac{1}{2n} \end{cases}$$

is infinite at $x = 0$.

Since $\lim_{n\to\infty} \delta_n(x)$, is real valued,

11.2 $\lim_{n\to\infty} \delta_n(x)$, **is not equal to the hyper-real** $\delta(x)$

11.3 $\text{P.V.}\int_{x=-\infty}^{x=\infty} \lim_{n\to\infty} \delta_n(x)dx = 0$ *in the Calculus of Limits*

Proof: Consider the Riemann Integral,

$$\int_{x=-\infty}^{x=\infty} \lim_{n\to\infty} \delta_n(x)dx.$$

Since $\lim_{n\to\infty} \delta_n(x)$ is infinite at $x = 0$, then, as the norm of the partitions decreases to zero, the Riemann Sum will include an undefined term of the form

$$0 \times \infty,$$

and the Riemann integral will not exist.

However, the principal value of the integral exists since

$$\int_{x=-\infty}^{x=-\frac{1}{n}} \lim_{n\to\infty} \delta_n(x)dx + \int_{x=\frac{1}{n}}^{x=\infty} \lim_{n\to\infty} \delta_n(x)dx \to 0, \text{ as } n \to \infty.$$

Thus, the Cauchy Principal Value of the Integral is

$$\text{P.V.} \int_{x=-\infty}^{x=\infty} \lim_{n\to\infty} \delta_n(x)dx = 0. \square$$

However, for any $n = 1,2,3,...$

$$\int_{x=-\infty}^{x=\infty} \delta_n(x)dx = 1,$$

and

$$\lim_{n\to\infty} \int_{x=-\infty}^{x=\infty} \delta_n(x)dx = 1$$

Consequently, in the Calculus of Limits we have

11.3 $$\text{P.V.} \int_{x=-\infty}^{x=\infty} \lim_{n\to\infty} \delta_n(x)dx = 0.$$

$$\neq 1 = \lim_{n\to\infty} \int_{x=-\infty}^{x=\infty} \delta_n(x)dx.$$

11.4 *In the Calculus of Limits, the Impulse function does not have the impact of each of the* $\delta_n(x)$.

To rectify this, we need to define Integration in Infinitesimal Calculus.

That integration will apply to the hyper-real Delta Function $\delta(x)$

The ill-defined $\lim_{n\to\infty} \delta_n(x)$ cannot be saved, by means of Infinitesimal calculus.

12.

Hyper-real Integral

Let $f(x)$ be a hyper-real function, defined on an interval that may be unbounded. $f(x)$ need not be continuous at any point, but must take a hyper-real value at any point.

This means that $f(x)$ values may be infinite hyper-reals, but never ∞. We have no use for ∞ as an $f(x)$ value.

The Hyper-Real Integral of $f(x)$ over the interval is the sum of the areas of the infinitely many rectangles with base $[x - \frac{dx}{2}, x + \frac{dx}{2}]$, and height $f(x)$.

That Sum may take infinite hyper-real values, such as $\frac{1}{dx}$, but may not equal to ∞.

Thus, the Hyper-real Integral of the function $f(x) = \dfrac{1}{|x|}$ over $[-1,1]$ diverges.

12.1 Hyper-real Integral Definition

Let $f(x)$ be a Hyper-real function on the interval $[a,b]$.

$f(x)$ may take infinite hyper-real values, and need not be bounded. At each

$$a \le x \le b,$$

there is a rectangle with base $[x - \frac{dx}{2}, x + \frac{dx}{2}]$, height $f(x)$, and area

$$f(x)dx.$$

We form the Sum of all the areas for the x's that start at $x = a$, and end at $x = b$,

$$\sum_{x \in [a,b]} f(x)dx.$$

If for any infinitesimal dx, the Integration Sum has the same hyper-real value, then $f(x)$ is Hyper-Real integrable over the interval $[a,b]$, and the Sum is the Hyper-real Integral of $f(x)$ from , to $x = b$, $\int\limits_{x=a}^{x=b} f(x)dx$.

If the hyper-real is infinite, it equals $\int\limits_{x=a}^{x=b} f(x)dx$

If the hyper-real is finite, its constant part is $\int\limits_{x=a}^{x=b} f(x)dx$.□

12.2 The countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers, $Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty.$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[a,b]$, and the Integration Sum has countably many terms.

Thus, while we do not sequence the real numbers in the interval, the summation is guaranteed to be over countably many $f(x)dx$.$\square$

13.

Lower Integral, Upper Integral

The Lower Integral is the Integration Sum where $f(x)$ is replaced by its lowest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

13.1
$$\sum_{x \in [a,b]} \left(\inf_{x - \frac{dx}{2} \leq t \leq x + \frac{dx}{2}} f(t) \right) dx$$

The Upper Integral is the Integration Sum where $f(x)$ is replaced by its largest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

13.2
$$\sum_{x \in [a,b]} \left(\sup_{x - \frac{dx}{2} \leq t \leq x + \frac{dx}{2}} f(t) \right) dx$$

If the integral is a finite hyper-real, we have

13.3 *A Hyper-real function has a finite Hyper-real Integral if and only if its upper integral and its lower integral are finite, and differ by an infinitesimal.*

13.4 *Hyper-real* $f(x)$ *Continuous on* $[a,b]$ *is integrable*

Proof:

$f(x)$ Continuous on $[a,b]$, is uniformly continuous on $[a,b]$.

Therefore, there is one infinitesimal $\eta = \eta(dx)$, that depends only on dx, so that

$$\sup_{x-\frac{dx}{2}\le t\le x+\frac{dx}{2}} f(t) - \inf_{x-\frac{dx}{2}\le t\le x+\frac{dx}{2}} f(t) \le \eta$$

Thus, for any dx we have

$$\sum_{x\in[a,b]}\left(\sup_{x-\frac{dx}{2}\le t\le x+\frac{dx}{2}} f(t)\right)dx - \sum_{x\in[a,b]}\left(\inf_{x-\frac{dx}{2}\le t\le x+\frac{dx}{2}} f(t)\right)dx =$$

$$= \sum_{x\in[a,b]}\left(\sup_{x-\frac{dx}{2}\le t\le x+\frac{dx}{2}} f(t) - \inf_{x-\frac{dx}{2}\le t\le x+\frac{dx}{2}} f(t)\right)dx$$

$$\le \eta \sum_{x\in[a,b]} dx$$

$$= \eta(b-a)$$

Consequently, the lower integral, and the upper integral differ by an infinitesimal, and by 12.3, $f(x)$ is integrable.□

14.

Oscillation Integral

14.1 The Oscillation of a bounded $f(x)$

Let $f(x)$ be bounded on the interval $[a,b]$.

The oscillation of $f(x)$ over the interval $[x-\frac{dx}{2}, x+\frac{dx}{2}]$ is

$$\omega(f)\Big|_{x-\frac{dx}{2}}^{x+\frac{dx}{2}} = \sup_{x-\frac{dx}{2}\le t\le x+\frac{dx}{2}} f(t) - \inf_{x-\frac{dx}{2}\le s\le x+\frac{dx}{2}} f(s)$$

14.2 The Oscillation Integral of a bounded $f(x)$

If for any infinitesimal dx, the infinite sum

$$\sum_{x\in[a,b]} \left(\omega(f)\Big|_{x-\frac{dx}{2}}^{x+\frac{dx}{2}}\right) dx$$

equals a unique finite hyper-real number, the constant part of the hyper-real infinite sum is the **Oscillations Integral** over the interval $[a,b]$.

14.3 The Total Oscillation of $f(x)$ over $[a,b]$.

$$\Omega(f)\Big|_a^b = \sum_{x\in[a,b]} \omega(f)\Big|_{x-\frac{dx}{2}}^{x+\frac{dx}{2}}.$$

is the **Total Oscillation** of $f(x)$ over $[a,b]$.

14.4 *The Oscillation Integral equals* $\Omega(f)\Big|_a^b dx$

In the Calculus of Limits, the **Total Variation** of $f(x)$ over $[a,b]$ is defined by

14.5 $V_a^b(f) = \sup\sum_{i=1}^{n}\left|f(x_i) - f(x_{i-1})\right|$,

where the sup is taken over all partitions of $[a,b]$.

Then, it is well-known [Randolph] that if $f(x)$ is bounded, its total oscillation equals its total variation over $[a,b]$. That is,

14.6 $f(x)$ *bounded over* $[a,b] \Rightarrow \Omega(f)\Big|_a^b = V_a^b(f)$.

And, it is well-known [Randolph] that

14.7 $f(x)$ *bounded over* $[a,b]$

is integrable from a *to* $b \Leftrightarrow V(f)\Big|_a^b$ *is bounded*

In Infinitesimal Calculus, the Total Variation definition is identical to the Total Oscillation definition.

Thus, we have

14.8 1st Oscillation Condition for Integrability

Let $f(x)$ be Hyper-real bounded function, then the following are equivalent

I $f(x)$ *is Hyper-real Integrable from* a *to* b

II $\Omega(f)\Big|_a^b$ *is bounded*

III *For any infinitesimal* dx*,* $\Omega(f)\Big|_a^b dx$ *is an infinitesimal.*

Condition III is the infinitesimal Calculus version of Riemann's first oscillation condition, [Riemann], or [Dan3].

Proof: For $f(x)$ bounded,

$$f(x) \text{ is integrable}$$

$$\Leftrightarrow V_a^b(f) \text{ is bounded}$$

$$\Leftrightarrow \Omega(f)\Big|_a^b \text{ is bounded}$$

$$\Leftrightarrow \Omega(f)\Big|_a^b dx \text{ is an infinitesimal}$$

$$\Leftrightarrow \text{III. } \square$$

Riemann derived another oscillation condition for integrability, [Riemann], or [Dan3].

To write its Infinitesimal Calculus version we define

14.9 The Oscillation Sum $s(o_*)$

Fix an infinitesimal o_*,

$$s(o_*) = \text{total length of the intervals in which } \Omega(f)\Big|_a^b \geq o_*$$

14.10 2nd Oscillation Condition for Integrability

Let $f(x)$ *be Hyper-real bounded. Then*

A. $f(x)$ *is Hyper-real integrable from* a *to* b

$\Leftrightarrow$ B. *For any infinitesimal* o_*, $s(o_*)$ *is infinitesimal.*

Since infinitesimals exist only on the Hyper-real line, and have no realizations on the real line, Condition B should not lead to measuring infinitesimal length on the real line.

The allusion of sets of measure zero to that, indicates a conceptual deficiency in Lebesgue Measure theory.

[Dan4], further discusses Lebesgue Measure, and Integration.

15.

Integral of Monotonic Function

For a monotonic function we obtain,

15.1 *A bounded monotonic Hyper-real function on the interval* $[a,b]$ *is integrable. Its Hyper-real Integral is finite, and it equals the real part of its lower, or upper integral*.

Proof:

For a monotonic increasing function, the Lower integral is the Integration Sum

$$\sum_{x\in[a,b]}\left(\inf_{x-\frac{dx}{2}\le t\le x+\frac{dx}{2}} f(t)\right)dx\,,$$

and the Upper Integral is the Infinite Riemann Sum

$$\sum_{x\in[a,b]}\left(\sup_{x-\frac{dx}{2}\le t\le x+\frac{dx}{2}} f(t)\right)dx$$

Therefore, For a bounded monotonic Hyper-real function, the difference between its upper integral, and its lower integral is the infinitesimal

$$(f(b)-f(a))dx\,.\square$$

16.

Hyper-real Integration of $\delta(x)$

16.1 Integral of $\delta(x)$

Let
$$\delta(x) = \begin{cases} \dfrac{1}{dx}, & -\dfrac{dx}{2} \le x \le \dfrac{dx}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$= \frac{1}{dx}\chi_{\left[-\frac{dx}{2}, \frac{dx}{2}\right]}$$

Then $$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1,$$

Proof: The only term in the Integration Sum is $\dfrac{1}{dx}dx = 1$.

Both the upper integral, and the lower integral are equal to

$\dfrac{1}{dx}dx = 1.\square$

17.

The Fundamental Theorem of the Infinitesimal Calculus

The Fundamental Theorem of Calculus guarantees that Integration and Differentiation are well defined inverse operations, that when applied consecutively yield the original function.

It is well known to hold in the Calculus of Limits under given conditions.

In the Infinitesimal Calculus, the Fundamental Theorem holds with almost no conditions.

17.1 *Let* $f(x)$ *be Hyper-real Integrable on* $[a,b]$

$$\textit{Then, for any } x \in [a,b], \qquad \text{p.v.D}\int_{u=a}^{u=x} f(u)du = f(x)$$

Proof:

$$\text{p.v.D}\int_{u=a}^{u=x} f(u)du = \frac{\displaystyle\int_{u=a}^{u=x+\frac{dx}{2}} f(u)du - \int_{u=a}^{u=x-\frac{dx}{2}} f(u)du}{dx}$$

$$= \frac{\displaystyle\sum_{u\in[a,x+\frac{dx}{2}]} f(u)du - \sum_{u\in[a,x-\frac{dx}{2}]} f(u)du}{dx}$$

$$= \frac{f(x)dx}{dx}$$

$$= f(x).\square$$

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Correction to Rationals Countability and Cantor's Proof

Gauge Institute Journal, February 2006

Professor Boyko Banchev pointed to my error in this paper, in an email that says

"…your statement that Cantor

> '…uses a mapping that is not one-one. Thus, the countability of the rationals was not proved by Cantor'

is wrong. Cantor's function is an immediate application of the triangular number formula (counting the sum of the lengths of the first $\mu + \nu - 2$ diagonals), and it is certainly a valid bijection from $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

Your hand calculation on pp.2-3 is incorrect with respect to the values of λ."

Indeed, Cantor wrote,

"*…the element* (μ, ν) *comes at the* λ *th place, where*

$$(9) \qquad \lambda = \mu + \frac{(\mu+\nu-1)(\mu+\nu-2)}{2}.$$

The variable λ *takes every numerical value* $1, 2, 3, \ldots$, *once. Consequently, by means of (9), a reciprocally univocal relation subsists between the aggregates* $\{\nu\}$ *and* $\{(\mu, \nu)\}$."

and we have

$$\mu_1 = 1, \quad \nu_1 = 1, \quad \lambda_1 = 1, \quad \text{and} \quad (1,1) \to 1$$

$$\mu_2 = 1, \quad \nu_2 = 2, \quad \lambda_2 = 2, \quad \text{and} \quad (1,2) \to 2$$

$$\mu_3 = 2, \quad \nu_3 = 1, \quad \lambda_3 = 3, \quad \text{and} \quad (2,1) \to 3$$

$$\mu_4 = 1, \quad \nu_4 = 3, \quad \lambda_4 = 4, \quad \text{and} \quad (1,3) \to 4$$

$$\mu_5 = 2, \quad \nu_5 = 2, \quad \lambda_5 = 5, \quad \text{and} \quad (2,2) \to 5$$

$$\mu_6 = 3, \quad \nu_6 = 1, \quad \lambda_6 = 6, \quad \text{and} \quad (3,1) \to 6$$

$$\mu_7 = 1, \quad \nu_7 = 4, \quad \lambda_7 = 7, \quad \text{and} \quad (1,4) \to 7$$

$$\mu_8 = 2, \quad \nu_8 = 3, \quad \lambda_8 = 8, \quad \text{and} \quad (2,3) \to 8$$

$$\mu_9 = 3, \quad \nu_9 = 2, \quad \lambda_9 = 9, \quad \text{and} \quad (3,2) \to 9$$

$$\mu_{10} = 4, \quad \nu_{10} = 1, \quad \lambda_{10} = 10, \quad \text{and} \quad (4,1) \to 10$$

The mapping is one-one because if $\lambda_j = \lambda_k$ then,

$$\mu_j + \frac{(\mu_j + \nu_j - 1)(\mu_j + \nu_j - 2)}{2} = \mu_k + \frac{(\mu_k + \nu_k - 1)(\mu_k + \nu_k - 2)}{2},$$

$$(\mu_j - \mu_k)(\mu_j + \mu_k - 1) + (\nu_j - \nu_k)(\nu_j + \nu_k - 3) + 2(\mu_j\nu_j - \mu_k\nu_k) = 0.$$

Since each of the three terms is independent of the other two terms, each is zero. Then,

$$(\mu_j - \mu_k)(\mu_j + \mu_k - 1) = 0 \Rightarrow \mu_j = \mu_k,$$

since $(\mu_j + \mu_k - 1) \geq 1$, for any j, and k.

$$\mu_j\nu_j - \mu_k\nu_k = 0 \Rightarrow \mu_j\nu_j = \mu_k\nu_k \Rightarrow \nu_j = \nu_k,$$

since $\mu_j = \mu_k$, for any j, and k.□

The mapping that we propose in the paper "Rationals Countability and Cantor's Proof" is one-one as well.