

Regarding Response to the American Mathematical Monthly Confusion about Circular Motion

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Introduction

In a circular motion,

$$\vec{r} = \begin{bmatrix} a \cos \omega t \\ a \sin \omega t \end{bmatrix},$$

where $\omega =$ angular velocity. Then,

$$\vec{r}' = \begin{bmatrix} -\omega a \sin \omega t \\ \omega a \cos \omega t \end{bmatrix},$$

$$\vec{r}'' = \begin{bmatrix} -\omega^2 a \cos \omega t \\ -\omega^2 a \sin \omega t \end{bmatrix} = -\omega^2 \vec{r}.$$

The acceleration is radial, directed into the center.

The term scalar acceleration is reserved to

$$|\vec{r}''| = \omega^2 r.$$

Confused about it, the American Mathematical Monthly [Saari, p.409] has

*“If $\vec{r}(t)$ is the vector position of a planet with mass m ,
relative to the sun with mass M ,*

$r = |\vec{r}|$ is the vector's length, and

G is the gravitational constant,

then, Newton's inverse-square force law is

$$m\vec{r}'' = -G \frac{Mm}{r^2} \frac{\vec{r}}{r}, \quad (1)$$

where \vec{r}'' is the acceleration.

To find the radial acceleration r'' , differentiate the scalar product $r^2 = (\vec{r}, \vec{r})$ twice. Using

$$\vec{r}^2 \vec{v}^2 = (\vec{r} \cdot \vec{v})^2 + (\vec{r} \times \vec{v})^2 = r^2 v^2 + r^2 v_{rot}^2 \quad (*)$$

where $\vec{v} = \dot{\vec{r}}$ is the velocity, and v_{rot} is the rotational velocity, it follows by substitution into equation 1 that the scalar acceleration satisfies

$$mr'' = -G \frac{Mm}{r^2} + \frac{mv_{rot}^2}{r} \quad (2)$$

Earth orbit is essentially circular, which means that

$$r'' \approx 0, \quad (**)$$

or (from equation 2) that the sun's mass is

$$M \approx \frac{rv_{rot}^2}{G} \quad (3).$$

In [Dan], we observed that for the Monthly, r'' means $|\vec{r}''|$.

And $r'' = 0$ means $|\vec{r}''| = 0$.

Thus, according to the Monthly, $r'' = 0 \Leftrightarrow \vec{r}'' = 0$.

In particular, their statement

To find the radial acceleration r'' , differentiate the scalar product $r^2 = (\vec{r}, \vec{r})$ twice.

leads precisely to $|\vec{r}''|$.

That is, the Monthly leaves no doubt that by r'' it means $|\vec{r}''|$.

But we received comments indicating that the Monthly's confusion of r'' with $|\vec{r}''|$ is contagious.

The comments require a further review of the Monthly's claims:

1.

The Monthly's Claim that the Gravitational Force is Zero

By

$$r'' ,$$

the Monthly refers to the scalar acceleration

$$|\vec{r}''| = \omega^2 r .$$

Indeed, the Monthly proposed derivation leads precisely to it:

$$\vec{r} \cdot \vec{r} = a^2 ,$$

$$2\vec{r} \cdot \vec{r}' = 0 ,$$

$$(\vec{r}')^2 + \vec{r} \cdot \vec{r}'' = 0 ,$$

$$|\vec{r}''| = -\frac{(\vec{r}')^2}{r} = -\frac{v^2}{r} = -\omega^2 r .$$

Therefore, $r'' = 0$ in their Equation (**) means

$$|\vec{r}''| = 0.$$

Hence,

$$m\vec{r}'' = 0.$$

Thus, by their Equation (1),

$$0 = -G \frac{Mm}{r^2} \frac{\vec{r}}{r}$$

And the Gravitational Force is Zero

That claim follows also from their statement

the scalar acceleration satisfies

$$mr'' = -G \frac{Mm}{r^2} + \frac{mv_{rot}^2}{r} \quad (2)$$

It is clear that r'' is their abuse of notation for the scalar acceleration $|\vec{r}''|$.

Since $|\vec{r}''| = \omega^2 r$, Equation 2 says

$$m\omega^2 r = -G \frac{Mm}{r^2} + \frac{mv_{rot}^2}{r}$$

Substituting $v_{rot} = \omega r$,

$$m\omega^2 r = -G \frac{Mm}{r^2} + \frac{m(\omega r)^2}{r},$$

$$0 = -G \frac{Mm}{r^2}.$$

And again, the Gravitational Force is Zero

2.

The Monthly's Claims that the Earth does not orbit the Sun, and the Sun's Mass is Zero

The Monthly's poor conception of vectors, leads to further abuse of notation in the equation

$$\vec{r}^2\vec{v}^2 = (\vec{r} \cdot \vec{v})^2 + (\vec{r} \times \vec{v})^2 = r^2r'^2 + r^2v_{rot}^2 \quad (*)$$

Since

$$r' = 0,$$

then, again, by r' , the Monthly means $|\vec{r}'|$, where

$$\vec{r}' = \vec{v}.$$

That is,

$$\vec{r}^2\vec{v}^2 = r^2r'^2 + r^2v_{rot}^2,$$

means

$$\vec{r}^2\vec{v}^2 = r^2v^2 + r^2v_{rot}^2.$$

But since $\vec{r}^2\vec{v}^2 = r^2v^2$, we have

$$0 = r^2v_{rot}^2,$$

$$0 = v_{rot}.$$

And the Earth does not orbit the Sun

Furthermore, from Equation 3, $M \approx \frac{rv_{rot}^2}{G} = \frac{r \cdot 0}{G} = 0,$

That is, The Sun's Mass is Zero

Finally, the Monthly is ignorant of the fact that in Circular Motion

$$\vec{v} \perp \vec{r},$$

$$\vec{r} \cdot \vec{v} = 0,$$

$$|\vec{v} \times \vec{r}| = vr$$

and the equation $\vec{r}^2 \vec{v}^2 = (\vec{r} \cdot \vec{v})^2 + (\vec{r} \times \vec{v})^2$ is the redundant

$$\vec{r}^2 \vec{v}^2 = r^2 v^2.$$

References

[Dan] H. Vic Dannon, ["American Mathematical Monthly claims that in a Circular Motion, the Radial Acceleration is zero, there is no Gravitational Force, the Earth does not orbit the Sun, and the Sun's Mass is Zero"](#) Gauge Institute Journal Of Math and Physics, Vol.11, No 3, August 2015. pp. 81-86.

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[Spiegel], Murray Spiegel, Theoretical Mechanics, Schaum's Outline, p.116.