Infinite Series with Infinite Hyper-real Sum

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Abstract: "Divergent to Infinity" Series in the Calculus of Limits, converge to infinite Hyper-reals in Infinitesimal Calculus. Examples of such Series are the Harmonic Series, and Kernels of Fourier Series, Fejer-Cesaro Series, Poisson-Abel Integral, Fourier-Bessel Series, ν-Bessel Series, Legendre Series, Hermit Series, Laguerre Series, and Chebyshev Series.

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Contents

Introduction

The sum of the Series

$$1+1+1+...$$

generates in the Calculus of Limits the same panic that

zero

invoked in Greek Mathematics.

However, the partial sums of the series are the sequence

$$1,2,3,.... = \left\langle n \right\rangle,$$

which is an infinite hyper-real number.

In Infinitesimal Calculus, it is the value assigned to the Hyper-real Delta Function over a hyper-real interval of infinitesimal size

$$\left\langle \tfrac{1}{n} \right\rangle.$$

We show here that "Divergent to Infinity" Series in the Calculus of Limits, converge to an infinite Hyper-real in Infinitesimal Calculus.

Non-trivial examples of such Series are the Harmonic Series, and <u>Kernels</u> of Fourier Series, Fejer-Cesaro Series, Poisson-Abel Integral, Fourier-Bessel Series, ν-Bessel Series, Legendre Series, Hermit Series, Laguerre Series, and Chebyshev Series.

1.

Hyper-real Line

Each real number α can be represented by a Cauchy sequence of rational numbers, $(r_1, r_2, r_3, ...)$ so that $r_n \to \alpha$.

The constant sequence $(\alpha, \alpha, \alpha, ...)$ is a constant hyper-real.

In [Dan2] we established that,

1. Any totally ordered set of positive, monotonically decreasing to zero sequences $(\iota_1, \iota_2, \iota_3, ...)$ constitutes a family of infinitesimal hyper-reals.
2. The infinitesimals are smaller than any real number, yet strictly greater than zero.
3. Their reciprocals $\left(\frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3}, ...\right)$ are the infinite hyper-reals.
4. The infinite hyper-reals are greater than any real number, yet strictly smaller than infinity.
5. The infinite hyper-reals with negative signs are smaller than any real number, yet strictly greater than $-\infty$.
6. The sum of a real number with an infinitesimal is a non-constant hyper-real.
7. The Hyper-reals are the totality of constant hyper-reals, a family of infinitesimals, a family of infinitesimals with

negative sign, a family of infinite hyper-reals, a family of infinite hyper-reals with negative sign, and non-constant hyper-reals.

8. The hyper-reals are totally ordered, and aligned along a line: the Hyper-real Line.
9. That line includes the real numbers separated by the non-constant hyper-reals. Each real number is the center of an interval of hyper-reals, that includes no other real number.
10. In particular, zero is separated from any positive real by the infinitesimals, and from any negative real by the infinitesimals with negative signs, $-dx$.
11. Zero is not an infinitesimal, because zero is not strictly greater than zero.
12. We do not add infinity to the hyper-real line.
13. The infinitesimals, the infinitesimals with negative signs, the infinite hyper-reals, and the infinite hyper-reals with negative signs are semi-groups with respect to addition. Neither set includes zero.
14. The hyper-real line is embedded in $\mathbb{R}^{\infty}$, and is not homeomorphic to the real line. There is no bi-continuous one-one mapping from the hyper-real onto the real line.

15. In particular, there are no points on the real line that can be assigned uniquely to the infinitesimal hyper-reals, or to the infinite hyper-reals, or to the non-constant hyper-reals.

16. No neighbourhood of a hyper-real is homeomorphic to an $\mathbb{R}^n$ ball. Therefore, the hyper-real line is not a manifold.

17. The hyper-real line is totally ordered like a line, but it is not spanned by one element, and it is not one-dimensional.

2.

The Infinite Hyper-Reals

In [Dan2], we established that

1. $\frac{1}{\iota} = \left\langle \frac{1}{\iota_1}, \frac{1}{\iota_2}, \frac{1}{\iota_3} ..... \right\rangle$

   is the infinite hyper-real reciprocal to $\iota = \langle \iota_1, \iota_2, \iota_3, ... \rangle$

2. If the family of infinitesimals is $\left\langle \frac{1}{n} \right\rangle$, $\left\langle \frac{1}{n^2} \right\rangle \left\langle \frac{1}{n^3} \right\rangle$,...

   the associated family of infinite hyper-reals are

$$< n >, < n^2 >, < n^3 > ...$$

3. The order between infinite hyper-reals is defined by

   $\frac{1}{\iota} < \frac{1}{o}$ means that for all $n > n_0$, $o_n < \iota_n$

   $\frac{1}{\iota} = \frac{1}{o}$ means that for all $n > n_0$, $\iota_n = o_n$

4. $\langle n \rangle < \langle n^2 \rangle < \langle n^3 \rangle < ...$

5. The infinite hyper-reals are Non-Archimedean

6. The infinite hyper-reals are totally ordered

7. Positive Infinite hyper-reals are strictly greater than any real number, but strictly smaller than ∞.

8. Negative Infinite Hyper-reals are smaller than any real number, but strictly greater than $-\infty$

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers, $Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty.$$

However, as we established in [Dan2],

9. None of the infinite hyper-reals equals the constant hyper-real infinity

$$\left\langle \infty, \infty, \infty, ... \right\rangle,$$

and none may be identified with it.

10. For any infinitesimal ι,

$$\frac{1}{\iota} < \frac{1}{\iota^2} < .... < \frac{1}{\iota^n} < ... < \infty,$$

$$-\frac{1}{\iota} > -\frac{1}{\iota^2} > .... > -\frac{1}{\iota^n} > ... > -\infty.$$

The distinction between infinite hyper-reals and Infinity is crucial to the definition of the sum of an infinite series that in the Calculus of Limits is divergent to infinity. An Infinite Series that diverges to infinity in the Calculus of Limits sums up to an infinite hyper-real in Infinitesimal Calculus.

3.

Integral of a Hyper-real Function

In [Dan3], we defined the integral of a Hyper-real Function.

Let $f(x)$ be a hyper-real function on the interval $[a,b]$.

The interval may not be bounded.

$f(x)$ may take infinite hyper-real values, and need not be bounded.

At each

$$a \le x \le b,$$

there is a rectangle with base $[x - \frac{dx}{2}, x + \frac{dx}{2}]$, height $f(x)$, and area

$$f(x)dx.$$

We form the **Integration Sum** of all the areas for the x's that start at $x = a$, and end at $x = b$,

$$\sum_{x \in [a,b]} f(x)dx.$$

If for any infinitesimal dx, the Integration Sum has the same hyper-real value, then $f(x)$ is integrable over the interval $[a,b]$.

Then, we call the Integration Sum the integral of $f(x)$ from $x = a$, to $x = b$, and denote it by

$$\int_{x=a}^{x=b} f(x)dx\,.$$

If the hyper-real is infinite, then it is the integral over $[a,b]$,

If the hyper-real is finite,

$$\int_{x=a}^{x=b} f(x)dx = \text{ real part of the hyper-real}\,.\Box$$

3.1 The countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers, $Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty.$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[a,b]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(x)dx$.

The Lower Integral is the Integration Sum where $f(x)$ is replaced by its lowest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

3.2
$$\sum_{x \in [a,b]} \left(\inf_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

The Upper Integral is the Integration Sum where $f(x)$ is replaced by its largest value on each interval $[x - \frac{dx}{2}, x + \frac{dx}{2}]$

3.3
$$\sum_{x \in [a,b]} \left(\sup_{x - \frac{dx}{2} \le t \le x + \frac{dx}{2}} f(t) \right) dx$$

If the integral is a finite hyper-real, we have

3.4 *A hyper-real function has a finite integral if and only if its upper integral and its lower integral are finite, and differ by an infinitesimal.*

4.

Delta Function

In [Dan4], we defined the Delta Function, and established its properties

1. The Delta Function is a hyper-real function defined from the hyper-real line into the set of two hyper-reals $\left\{0, \frac{1}{dx}\right\}$. The hyper-real 0 is the sequence $\langle 0, 0, 0, ...\rangle$. The infinite hyper-real $\frac{1}{dx}$ depends on our choice of dx.
2. We will usually choose the family of infinitesimals that is spanned by the sequences $\left\langle \frac{1}{n} \right\rangle, \left\langle \frac{1}{n^2} \right\rangle, \left\langle \frac{1}{n^3} \right\rangle, \ldots$ It is a semigroup with respect to vector addition, and includes all the scalar multiples of the generating sequences that are non-zero. That is, the family includes infinitesimals with negative sign. Therefore, $\frac{1}{dx}$ will mean the sequence $\langle n \rangle$. Alternatively, we may choose the family spanned by the sequences $\left\langle \frac{1}{2^n} \right\rangle, \left\langle \frac{1}{3^n} \right\rangle, \left\langle \frac{1}{4^n} \right\rangle, \ldots$ Then, $\frac{1}{dx}$ will mean the

sequence $\left\langle 2^n \right\rangle$. Once we determined the basic infinitesimal dx, we will use it in the Infinite Riemann Sum that defines an Integral in Infinitesimal Calculus.

3. The Delta Function is strictly smaller than ∞

4. We define, $\delta(x) \equiv \dfrac{1}{dx}\chi_{[-\frac{dx}{2},\frac{dx}{2}]}(x)$,

   where $\chi_{[-\frac{dx}{2},\frac{dx}{2}]}(x) = \begin{cases} 1, x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right] \\ 0, \text{otherwise} \end{cases}$.

5. Hence,

   - for $x < 0$, $\delta(x) = 0$
   - at $x = -\dfrac{dx}{2}$, $\delta(x)$ jumps from 0 to $\dfrac{1}{dx}$,
   - for $x \in \left[-\frac{dx}{2}, \frac{dx}{2}\right]$, $\delta(x) = \dfrac{1}{dx}$.
   - at $x = 0$, $\delta(0) = \dfrac{1}{dx}$
   - at $x = \dfrac{dx}{2}$, $\delta(x)$ drops from $\dfrac{1}{dx}$ to 0.
   - for $x > 0$, $\delta(x) = 0$.
   - $x\delta(x) = 0$

6. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle \chi_{[-\frac{1}{2},\frac{1}{2}]}(x), 2\chi_{[-\frac{1}{4},\frac{1}{4}]}(x), 3\chi_{[-\frac{1}{6},\frac{1}{6}]}(x) ... \right\rangle$

7. If $dx = \left\langle \frac{2}{n} \right\rangle$, $\delta(x) = \left\langle \dfrac{1}{2\cosh^2 x}, \dfrac{2}{2\cosh^2 2x}, \dfrac{3}{2\cosh^2 3x}, ... \right\rangle$

8. If $dx = \left\langle \frac{1}{n} \right\rangle$, $\delta(x) = \left\langle e^{-x}\chi_{[0,\infty)}, 2e^{-2x}\chi_{[0,\infty)}, 3e^{-3x}\chi_{[0,\infty)}, \ldots \right\rangle$

9. $$\int_{x=-\infty}^{x=\infty} \delta(x)dx = 1.$$

10. $$\delta(\xi - x) = \frac{1}{2\pi} \int_{k=-\infty}^{k=\infty} e^{-ik(\xi - x)}dk$$

5.

Periodic Delta Function $\delta_{Periodic}(\xi - x)$

5.1 Periodic Delta Function Definition

$$\delta_{Periodic}(\xi - x) = ... + \delta(\xi - x + 2) + \delta(\xi - x) + \delta(\xi - x - 2) + ...$$

is a periodic hyper-real Delta function, with period $T = 2$.

5.2 Fourier Transform of $\delta_{Periodic}(x)$

$$\mathcal{F}\left\{\delta_{Periodic}(x)\right\} = ... + e^{-i4\pi\nu} + 1 + e^{i4\pi\nu} + ...$$

Proof: $\mathcal{F}\left\{\delta_{Periodic}(x)\right\} = ... + \mathcal{F}\left\{\delta(x+2)\right\} + \mathcal{F}\left\{\delta(x)\right\} + \mathcal{F}\left\{\delta(x-2)\right\} + ...$

$$= ... + \int_{x=-\infty}^{x=\infty} \delta(x+2)e^{-i2\pi\nu x}dx + \int_{x=-\infty}^{x=\infty} \delta(x)e^{-i2\pi\nu x}dx + \int_{x=-\infty}^{x=\infty} \delta(x-2)e^{-i2\pi\nu x}dx + ...$$

$$= ... + e^{2\pi i 2\nu} + 1 + e^{-2\pi i 2\nu} + ...$$

5.3 Fourier Integral Theorem for $\delta_{Periodic}(x)$

$$\mathcal{F}^{-1}\mathcal{F}\left\{\delta_{Periodic}(x)\right\} = \delta_{Periodic}(x)$$

Proof: $\mathcal{F}^{-1}\mathcal{F}\left\{\delta_{Periodic}(x)\right\} = ... + \mathcal{F}^{-1}\left\{e^{2\pi i 2\nu}\right\} + \mathcal{F}^{-1}\left\{1\right\} + \mathcal{F}^{-1}\left\{e^{-2\pi i 2\nu}\right\} + ...$

$$= ... + \int_{\nu=-\infty}^{\nu=\infty} e^{i2\pi\nu 2}e^{i2\pi\nu x}d\nu + \int_{\nu=-\infty}^{\nu=\infty} e^{i2\pi\nu x}d\nu + \int_{\nu=-\infty}^{\nu=\infty} e^{-i2\pi\nu 2}e^{i2\pi\nu x}d\nu + ...$$

$$= ... + \delta(x+2) + \delta(x) + \delta(x-2) + ...$$

6.

Convergent Sequences

In Infinitesimal Calculus, the sequence a_n that converges to a finite number a,

$$a_n \to a,$$

represents the finite hyper-real number $\langle a, a, a, ... \rangle$.

We define

6.1 Sequence Convergence to a finite hyper-real a

$$a_n \to a \ \ \textit{iff} \ \ \langle a_n \rangle - \langle a \rangle = \text{infinitesimal}.$$

In the Calculus of Limits, the sequence

$$1, 2, 3, ...$$

and the sequence

$$1^2, 2^2, 3^2, ...$$

"diverge to infinity".

In Infinitesimal Calculus,

the first sequence represents the infinite hyper-real $\langle n \rangle$,

and the second the infinite hyper-real $\langle n^2 \rangle$.

Neither Infinite hyper-real equals infinity. In fact,

$$\left\langle n \right\rangle < \left\langle n^2 \right\rangle < \infty$$

We define

6.2 Sequence Convergence to an infinite hyper-real A

$a_n \to A$ *iff* $\left\langle a_n \right\rangle$ represents the infinite hyper-real A.

7.

Convergent Series

The Convergence of an infinite Series to a finite sum s, is the convergence of its Partial Sums Sequence

$$a_1 + ... + a_n \to s.$$

We define

7.1 Series Convergence to a finite hyper-real s

$$a_1 + a_2 + ... \to s \quad \textit{iff} \quad \left\langle a_1 + ... + a_n \right\rangle - \left\langle s \right\rangle = \text{infinitesimal}.$$

In the Calculus of Limits, the Infinite Series

$$1 + 1 + 1 + ...$$

and the Infinite Series

$$1 + 2 + 3 + ...$$

both "diverge to infinity".

In fact,

these series don't sum up to infinity

In Infinitesimal Calculus,

the first series sums up to the infinite hyper-real number $\left\langle n \right\rangle$,

and the second to the infinite hyper-real number $\left\langle \frac{1}{2}n(n+1) \right\rangle$.

Neither infinite hyper-real equals infinity.

We have

$$\langle n \rangle < \left\langle \tfrac{1}{2} n(n+1) \right\rangle < \infty .$$

Thus, the Calculus of Limits conclusion of "divergence to infinity" is erroneous.

We define

7.2 Series Convergence to an Infinite Hyper-real S

$a_1 + a_2 + ... \to S$ *iff*

$\langle a_1 + ... + a_n \rangle$ represents the infinite hyper-real S.

7.3 The Harmonic Series Sum

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + ... + \tfrac{1}{n} + ... = \langle \gamma + \log n \rangle ,$$

where γ *is Euler's Constant.*

Proof: The infinite series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + ... + \tfrac{1}{n} + ...$$

sums up to the infinite hyper-real represented by

$$\left\langle 1 + \tfrac{1}{2} + \tfrac{1}{3} + ... + \tfrac{1}{n} \right\rangle$$

Euler showed that

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + ... + \tfrac{1}{n} - \log n \to \gamma .$$

Thus, $\langle \gamma + \log n \rangle$ represents the infinite hyper-real sum of the harmonic series.$\square$

8.

Dirichlet Kernel

The partial sums of the Fourier Series associated with $f(x)$,

$$\int_{\xi=-1}^{\xi=1} f(\xi)\underbrace{\left\{\tfrac{1}{2}e^{-in\pi(\xi-x)}+...+\tfrac{1}{2}e^{-i\pi(\xi-x)}+\tfrac{1}{2}+\tfrac{1}{2}e^{i\pi(\xi-x)}+...+\tfrac{1}{2}e^{in\pi(\xi-x)}\right\}}_{\text{Dirichlet Sequence}}d\xi\,,$$

give rise to the Dirichlet Sequence.

The limit of the Dirichlet Sequence is the Dirichlet Kernel,

$$D_{irichlet}(\xi-x)=...+\tfrac{1}{2}e^{-i2\pi(\xi-x)}+\tfrac{1}{2}e^{-i\pi(\xi-x)}+\tfrac{1}{2}+\tfrac{1}{2}e^{i\pi(\xi-x)}+\tfrac{1}{2}e^{i2\pi(\xi-x)}+....$$

$$=\frac{1}{2}+\cos\pi(x-\xi)+\cos 2\pi(x-\xi)+...,$$

$$=\lim_{n\to\infty}\frac{\sin(n+\frac{1}{2})\pi(x-\xi)}{2\sin\frac{1}{2}\pi(x-\xi)}.$$

In the Calculus of Limits, the Dirichlet Kernel diverges to infinity for any $x-\xi=2m$, and vanishes for any $x-\xi\neq 2m$. [Dan5].

We established, [Dan5], that in Infinitesimal Calculus, the Dirichlet Kernel is a Hyper-real Infinite Series represented by

$$\left\langle\frac{\sin(n+\frac{1}{2})\pi(\xi-x)}{2\sin\frac{1}{2}\pi(\xi-x)}\right\rangle=$$

$$=\left\langle\tfrac{1}{2}+\cos\pi(\xi-x)+\cos 2\pi(\xi-x)+...+\cos n\pi(\xi-x)\right\rangle$$

$$=\left\langle\tfrac{1}{2}e^{-in\pi(\xi-x)}+..+\tfrac{1}{2}e^{-i\pi(\xi-x)}+\tfrac{1}{2}+\tfrac{1}{2}e^{i\pi(\xi-x)}+..+\tfrac{1}{2}e^{in\pi(\xi-x)}\right\rangle$$

The Dirichlet Kernel sums up to a Periodic Delta Function, with a period $T = 2$.

$$D_{irichlet}(\xi - x) = \begin{cases} \left\langle \frac{1}{2} + n \right\rangle, & \xi - x = 2m \\ 0, & \xi - x \neq 2m \end{cases}$$

$$= ... + \delta(\xi - x + 2) + \delta(\xi - x) + \delta(\xi - x - 2) + ...$$

Thus,

8.1 Dirichlet Kernel Summation

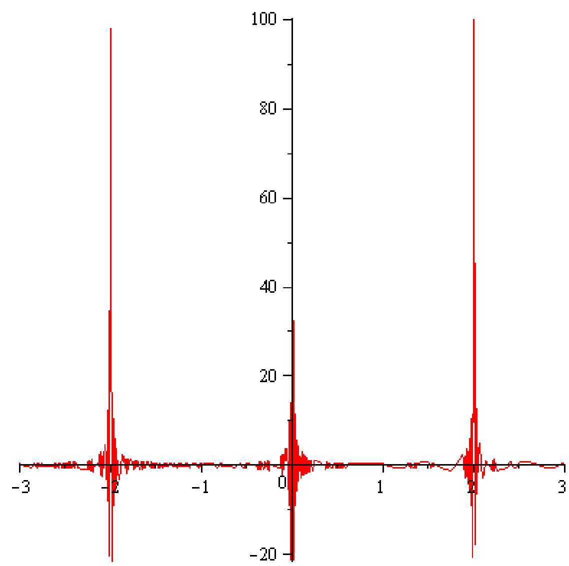
$$... + \tfrac{1}{2}e^{-i2\pi(\xi - x)} + \tfrac{1}{2}e^{-i\pi(\xi - x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi - x)} + \tfrac{1}{2}e^{i2\pi(\xi - x)} + ... =$$

$$= \frac{1}{2} + \cos\pi(x - \xi) + \cos 2\pi(x - \xi) + ...$$

$$= ... + \delta(\xi - x + 2) + \delta(\xi - x) + \delta(\xi - x - 2) + ...$$

8.2 In Maple

$plot\left(\dfrac{\sin\left(\pi\dfrac{201\,x}{2}\right)}{2\,\sin\left(\pi\dfrac{x}{2}\right)}, x = -3..3\right)$ plots the spikes at $x = 0$, $x = -2$, $x = 2$.



9.

Fejer Kernel

The partial sums of the Fourier Series associated with $f(x)$ are

$$\mathcal{S}_n\left\{f(x)\right\} = \int_{\xi=-1}^{\xi=1} f(\xi)\underbrace{\left\{\tfrac{1}{2}e^{-in\pi(\xi-x)} + ... + \tfrac{1}{2}e^{-i\pi(\xi-x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi-x)} + ... + \tfrac{1}{2}e^{in\pi(\xi-x)}\right\}}_{\text{Dirichlet Sequence}}d\xi\,.$$

The Fejer Summation partial sums are the Arithmetic Means

$$\mathcal{F}_{ej}\mathcal{S}_n\left\{f(x)\right\} = \frac{\mathcal{S}_0\left\{f(x)\right\} + \mathcal{S}_1\left\{f(x)\right\} + ... + \mathcal{S}_n\left\{f(x)\right\}}{n+1}$$

$$= \int_{\xi=-1}^{\xi=1} f(\xi)\underbrace{\frac{1}{n+1}\{(n+1)\tfrac{1}{2} + n\cos[\pi(\xi-x)] + ... + \cos[\pi n(\xi-x)]\}}_{\text{Fejer Sequence}}d\xi\,.$$

The limit of the Fejer Sequence is the Fejer Kernel,

$$F_{ejer}(\xi - x) = \lim_{m\to\infty}\frac{1}{2m}\frac{\sin^2\frac{1}{2}m\pi(\xi-x)}{\sin^2\frac{1}{2}\pi(\xi-x)}$$

$$= \lim_{m\to\infty}\left\{\tfrac{1}{2} + \tfrac{m-1}{m}\cos\pi(\xi-x) + ... + \tfrac{m-(m-1)}{m}\cos(m-1)\pi(\xi-x)\right\}$$

The Fejer Kernel diverges to infinity for any $x - \xi = 2m$, and vanishes for any $x - \xi \neq 2m$,[Dan6].

In [Dan6], we established that in Infinitesimal Calculus, the Fejer Kernel is a Hyper-real Infinite Series represented by

$$\left\langle \frac{1}{2n} \frac{\sin^2 \frac{1}{2} n\pi(\xi - x)}{\sin^2 \frac{1}{2}\pi(\xi - x)} \right\rangle =$$

$$= \left\langle \tfrac{1}{2} + \tfrac{n-1}{n}\cos\pi(\xi - x) + ... + \tfrac{n-(n-1)}{n}\cos(n-1)\pi(\xi - x) \right\rangle$$

The Fejer Kernel sums up to a Periodic Delta Function, with a period $T = 2$.

$$F_{ejer}(\xi - x) = \begin{cases} \left\langle \frac{1}{2} n \right\rangle, & \xi - x = 2m \\ 0, & \xi - x \neq 2m \end{cases}$$

$$= ... + \delta(\xi - x + 2) + \delta(\xi - x) + \delta(\xi - x - 2) + ...$$

Thus,

9.1 Fejer Kernel Summation

Let $\frac{1}{2}N = \frac{1}{dx}$ be an infinite Hyper-real.

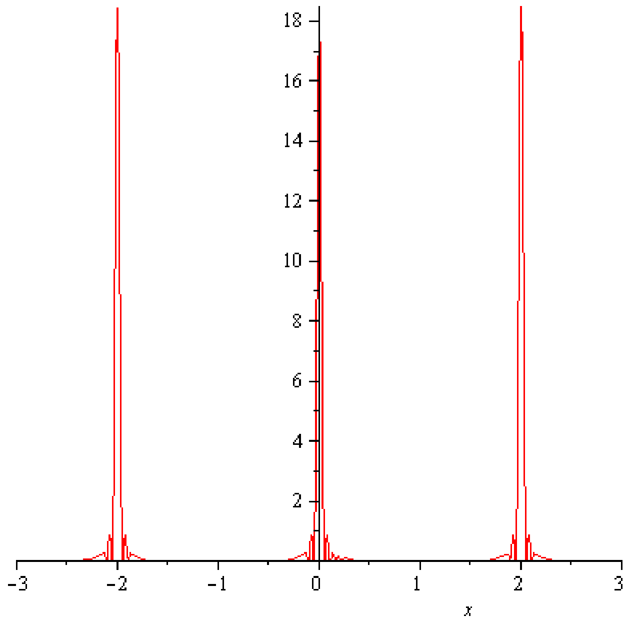
The Hyper-real Fejer Kernel is

$$\tfrac{1}{2} + \tfrac{N-1}{N}\cos\pi(\xi - x) + ... + \tfrac{N-(N-1)}{N}\cos(N-1)\pi(\xi - x) =$$

$$= ... + \delta(\xi - x + 2) + \delta(\xi - x) + \delta(\xi - x - 2) + ...$$

9.2 In Maple

$$plot\left(\frac{\sin^2\left(\pi \frac{37\,x}{2}\right)}{2\cdot 37\ \sin^2\left(\pi \frac{x}{2}\right)}, x = -3..3 \right)$$ plots the spikes at $x = 0$, $x = -2$, $x = 2$


18
16
14
12
10
8
6
4
2
-3
-2
-1
0
1
2
3
x

10.

Poisson Kernel

The partial sums of the Fourier Series associated with $f(x)$ are

$$\mathcal{S}_n\left\{f(x)\right\} = \int_{\xi=-1}^{\xi=1} f(\xi)\underbrace{\left\{\tfrac{1}{2}e^{-in\pi(\xi-x)} + ... + \tfrac{1}{2}e^{-i\pi(\xi-x)} + \tfrac{1}{2} + \tfrac{1}{2}e^{i\pi(\xi-x)} + ... + \tfrac{1}{2}e^{in\pi(\xi-x)}\right\}}_{\text{Dirichlet Sequence}}d\xi\,.$$

To Apply Abel's summation to Fourier Series, Poisson considered

$$\lim_{r\uparrow 1}\left\{.. + r^n c_{-n}e^{i(-n)\pi x} + .. + rc_{-1}e^{i(-1)\pi x} + c_0 + rc_1 e^{i(1)\pi x} + .. + r^n c_n e^{i(n)\pi x} + ..\right\} =$$

$$= \lim_{r\uparrow 1}\int_{\xi=-1}^{\xi=1} f(\xi)\tfrac{1}{2}\left\{.. + r^n e^{-in\pi(\xi-x)} + .. + re^{-i\pi(\xi-x)} + 1 + re^{i\pi(\xi-x)} + .. + r^n e^{in\pi(\xi-x)} + ..\right\}d\xi$$

$$= \int_{\xi=-1}^{\xi=1} f(\xi)\underbrace{\lim_{r\uparrow 1}\tfrac{1}{2}\left\{.. + r^n e^{-in\pi(\xi-x)} + .. + re^{-i\pi(\xi-x)} + 1 + re^{i\pi(\xi-x)} + .. + r^n e^{in\pi(\xi-x)} + ..\right\}}_{\text{Poisson Kernel}}d\xi$$

Since

$$\tfrac{1}{2}\left\{.. + r^2e^{-2i\varphi} + re^{-i\varphi} + 1 + re^{i\varphi} + r^2e^{2i\varphi} + ..\right\} = \tfrac{1}{2}\frac{1-r^2}{1+r^2-2r\cos\varphi},$$

The Poisson Kernel is

$$\mathcal{P}_{oisson}(\xi - x) = \lim_{r\uparrow 1}\tfrac{1}{2}\frac{1-r^2}{1-2r\cos\pi(\xi-x)+r^2}\,.$$

In the Calculus of Limits, the Poisson Kernel diverges to infinity for any $x - \xi = 2m$, and vanishes for any $x - \xi \neq 2m$, [Dan7].

We established, [Dan7], that in Infinitesimal Calculus, the Poisson Kernel is the Hyper-real Infinite Series

$$\tfrac{1}{2}\left\{..+r^n e^{-in\pi(\xi-x)}+..+re^{-i\pi(\xi-x)}+1+re^{i\pi(\xi-x)}+..+r^n e^{in\pi(\xi-x)}+..\right\}\Big|_{r=1-dr}.$$

It sums up to

$$\tfrac{1}{2}\frac{1-r^2}{1-2r\cos\pi(\xi-x)+r^2}\Bigg|_{r=1-dr}=\tfrac{1}{2}\frac{dr(2-dr)}{2(1-dr)(1-\cos\pi[\xi-x])+(dr)^2}$$

This is a Periodic Delta Function, with a period $T=2$.

$$P_{oisson}(\xi-x)=\begin{cases}\langle k-1\rangle, \xi-x=2m\\ \quad 0, \qquad \xi-x\neq 2m\end{cases}$$

$$=...+\delta(\xi-x+2)+\delta(\xi-x)+\delta(\xi-x-2)+....$$

Thus,

10.1 Poisson Kernel Summation

The Hyper-real Poisson Kernel is

$$\tfrac{1}{2}\left\{..+r^n e^{-in\pi(\xi-x)}+..+re^{-i\pi(\xi-x)}+1+re^{i\pi(\xi-x)}+..+r^n e^{in\pi(\xi-x)}+..\right\}\Big|_{r=1-dr}=$$

$$=\tfrac{1}{2}\frac{dr(2-dr)}{2(1-dr)(1-\cos\pi[\xi-x])+(dr)^2}$$

$$=...+\delta(\xi-x+2)+\delta(\xi-x)+\delta(\xi-x-2)+...$$

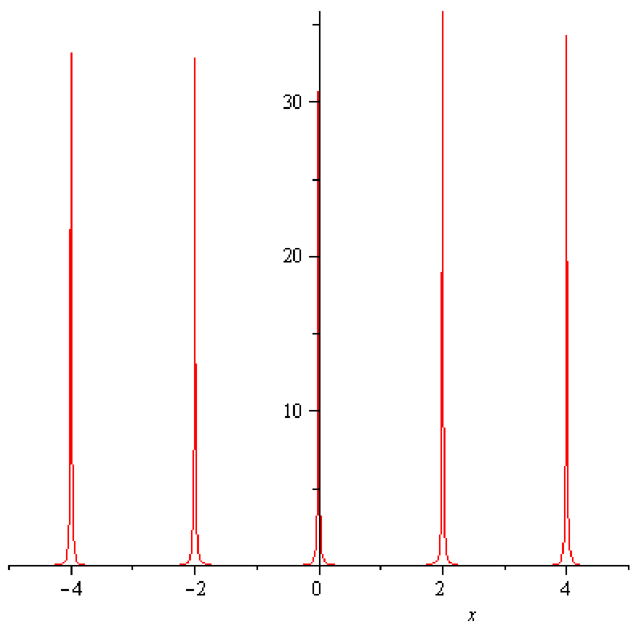
To plot it, let

$$r_k=1-\tfrac{1}{k}$$

For $k = 37$,

$$plot\left(\frac{36}{74\cdot 36 + 1 - 74\cdot 36\cos(\pi x)},\ x = -5..5\right)$$

plots the spikes at $x = 0$, $x = -2$, $x = 2$, $x = -4$, $x = 4$



11.

Legendre Kernel

The partial sums of the Expansion of $f(x)$ in Legendre Polynomials,

$$\int_{\xi=-1}^{\xi=1} f(\xi)\underbrace{\left\{\tfrac{1}{2}P_0(\xi)P_0(x)+\tfrac{3}{2}P_1(\xi)P_1(x)+...+\tfrac{2n+1}{2}P_n(\xi)P_n(x)\right\}}_{\text{Legendre Sequence}}d\xi\,,$$

give rise to the Legendre Sequence.

The limit of the Legendre Sequence is the Legendre Kernel,

$$\tfrac{1}{2}P_0(\xi)P_0(x)+\tfrac{3}{2}P_1(\xi)P_1(x)+...+\tfrac{2\cdot n+1}{2}P_n(\xi)P_n(x)+...$$

The Legendre Kernel diverges to infinity for any $x-\xi=2m$, and vanishes for any $x-\xi\neq 2m$. [Dan8].

Let

$$x=\cos\theta_x\,,\quad \xi=\cos\theta_\xi\,,$$

$$\langle n\rangle \text{ an infinite Hyper-real.}$$

We established, [Dan8], that in Infinitesimal Calculus, the Legendre Kernel is a Hyper-real Infinite Series that sums up to a Periodic Delta Function, with a period $T=2$.

$$\mathcal{L}_{egendre}(\theta_\xi-\theta_x)=\begin{cases}\langle n\rangle, & \theta_\xi-\theta_x=2\pi m\\ 0, & \theta_\xi-\theta_x\neq 2\pi m\end{cases}$$

$$= ... + \delta(\theta_\xi - \theta_x + 2\pi) + \delta(\theta_\xi - \theta_x) + \delta(\theta_\xi - \theta_x - 2\pi) + ...$$

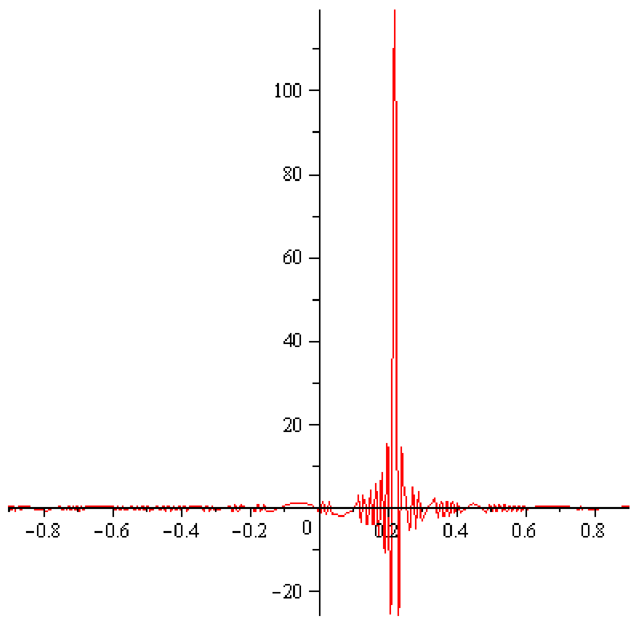
Thus,

11.1 Legendre Kernel Summation

$$\tfrac{1}{2}P_0(\cos\theta_\xi)P_0(\cos\theta_x) + \tfrac{3}{2}P_1(\cos\theta_\xi)P_1(\cos\theta_x) + \tfrac{5}{2}P_2(\cos\theta_\xi)P_2(\cos\theta_x) + ... =$$

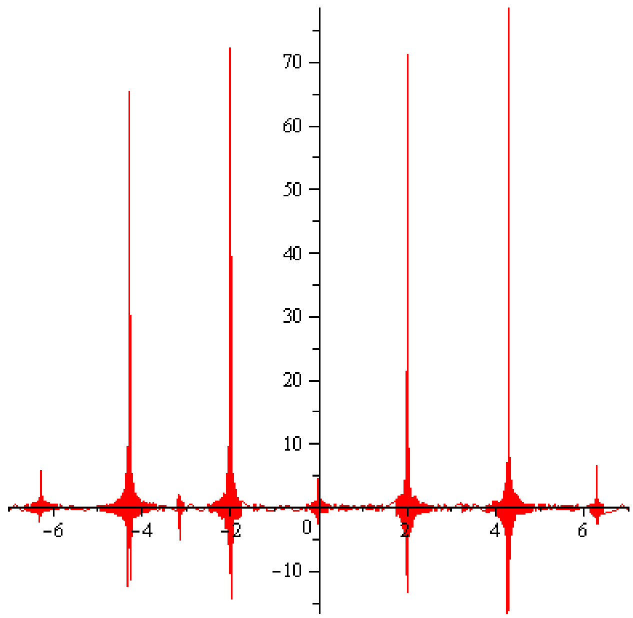
$$= ... + \delta(\theta_\xi - \theta_x + 2\pi) + \delta(\theta_\xi - \theta_x) + \delta(\theta_\xi - \theta_x - 2\pi) + ...$$

11.2 In Maple, plot($\sum_{i=0}^{365} \frac{2i+1}{2}$*LegendreP(i, 0.22)*LegendreP(i, x), x = -.9 .. .9)



For $\theta = \arccos(x)$, the pulses are periodic

11.3 In Maple, plot($\sum_{i=0}^{223} \frac{2i+1}{2}$*LegendreP(i, cos(2))*LegendreP(i, cos(θ)), θ = -7 .. 7)


70
60
50
40
30
20
10
0
-10
-6
-4
-2
2
4
6

12.

Hermite Kernel

The partial sums of the Expansion of $f(x)$ in Hermite Functions,

$$\int_{\xi=-\infty}^{\xi=\infty} f(\xi) \underbrace{\frac{1}{\sqrt{\pi}} e^{-\xi^2} \left\{ H_0(\xi)H_0(x) + ... + \frac{1}{2^n n!} H_n(\xi)H_n(x) \right\}}_{\text{Hermite Sequence}} d\xi ,$$

give rise to the Hermite Sequence.

The limit of the Hermite Sequence is the Hermite Kernel,

$$\frac{1}{\sqrt{\pi}} e^{-\xi^2} \left\{ H_0(\xi)H_0(x) + ... + \frac{1}{2^n n!} H_n(\xi)H_n(x) + ... \right\}$$

The Hermite Kernel diverges to infinity for any $x = \xi$, and vanishes for any $x \neq \xi$. [Dan9].

Let

$$\left\langle \sqrt{n} \right\rangle \text{ be an infinite Hyper-real.}$$

We established, [Dan9], that in Infinitesimal Calculus, the Hermite Kernel is a Hyper-real Infinite Series that sums up to a Delta Function.

$$\mathcal{H}_{ermite}(\xi - x) = \begin{cases} \left\langle \sqrt{n} \right\rangle & ,\xi = x \\ 0 & ,\ \xi \neq x \end{cases}$$

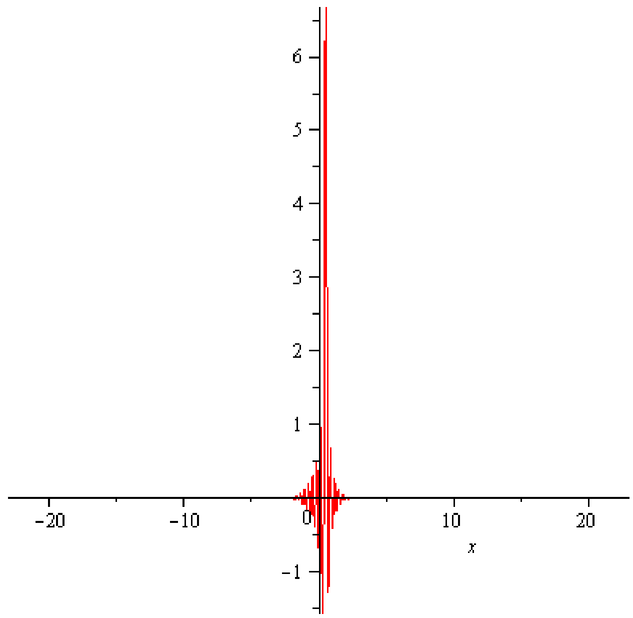
$$= \delta(\xi - x).$$

Thus,

12.1 Hermite Kernel Summation

$$\frac{1}{\sqrt{\pi}} e^{-\xi^2} \left\{ H_0(\xi)H_0(x) + ... + \frac{1}{2^n n!} H_n(\xi)H_n(x) + ... \right\} = \delta(\xi - x)$$

12.2 In Maple, $plot(\sum_{i=0}^{223} \frac{1}{\sqrt{\pi}} e^{-x^2} \frac{1}{2^i i!} * HermiteH(i,.5) * HermiteH(i,x), x = -23..23)$



13.

Laguerre Kernel

The partial sums of the Expansion of $f(x)$ in Laguerre Functions,

$$\int_{\xi=0}^{\xi=\infty} f(\xi)\underbrace{e^{-\xi}\left\{L_0(\xi)L_0(x)+L_1(\xi)L_1(x)+...+L_n(\xi)L_n(x)\right\}}_{\text{Laguerre Sequence}}d\xi\,,$$

give rise to the Laguerre Sequence.

The limit of the Laguerre Sequence is the Laguerre Kernel,

$$e^{-\xi}\left\{L_0(\xi)L_0(x)+L_1(\xi)L_1(x)+...+L_n(\xi)L_n(x)+...\right\}.$$

The Laguerre Kernel diverges to infinity for any $x=\xi$, and vanishes for any $x\neq\xi$. [Dan10].

We established, [Dan10], that in Infinitesimal Calculus, the Laguerre Kernel is a Hyper-real Infinite Series that sums up to a Delta Function.

$$\mathcal{L}_{aguerre}(\xi-x)=\begin{cases}\left\langle n\right\rangle & ,\xi=x\\ 0 & ,\xi\neq x\end{cases}$$

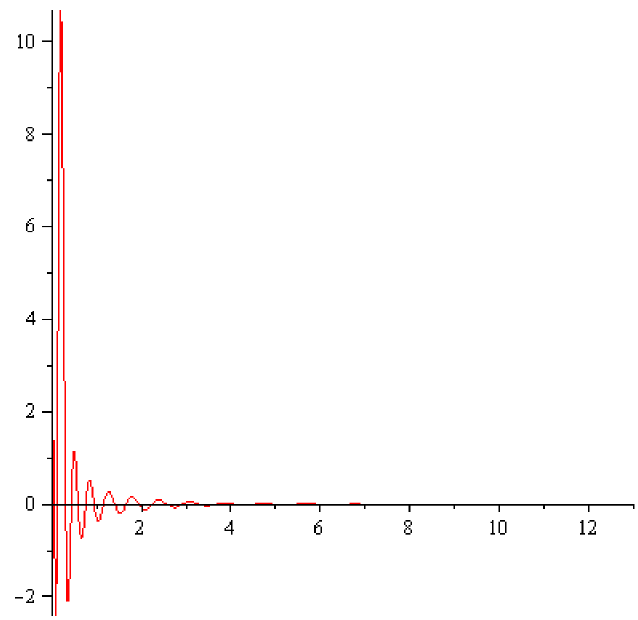
$$=\delta(\xi-x).$$

Thus,

13.1 Laguerre Kernel Summation

$$e^{-\xi}\left\{L_0(\xi)L_0(x)+L_1(\xi)L_1(x)+...+L_n(\xi)L_n(x)+...\right\}=\delta(\xi-x)$$

13.2 **In Maple,** plot($\sum_{i=0}^{223} e^{-x}$LaguerreL(i, 0.2)))*LaguerreL(i, x), x = .1 .. 23)



14.

Chebyshev Kernel

The partial sums of the Expansion of $f(x)$ in Chebyshev Polynomials,

$$\int_{\xi=-1}^{\xi=1} f(\xi)\underbrace{\tfrac{2}{\pi}\tfrac{1}{\sqrt{1-x^2}}\left\{\tfrac{1}{2}+T_1(\xi)PT_1(x)+..T_n(\xi)T_n(x)\right\}}_{\text{Chebyshev Sequence}}d\xi\,,$$

give rise to the Chebyshev Sequence.

The limit of the Chebyshev Sequence is the Chebyshev Kernel,

$$\tfrac{2}{\pi}\tfrac{1}{\sqrt{1-\xi^2}}\left\{\tfrac{1}{2}+T_1(\xi)T_1(x)+...+T_n(\xi)T_n(x)+...\right\}$$

The Chebyshev Kernel diverges to infinity for any $x-\xi=2m$, and vanishes for any $x-\xi\neq 2m$. [Dan11].

Let

$$x=\cos\theta_x,\quad \xi=\cos\theta_\xi,$$

$\langle n\rangle$ an infinite Hyper-real.

We established, [Dan11], that in Infinitesimal Calculus, the Chebyshev Kernel is a Hyper-real Infinite Series that sums up to a Periodic Delta Function, with a period $T=2$.

$$\mathcal{C}_{hebyshev}(\theta_\xi-\theta_x)=\begin{cases}\langle n\rangle, & \theta_\xi-\theta_x=2\pi m\\ 0, & \theta_\xi-\theta_x\neq 2\pi m\end{cases}$$

$$= \ldots + \delta(\theta_\xi - \theta_x + 2\pi) + \delta(\theta_\xi - \theta_x) + \delta(\theta_\xi - \theta_x - 2\pi) + \ldots$$

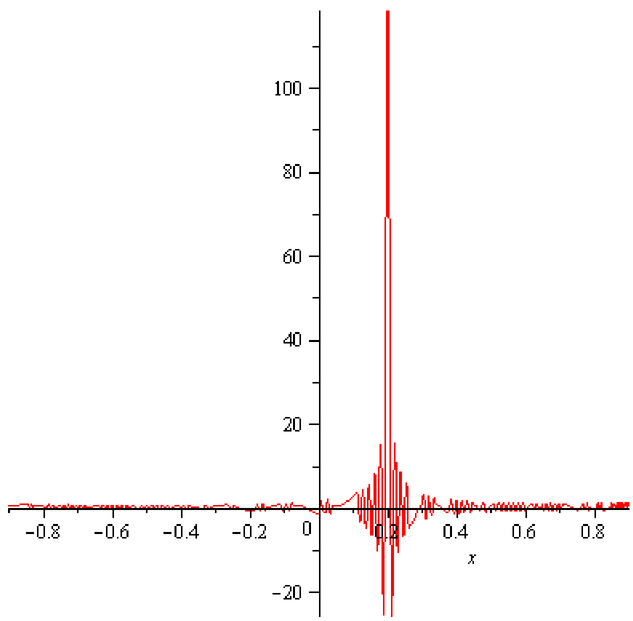
Thus,

14.1 Chebyshev Kernel Summation

$$\frac{2}{\pi}\frac{1}{\sqrt{1-\xi^2}}\left\{\frac{1}{2} + T_1(\xi)T_1(x) + \ldots + T_n(\xi)T_n(x) + \ldots\right\} =$$

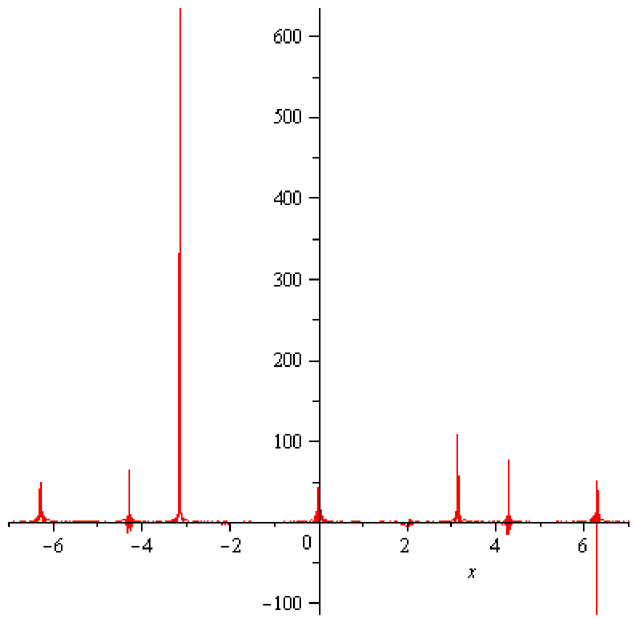
$$= \ldots + \delta(\theta_\xi - \theta_x + 2\pi) + \delta(\theta_\xi - \theta_x) + \delta(\theta_\xi - \theta_x - 2\pi) + \ldots$$

14.2 In Maple, plot($\sum_{i=0}^{365} \frac{2}{\pi}\frac{1}{\sqrt{1-x^2}}$ChebyshevT(i, 0.2)*ChebyshevT(i, x), x = -.9 .. .9)



For $\theta = \arccos(x)$, the pulses are periodic

14.3 In Maple, plot($\sum_{i=0}^{222} \frac{2}{\pi}\frac{1}{\sqrt{1-\cos^2(\theta)}}$ChebyshevT(i, cos(2))*ChebyshevT(i, cos(θ)), θ = -7 .. 7)


600
500
400
300
200
100
-100
-6
-4
-2
0
2
4
6
x

15.

Fourier-Bessel Kernel

Let

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots$$

be the zeros of the Bessel Function $J_0(x)$.

The partial sums of the Fourier-Bessel Series associated with $f(x)$,

$$\int_{\xi=0}^{\xi=1} f(\xi) \underbrace{2\xi\left\{\frac{J_0(\lambda_1\xi)J_0(\lambda_1 x)}{J_1^2(\lambda_1)} + .. + \frac{J_0(\lambda_k\xi)J_0(\lambda_k x)}{J_1^2(\lambda_k)}\right\}}_{\text{Bessel Sequence}} d\xi\,,$$

give rise to the Bessel Sequence.

The limit of the Bessel Sequence is the Bessel Kernel,

$$2\xi\left\{\frac{J_0(\lambda_1\xi)J_0(\lambda_1 x)}{J_1^2(\lambda_1)} + \frac{J_0(\lambda_2\xi)J_0(\lambda_2 x)}{J_1^2(\lambda_2)} + \frac{J_0(\lambda_3\xi)J_0(\lambda_3 x)}{J_1^2(\lambda_3)} + \ldots\right\}$$

$$= \sqrt{\tfrac{\xi}{x}}\left\{\tfrac{1}{2} + \cos\pi(\xi - x) + \cos 2\pi(\xi - x) + \ldots\right\}$$

$$= \sqrt{\tfrac{\xi}{x}}\,\tfrac{1}{2}\left\{\ldots + e^{-2i\pi(\xi-x)} + e^{-i\pi(\xi-x)} + 1 + e^{i\pi(\xi-x)} + e^{2i\pi(\xi-x)} + \ldots\right\}.$$

The Bessel Kernel diverges to infinity for any $x - \xi = 2m$, and vanishes for any $x - \xi \neq 2m$. [Dan12]

We established, [Dan12], that in Infinitesimal Calculus, the Bessel Kernel is a Hyper-real Infinite Series

$$\mathcal{B}_{essel}(\xi - x) = 2\xi\left\{\frac{J_0(\lambda_1\xi)J_0(\lambda_1 x)}{J_1^2(\lambda_1)} + \frac{J_0(\lambda_2\xi)J_0(\lambda_2 x)}{J_1^2(\lambda_2)} + ...\right\}$$

$$= \sqrt{\tfrac{\xi}{x}}\left\{\tfrac{1}{2} + \cos\pi(\xi - x) + \cos 2\pi(\xi - x) + ...\right\}$$

$$= \sqrt{\tfrac{\xi}{x}}\,\tfrac{1}{2}\left\{... + e^{-2i\pi(\xi - x)} + e^{-i\pi(\xi - x)} + 1 + e^{i\pi(\xi - x)} + e^{2i\pi(\xi - x)} + ...\right\}$$

The Bessel Kernel sums up to a Periodic Delta Function, with a period $T = 2$.

$$\mathcal{B}_{essel}(\xi - x) = \begin{cases} \sqrt{\tfrac{\xi}{x}}\left\langle \tfrac{1}{2} + n \right\rangle, & \xi - x = 2m \\ 0, & \xi - x \neq 2m \end{cases}$$

$$= \sqrt{\tfrac{\xi}{x}}\left\{... + \delta(\xi - x + 2) + \delta(\xi - x) + \delta(\xi - x) + ...\right\}.$$

Thus,

15.1 Fourier-Bessel Kernel Summation

$$2\xi\left\{\frac{J_0(\lambda_1\xi)J_0(\lambda_1 x)}{J_1^2(\lambda_1)} + \frac{J_0(\lambda_2\xi)J_0(\lambda_2 x)}{J_1^2(\lambda_2)} + ...\right\} =$$

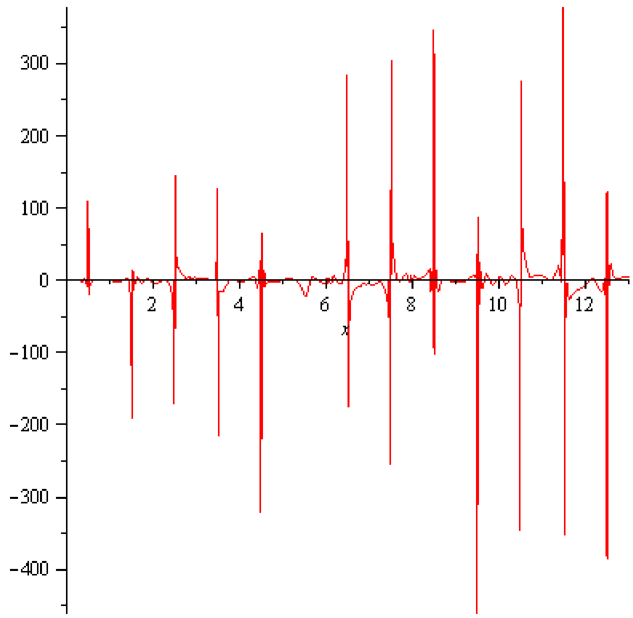
$$= \sqrt{\tfrac{\xi}{x}}\,\tfrac{1}{2}\left\{... + e^{-2i\pi(\xi - x)} + e^{-i\pi(\xi - x)} + 1 + e^{i\pi(\xi - x)} + e^{2i\pi(\xi - x)} + ...\right\}$$

$$= \sqrt{\tfrac{\xi}{x}}\left\{\tfrac{1}{2} + \cos\pi(\xi - x) + \cos 2\pi(\xi - x) + ...\right\}$$

$$= \sqrt{\tfrac{\xi}{x}}\left\{... + \delta(\xi - x + 2) + \delta(\xi - x) + \delta(\xi - x) + ...\right\}$$

15.2 **In Maple,**

$$plot\left(\sum_{i=1}^{111} 2x\,\frac{\mathrm{BesselJ}(0,\frac{\mathrm{BesselJZeros}(0,i)}{2})}{\left[\mathrm{BesselJ}(1,\mathrm{BesselJZeros}(0,i))\right]^2}\cdot \mathrm{BesselJ}(0,\mathrm{BesselJZeros}(0,i)*x), x = 0..13\right)$$



16.

ν-Bessel Kernel

Let

$$0 < \lambda_{\nu 1} < \lambda_{\nu 2} < \lambda_{\nu 3} < ...$$

be the zeros of the Bessel Function $J_\nu(x)$.

The partial sums of the ν-Bessel Series associated with $f(x)$,

$$\int_{\xi=0}^{\xi=1} f(\xi) \underbrace{2\xi \left\{ \frac{J_\nu(\lambda_{\nu 1}\xi) J_\nu(\lambda_{\nu 1}x)}{J_{\nu+1}^2(\lambda_{\nu 1})} + ... + \frac{J_\nu(\lambda_{\nu k}\xi) J_\nu(\lambda_{\nu k}x)}{J_{\nu+1}^2(\lambda_{\nu k})} \right\}}_{\nu\text{-Bessel Sequence}} d\xi,$$

give rise to the ν-Bessel Sequence.

The limit of the ν-Bessel Sequence is the ν-Bessel Kernel,

$$2\xi \left\{ \frac{J_\nu(\lambda_{\nu 1}\xi) J_\nu(\lambda_{\nu 1}x)}{J_{\nu+1}^2(\lambda_{\nu 1})} + \frac{J_\nu(\lambda_{\nu 2}\xi) J_\nu(\lambda_{\nu 2}x)}{J_{\nu+1}^2(\lambda_{\nu 2})} + \frac{J_\nu(\lambda_{\nu 3}\xi) J_\nu(\lambda_{\nu 3}x)}{J_{\nu+1}^2(\lambda_{\nu 3})} + ... \right\} =$$

$$= \sqrt{\tfrac{\xi}{x}} \left\{ \tfrac{1}{2} + \cos \pi(\xi - x) + \cos 2\pi(\xi - x) + ... \right\}$$

$$= \sqrt{\tfrac{\xi}{x}}\, \tfrac{1}{2} \left\{ ... + e^{-2i\pi(\xi - x)} + e^{-i\pi(\xi - x)} + 1 + e^{i\pi(\xi - x)} + e^{2i\pi(\xi - x)} + ... \right\}.$$

The ν-Bessel Kernel diverges to infinity for any $x - \xi = 2m$, and vanishes for any $x - \xi \neq 2m$. [Dan12]

We established, [Dan12], that in Infinitesimal Calculus, the ν-Bessel Kernel is a Hyper-real Infinite Series

$$\nu\mathcal{B}_{essel}(\xi - x) = 2\xi\left\{\frac{J_\nu(\lambda_{\nu 1}\xi)J_\nu(\lambda_{\nu 1}x)}{J^2_{\nu+1}(\lambda_{\nu 1})} + \frac{J_\nu(\lambda_{\nu 2}\xi)J_\nu(\lambda_{\nu 2}x)}{J^2_{\nu+1}(\lambda_{\nu 2})} + ...\right\}$$

$$= \sqrt{\tfrac{\xi}{x}}\left\{\tfrac{1}{2} + \cos\pi(\xi - x) + \cos 2\pi(\xi - x) + ...\right\}$$

$$= \sqrt{\tfrac{\xi}{x}}\,\tfrac{1}{2}\left\{... + e^{-2i\pi(\xi-x)} + e^{-i\pi(\xi-x)} + 1 + e^{i\pi(\xi-x)} + e^{2i\pi(\xi-x)} + ...\right\}$$

The ν-Bessel Kernel sums up to a Periodic Delta Function, with a period $T = 2$.

$$\nu\mathcal{B}_{essel}(\xi - x) = \begin{cases} \frac{\xi}{x}\left\langle \frac{1}{2} + n \right\rangle, & \xi - x = 2m \\ 0, & \xi - x \neq 2m \end{cases}$$

$$= \sqrt{\tfrac{\xi}{x}}\left\{... + \delta(\xi - x + 2) + \delta(\xi - x) + \delta(\xi - x) + ...\right\}.$$

Thus,

16.1 ν-Bessel Kernel Summation

$$2\xi\left\{\frac{J_\nu(\lambda_{\nu 1}\xi)J_\nu(\lambda_{\nu 1}x)}{J^2_{\nu+1}(\lambda_{\nu 1})} + \frac{J_\nu(\lambda_{\nu 2}\xi)J_\nu(\lambda_{\nu 2}x)}{J^2_{\nu+1}(\lambda_{\nu 2})} + \frac{J_\nu(\lambda_{\nu 3}\xi)J_\nu(\lambda_{\nu 3}x)}{J^2_{\nu+1}(\lambda_{\nu 3})} + ...\right\} =$$

$$= \sqrt{\tfrac{\xi}{x}}\,\tfrac{1}{2}\left\{... + e^{-2i\pi(\xi-x)} + e^{-i\pi(\xi-x)} + 1 + e^{i\pi(\xi-x)} + e^{2i\pi(\xi-x)} + ...\right\}$$

$$= \sqrt{\tfrac{\xi}{x}}\left\{\tfrac{1}{2} + \cos\pi(\xi - x) + \cos 2\pi(\xi - x) + ...\right\}$$

$$= \sqrt{\tfrac{\xi}{x}}\left\{... + \delta(\xi - x + 2) + \delta(\xi - x) + \delta(\xi - x) + ...\right\}$$

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