The Curvature of Space-Time

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Abstract: The Curvature of space-time measures the curving of space-time in the x, y, z, t embedding space.

The curvature is

$$\kappa = \frac{\text{The Second Fundamental Form of Space-time}}{\text{The First Fundamental Form of Space-time}}.$$

To compute the Second Fundamental Form for a two parameter surface in a 3-dimensional space, the Normal is taken as the cross product of the two tangent vectors along the two parameters. However, the cross product is not defined for the four dimensional tangent vectors of a surface. And the curvature of the three parameter space-time embedded in the x, y, z, t space is unknown.

Since curvature is the most important characterization of a surface, one may expect to find the answer in Riemannian Geometry. But all Riemannian Geometry textbooks present examples of 2-surfaces in 3-dimensional spaces. None attempt to find the curvature of Space-time which is a 3-surface in the 4-dimensional x, y, z, t space.

They tell about Riemann's Curvature Tensor. But they do not know how to find the scalar[1] Curvature in a 4-dimensional space.

We find the normal to Space-time, and compute its curvature.

Space-time curvature is measured in volume units.

Keywords: Curvature, 2-surface, 3-surface, Metric Tensor, Normal to Surface, First Fundamental Form, Second Fundamental Form,

Contents

[1] "Total", or "Gaussian" Curvature

1.

Hyperbolic Hyperboloid in x, y, z

A hyperbolic hyperboloid is the x, y, z surface

$$x^2 + y^2 - z^2 = R^2$$

Parameterized, with angle ϕ, and angle θ, it is

$$\vec{x}(\phi, \theta) = \begin{pmatrix} R\cosh\theta\cos\phi \\ R\cosh\theta\sin\phi \\ iR\sinh\theta \end{pmatrix}.$$

The Tangent vectors at $\vec{x}$ are

$$\partial_\phi \vec{x} = \partial_\phi \begin{pmatrix} R\cosh\theta\cos\phi \\ R\cosh\theta\sin\phi \\ iR\sinh\theta \end{pmatrix} = \begin{pmatrix} -R\cosh\theta\sin\phi \\ R\cosh\theta\cos\phi \\ 0 \end{pmatrix}$$

and

$$\partial_\theta \vec{x} = \partial_\theta \begin{pmatrix} R\cosh\theta\cos\phi \\ R\cosh\theta\sin\phi \\ iR\sinh\theta \end{pmatrix} = \begin{pmatrix} R\sinh\theta\cos\phi \\ R\sinh\theta\sin\phi \\ iR\cosh\theta \end{pmatrix}$$

The Metric Tensor is

$$g_{ij} = (\partial_i \vec{x}) \cdot (\partial_j \vec{x})$$

$$= \begin{bmatrix} R^2\cosh^2\theta & 0 \\ 0 & R^2(\sinh^2\theta - \cosh^2\theta) \end{bmatrix}$$

$$= \begin{bmatrix} R^2\cosh^2\theta & 0 \\ 0 & -R^2 \end{bmatrix}$$

The First Fundamental Form is

$$I = \det(g_{ij}) = -R^4 \cosh^2 \theta$$

A normal vector to the surface at $\vec{x}$ is the cross product

$$(\partial_\phi \vec{x}) \times (\partial_\theta \vec{x}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -R\cosh\theta\sin\phi & R\cosh\theta\cos\phi & 0 \\ R\sinh\theta\cos\phi & R\sinh\theta\sin\phi & iR\cosh\theta \end{vmatrix}$$

$$= \begin{bmatrix} iR^2 \cosh^2\theta\cos\phi \\ iR^2\cosh^2\theta\sin\phi \\ -R^2\cosh\theta\sinh\theta \end{bmatrix}$$

$$= iR^2\cosh\theta \begin{bmatrix} \cosh\theta\cos\phi \\ \cosh\theta\sin\phi \\ i\sinh\theta \end{bmatrix}$$

The unit normal is

$$\vec{N} = \begin{bmatrix} \cosh\theta\cos\phi \\ \cosh\theta\sin\phi \\ i\sinh\theta \end{bmatrix}$$

$\vec{N}$ is perpendicular to the tangents vectors:

$$\vec{N} \cdot \partial_\phi \vec{x} = \begin{bmatrix} \cosh\theta\cos\phi \\ \cosh\theta\sin\phi \\ i\sinh\theta \end{bmatrix} \cdot \begin{pmatrix} -R\cosh\theta\sin\phi \\ R\cosh\theta\cos\phi \\ 0 \end{pmatrix} = 0,$$

and

$$\vec{N}\cdot\partial_\theta\vec{x} = \begin{bmatrix}\cosh\theta\cos\phi\\ \cosh\theta\sin\phi\\ i\sinh\theta\end{bmatrix}\cdot\begin{pmatrix}R\sinh\theta\cos\phi\\ R\sinh\theta\sin\phi\\ iR\cosh\theta\end{pmatrix} = 0\,.$$

For the matrix of the Second Fundamental Form,

$$\partial_{\phi\phi}\vec{x} = \partial_\phi(\partial_\phi\vec{x}) = \partial_\phi\begin{pmatrix}-R\cosh\theta\sin\phi\\ R\cosh\theta\cos\phi\\ 0\end{pmatrix} = \begin{pmatrix}-R\cosh\theta\cos\phi\\ -R\cosh\theta\sin\phi\\ 0\end{pmatrix}$$

$$\vec{N}\cdot\partial_{\phi\phi}\vec{x} = \begin{bmatrix}\cosh\theta\cos\phi\\ \cosh\theta\sin\phi\\ i\sinh\theta\end{bmatrix}\cdot\begin{pmatrix}-R\cosh\theta\cos\phi\\ -R\cosh\theta\sin\phi\\ 0\end{pmatrix} = -iR\cosh^2\theta$$

$$\partial_{\theta\theta}\vec{x} = \partial_\theta(\partial_\theta\vec{x}) = \partial_\theta\begin{pmatrix}R\sinh\theta\cos\phi\\ R\sinh\theta\sin\phi\\ iR\cosh\theta\end{pmatrix} = \begin{pmatrix}R\cosh\theta\cos\phi\\ R\cosh\theta\sin\phi\\ iR\sinh\theta\end{pmatrix}$$

$$\vec{N}\cdot\partial_{\theta\theta}\vec{x} = \begin{bmatrix}\cosh\theta\cos\phi\\ \cosh\theta\sin\phi\\ i\sinh\theta\end{bmatrix}\cdot\begin{pmatrix}R\cosh\theta\cos\phi\\ R\cosh\theta\sin\phi\\ iR\sinh\theta\end{pmatrix}$$

$$= R(\cosh^2\theta - \sinh^2\theta)$$

$$= R\,.$$

$$\partial_{\phi\theta}\vec{x} = \partial_\phi\begin{pmatrix}R\sinh\theta\cos\phi\\ R\sinh\theta\sin\phi\\ iR\cosh\theta\end{pmatrix} = \begin{pmatrix}-R\sinh\theta\sin\phi\\ R\sinh\theta\cos\phi\\ 0\end{pmatrix} = \partial_{\theta\phi}\vec{x}$$

$$\vec{N}\cdot\partial_{\phi\theta}\vec{x} = \begin{bmatrix}\cosh\theta\cos\phi\\ \cosh\theta\sin\phi\\ i\sinh\theta\end{bmatrix}\cdot\begin{pmatrix}-R\sinh\theta\sin\phi\\ R\sinh\theta\cos\phi\\ 0\end{pmatrix} = 0$$

The matrix for the Second Fundamental Form is

$$\vec{N} \cdot \partial_{ij}\vec{x} = \begin{pmatrix} -iR\cosh^2\theta & 0 \\ 0 & R \end{pmatrix}$$

The Second Fundamental Form is

$$\det(\vec{N} \cdot \partial_{ij}\vec{x}) = -iR^2\cosh^2\theta$$

The Curvature of the hyperbolic hyperboloid in x, y, z is

$$\frac{II}{I} = \frac{\det(\vec{N} \cdot \partial_{ij}\vec{x})}{\det(g_{ij})} = \frac{-iR^2\cosh^2\theta}{-R^4\cosh^2\theta} = \frac{i}{R^2}.\square$$

2.

Space time

Space time is a hyperbolic hyperboloid in x, y, z, t,

$$x^2 + y^2 + z^2 - (ct)^2 = R^2$$

Parameterized, with angle ϕ, angle θ, and angle ψ, it is

$$\vec{x}(\phi, \theta, \psi) = \begin{pmatrix} R\cosh\psi\cos\theta\cos\phi \\ R\cosh\psi\cos\theta\sin\phi \\ R\cosh\psi\sin\theta \\ iR\sinh\psi \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ ict \end{pmatrix}$$

The Tangent vectors at $\vec{x}$ are

$$\partial_\phi \vec{x} = \partial_\phi \begin{pmatrix} R\cosh\psi\cos\theta\cos\phi \\ R\cosh\psi\cos\theta\sin\phi \\ R\cosh\psi\sin\theta \\ iR\sinh\psi \end{pmatrix} = \begin{pmatrix} -R\cosh\psi\cos\theta\sin\phi \\ R\cosh\psi\cos\theta\cos\phi \\ 0 \\ 0 \end{pmatrix},$$

$$\partial_\theta \vec{x} = \partial_\theta \begin{pmatrix} R\cosh\psi\cos\theta\cos\phi \\ R\cosh\psi\cos\theta\sin\phi \\ R\cosh\psi\sin\theta \\ iR\sinh\psi \end{pmatrix} = \begin{pmatrix} -R\cosh\psi\sin\theta\cos\phi \\ -R\cosh\psi\sin\theta\sin\phi \\ R\cosh\psi\cos\theta \\ 0 \end{pmatrix},$$

and

$$\partial_\psi \vec{x} = \partial_\psi \begin{pmatrix} R\cosh\psi\cos\theta\cos\phi \\ R\cosh\psi\cos\theta\sin\phi \\ R\cosh\psi\sin\theta \\ iR\sinh\psi \end{pmatrix} = \begin{pmatrix} R\sinh\psi\cos\theta\cos\phi \\ R\sinh\psi\cos\theta\sin\phi \\ R\sinh\psi\sin\theta \\ iR\cosh\psi \end{pmatrix}$$

The Metric Tensor is

$$g_{ij} = (\partial_i \vec{x}) \cdot (\partial_j \vec{x})$$

$$= \begin{pmatrix} R^2 \cosh^2 \psi \cos^2 \theta & 0 & 0 \\ 0 & R^2 \cosh^2 \psi & 0 \\ 0 & 0 & R^2(\sinh^2 \psi - \cosh^2 \psi) \end{pmatrix}$$

$$= \begin{pmatrix} R^2 \cosh^2 \psi \cos^2 \theta & 0 & 0 \\ 0 & R^2 \cosh^2 \psi & 0 \\ 0 & 0 & -R^2 \end{pmatrix}$$

<u>The First Fundamental Form is</u>

$$\underline{I = \det(g_{ij}) = -R^6 \cosh^4 \psi \cos^2 \theta}$$

The unit normal,

$$\vec{N} = \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}$$

is Perpendicular to the tangents vectors:

$$0 = \vec{N} \cdot \partial_\phi \vec{x} = \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} \cdot \begin{pmatrix} -R \cosh \psi \cos \theta \sin \phi \\ R \cosh \psi \cos \theta \cos \phi \\ 0 \\ 0 \end{pmatrix}$$

$$= -AR \cosh \psi \cos \theta \sin \phi + BR \cosh \psi \cos \theta \cos \phi$$

Therefore, we set

$$A = \cosh \psi \cos \theta \cos \phi,$$
$$B = \cosh \psi \cos \theta \sin \phi,$$

and

$$\vec{N} = \begin{pmatrix} \cosh\psi\cos\theta\cos\phi \\ \cosh\psi\cos\theta\sin\phi \\ C \\ D \end{pmatrix}.$$

$$0 = \vec{N}\cdot\partial_\theta\vec{x} = \begin{pmatrix} \cosh\psi\cos\theta\cos\phi \\ \cosh\psi\cos\theta\sin\phi \\ C \\ D \end{pmatrix}\cdot\begin{pmatrix} -R\cosh\psi\sin\theta\cos\phi \\ -R\cosh\psi\sin\theta\sin\phi \\ R\cosh\psi\cos\theta \\ 0 \end{pmatrix}$$

$$= -R\cosh^2\psi\cos\theta\sin\theta + RC\cosh\psi\cos\theta$$

Therefore,

$$C = \cosh\psi\sin\theta\,.$$

$$0 = \vec{N}\cdot\partial_\psi\vec{x} = \begin{pmatrix} \cosh\psi\cos\theta\cos\phi \\ \cosh\psi\cos\theta\sin\phi \\ \cosh\psi\sin\theta \\ D \end{pmatrix}\cdot\begin{pmatrix} R\sinh\psi\cos\theta\cos\phi \\ R\sinh\psi\cos\theta\sin\phi \\ R\sinh\psi\sin\theta \\ iR\cosh\psi \end{pmatrix}$$

$$= \cosh\psi\sinh\psi + iD\cosh\psi\,.$$

Therefore,

$$D = i\sinh\psi$$

and

$$\vec{N} = \begin{pmatrix} \cosh\psi\cos\theta\cos\phi \\ \cosh\psi\cos\theta\sin\phi \\ \cosh\psi\sin\theta \\ i\sinh\psi \end{pmatrix}$$

For the matrix of the Second Fundamental Form,

$$\partial_{\phi\phi}\vec{x} = \partial_\phi(\partial_\phi\vec{x}) = \partial_\phi\begin{pmatrix} -R\cosh\psi\cos\theta\sin\phi \\ R\cosh\psi\cos\theta\cos\phi \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -R\cosh\psi\cos\theta\cos\phi \\ -R\cosh\psi\cos\theta\sin\phi \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{N}\cdot\partial_{\phi\phi}\vec{x} = \begin{pmatrix} \cosh\psi\cos\theta\cos\phi \\ \cosh\psi\cos\theta\sin\phi \\ \cosh\psi\sin\theta \\ i\sinh\psi \end{pmatrix}\cdot\begin{pmatrix} -R\cosh\psi\cos\theta\cos\phi \\ -R\cosh\psi\cos\theta\sin\phi \\ 0 \\ 0 \end{pmatrix}$$

$$= -R\cosh^2\psi\cos^2\theta$$

$$\partial_{\theta\theta}\vec{x} = \partial_\theta(\partial_\theta\vec{x}) = \partial_\theta\begin{pmatrix} -R\cosh\psi\sin\theta\cos\phi \\ -R\cosh\psi\sin\theta\sin\phi \\ R\cosh\psi\cos\theta \\ 0 \end{pmatrix} = \begin{pmatrix} -R\cosh\psi\cos\theta\cos\phi \\ -R\cosh\psi\cos\theta\sin\phi \\ -R\cosh\psi\sin\theta \\ 0 \end{pmatrix}$$

$$\vec{N}\cdot\partial_{\theta\theta}\vec{x} = \begin{pmatrix} \cosh\psi\cos\theta\cos\phi \\ \cosh\psi\cos\theta\sin\phi \\ \cosh\psi\sin\theta \\ i\sinh\psi \end{pmatrix}\cdot\begin{pmatrix} -R\cosh\psi\cos\theta\cos\phi \\ -R\cosh\psi\cos\theta\sin\phi \\ -R\cosh\psi\sin\theta \\ 0 \end{pmatrix}$$

$$= -R\cosh^2\psi$$

$$\partial_{\psi\psi}\vec{x} = \partial_\psi(\partial_\psi\vec{x}) = \partial_\psi\begin{pmatrix} R\sinh\psi\cos\theta\cos\phi \\ R\sinh\psi\cos\theta\sin\phi \\ R\sinh\psi\sin\theta \\ iR\cosh\psi \end{pmatrix} = \begin{pmatrix} R\cosh\psi\cos\theta\cos\phi \\ R\cosh\psi\cos\theta\sin\phi \\ R\cosh\psi\sin\theta \\ iR\sinh\psi \end{pmatrix}$$

$$\vec{N}\cdot\partial_{\psi\psi}\vec{x} = \begin{pmatrix} \cosh\psi\cos\theta\cos\phi \\ \cosh\psi\cos\theta\sin\phi \\ \cosh\psi\sin\theta \\ i\sinh\psi \end{pmatrix}\cdot\begin{pmatrix} R\cosh\psi\cos\theta\cos\phi \\ R\cosh\psi\cos\theta\sin\phi \\ R\cosh\psi\sin\theta \\ iR\sinh\psi \end{pmatrix},$$

$$= R(\cosh^2\psi - \sinh^2\psi),$$

$$= R.$$

$$\partial_\theta \partial_\phi \vec{x} = \partial_\theta \begin{pmatrix} -R\cosh\psi\cos\theta\sin\phi \\ R\cosh\psi\cos\theta\cos\phi \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} R\cosh\psi\sin\theta\sin\phi \\ -R\cosh\psi\sin\theta\cos\phi \\ 0 \\ 0 \end{pmatrix} = \partial_\phi \partial_\theta \vec{x}$$

$$\vec{N}\cdot\partial_{\phi\theta}\vec{x} = \begin{pmatrix} \cosh\psi\cos\theta\cos\phi \\ \cosh\psi\cos\theta\sin\phi \\ \cosh\psi\sin\theta \\ i\sinh\psi \end{pmatrix} \cdot \begin{pmatrix} R\cosh\psi\sin\theta\sin\phi \\ -R\cosh\psi\sin\theta\cos\phi \\ 0 \\ 0 \end{pmatrix} = 0.$$

$$\partial_\theta \partial_\psi \vec{x} = \partial_\theta \begin{pmatrix} R\sinh\psi\cos\theta\cos\phi \\ R\sinh\psi\cos\theta\sin\phi \\ R\sinh\psi\sin\theta \\ R\cosh\psi \end{pmatrix} = \begin{pmatrix} -R\sinh\psi\sin\theta\cos\phi \\ -R\sinh\psi\sin\theta\sin\phi \\ R\sinh\psi\cos\theta \\ 0 \end{pmatrix} = \partial_\psi \partial_\theta \vec{x}$$

$$\vec{N}\cdot\partial_{\theta\psi}\vec{x} = \begin{pmatrix} \cosh\psi\cos\theta\cos\phi \\ \cosh\psi\cos\theta\sin\phi \\ \cosh\psi\sin\theta \\ i\sinh\psi \end{pmatrix} \cdot \begin{pmatrix} -\sinh\psi\sin\theta\cos\phi \\ -\sinh\psi\sin\theta\sin\phi \\ \sinh\psi\cos\theta \\ 0 \end{pmatrix} = 0$$

$$\partial_\phi \partial_\psi \vec{x} = \partial_\phi \begin{pmatrix} R\sinh\psi\cos\theta\cos\phi \\ R\sinh\psi\cos\theta\sin\phi \\ R\sinh\psi\sin\theta \\ iR\cosh\psi \end{pmatrix} = \begin{pmatrix} -R\sinh\psi\cos\theta\sin\phi \\ R\sinh\psi\cos\theta\cos\phi \\ 0 \\ 0 \end{pmatrix} = \partial_\psi \partial_\phi \vec{x}$$

$$\vec{N}\cdot\partial_{\theta\psi}\vec{x} = \begin{pmatrix} \cosh\psi\cos\theta\cos\phi \\ \cosh\psi\cos\theta\sin\phi \\ \cosh\psi\sin\theta \\ i\sinh\psi \end{pmatrix} \cdot \begin{pmatrix} -R\sinh\psi\cos\theta\sin\phi \\ R\sinh\psi\cos\theta\cos\phi \\ 0 \\ 0 \end{pmatrix} = 0$$

The matrix for the Second Fundamental Form is

$$\vec{N} \cdot \partial_{ij}\vec{x} = \begin{pmatrix} -R\cosh^2\psi\cos^2\theta & 0 & 0 \\ 0 & -R\cosh^2\psi & 0 \\ 0 & 0 & R \end{pmatrix}$$

<u>The Second Fundamental Form is</u>

$$\underline{\det(\vec{N} \cdot \partial_{ij}\vec{x}) = R^3\cosh^4\psi\cos^2\theta}$$

<u>The Curvature of space time is</u>

$$\frac{II}{I} = \frac{\det(\vec{N} \cdot \partial_{ij}\vec{x})}{\det(g_{ij})} = \frac{R^3\cosh^4\psi\cos^2\theta}{-R^6\cosh^4\psi\cos^2\theta}$$

$$\boxed{= -\frac{1}{R^3}}. \square$$

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