The Curvature of a Four Dimensional Surface

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Abstract: The Curvature of a surface measures the curving of the surface in its embedding space.

If a Two-Parameters Surface is a product of two One-Parameter Surfaces, its Curvature is the product of their Curvatures.

Else, the curvature is

$$\kappa = \frac{\text{The Second Fundamental Form of the Surface}}{\text{The First Fundamental Form of the Surface}}.$$

To compute the Second Fundamental Form for a two parameter surface, the Normal is taken as the cross product of the three dimensional tangent vectors along the two parameters. However, the cross product is not defined for the four dimensional tangent vectors of a three parameter surface. And the curvature of three parameter surfaces remains unknown.

But the for four dimensional three parameter Sphere the Unit Normal lies along the position vector, directed towards the center. And for the four dimensional two parameter Klein Bottle we can obtain the four dimensional Normal as well.

Keywords: Curvature, Surface, 1-surface, 2-surface, 3-surface, Metric Tensor, Normal to Surface, First Fundamental Form, Second Fundamental Form,

Contents

1.

Circle of Radius R

The circle is $\vec{x}(\phi) = \begin{pmatrix} R\cos\phi \\ R\sin\phi \end{pmatrix}$.

The Tangent vector is $\partial_\phi \vec{x}(\phi) = \begin{pmatrix} -R\sin\phi \\ R\cos\phi \end{pmatrix}$

The Metric Tensor is

$$g_{11} = \partial_\phi \vec{x} \cdot \partial_\phi \vec{x} = R^2$$

The First Fundamental Form is

$$\det(g_{11}) = R^2$$

The Unit Normal at $\vec{x}$ lies along $-\vec{x}$

$$\vec{N} = \begin{bmatrix} -\cos\phi \\ -\sin\phi \end{bmatrix}.$$

$\vec{N}$ is perpendicular to the Tangent vector

$$\vec{N} \cdot \partial_\phi \vec{x} = \begin{bmatrix} -\cos\phi \\ -\sin\phi \end{bmatrix} \cdot \begin{pmatrix} -R\sin\phi \\ R\cos\phi \end{pmatrix} = 0$$

For the Second Fundamental Form,

$$\partial_{\phi\phi} \vec{x}(\phi) = \begin{pmatrix} -R\cos\phi \\ -R\sin\phi \end{pmatrix}$$

The Second Fundamental Form is

$$\vec{N} \cdot \partial_{\phi\phi} \vec{x}(\phi) = R$$

The circle's curvature is

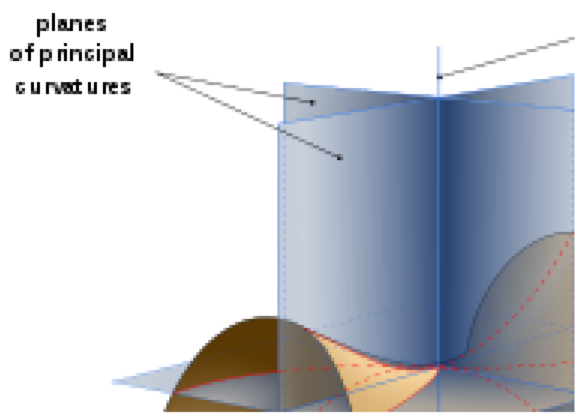
$$\kappa = \frac{\text{Second Fundamental Form}}{\text{First Fundamental Form}} = \frac{\det(\vec{N} \cdot \partial_{\phi\phi} \vec{x})}{\det(g_{11})} = \frac{R}{R^2} = \frac{1}{R}. \square$$

2.

2-Sphere in E^3

Each point of a differentiable two parameter surface has a family of normal planes. Each plane intersects the surface in a plane curve. The curvatures of the curves lie between

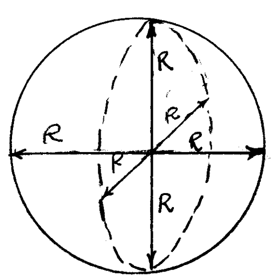
$$\kappa_{\min} \equiv \kappa_1, \text{ and } \kappa_{\max} \equiv \kappa_2 .$$



The Gaussian curvature of a two-dimensional surface is

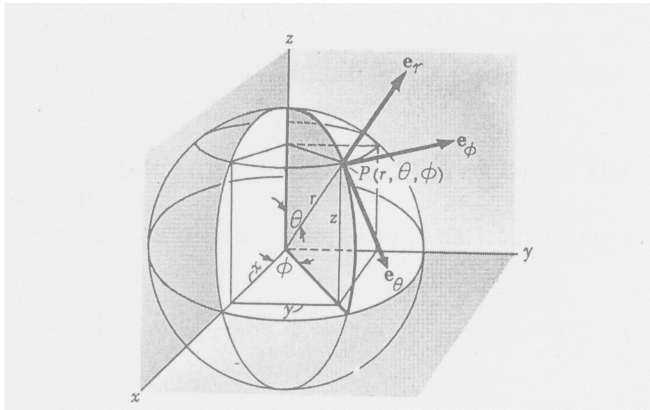
$$K = \kappa_1 \kappa_2 .$$

On a 2-sphere with radius R, any two perpendicularly intersecting great circles have the principal curvatures $\frac{1}{R}$.



The Gaussian Curvature of a 2-sphere of Radius R is $\dfrac{1}{R}\dfrac{1}{R} = \dfrac{1}{R^2}$.

A 2-sphere with Radius R, Azimuth angle ϕ, and polar angle θ,



Is parametrically given by

$$\vec{x}(\theta,\phi) = \begin{pmatrix} R\sin\theta\cos\phi \\ R\sin\theta\sin\phi \\ R\cos\theta \end{pmatrix}$$

The Tangent vectors at $\vec{x}$ are

$$\partial_\theta \vec{x} = \begin{pmatrix} R\cos\theta\cos\phi \\ R\cos\theta\sin\phi \\ -R\sin\theta \end{pmatrix}$$

and

$$\partial_\phi \vec{x} = \begin{pmatrix} -R\sin\theta\sin\phi \\ R\sin\theta\cos\phi \\ 0 \end{pmatrix}$$

The Metric Tensor is

$$g_{ij} = (\partial_i \vec{x}) \cdot (\partial_j \vec{x}) = \begin{bmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{bmatrix}$$

The First Fundamental Form is

$$\det(g_{ij}) = R^4\sin^2\theta$$

The Unit Normal at $\vec{x}$ lies along $-\vec{x}$

$$\vec{N} = \begin{pmatrix} -\sin\theta\cos\phi \\ -\sin\theta\sin\phi \\ -\cos\theta \end{pmatrix}$$

$\vec{N}$ is Perpendicular to the tangents vectors:

$$\vec{N}\cdot\partial_\theta\vec{x} = \begin{pmatrix} -\sin\theta\cos\phi \\ -\sin\theta\sin\phi \\ -\cos\theta \end{pmatrix}\cdot\begin{pmatrix} R\cos\theta\cos\phi \\ R\cos\theta\sin\phi \\ -R\sin\theta \end{pmatrix} = 0,$$

and

$$\vec{N}\cdot\partial_\phi\vec{x} = \begin{pmatrix} -\sin\theta\cos\phi \\ -\sin\theta\sin\phi \\ -\cos\theta \end{pmatrix}\cdot\begin{pmatrix} -R\sin\theta\sin\phi \\ R\sin\theta\cos\phi \\ 0 \end{pmatrix} = 0.$$

For the matrix of the Second Fundamental Form,

$$\partial_{\theta\theta}\vec{x} = \partial_\theta\begin{pmatrix} R\cos\theta\cos\phi \\ R\cos\theta\sin\phi \\ -R\sin\theta \end{pmatrix} = \begin{pmatrix} -R\sin\theta\cos\phi \\ -R\sin\theta\sin\phi \\ -R\cos\theta \end{pmatrix}$$

$$\vec{N}\cdot\partial_{\theta\theta}\vec{x} = \begin{pmatrix} -\sin\theta\cos\phi \\ -\sin\theta\sin\phi \\ -\cos\theta \end{pmatrix}\cdot\begin{pmatrix} -R\sin\theta\cos\phi \\ -R\sin\theta\sin\phi \\ -R\cos\theta \end{pmatrix} = R$$

$$\partial_{\phi\theta}\vec{x} = \partial_\phi\begin{pmatrix} R\cos\theta\cos\phi \\ R\cos\theta\sin\phi \\ -R\sin\theta \end{pmatrix} = \begin{pmatrix} -R\cos\theta\sin\phi \\ R\cos\theta\cos\phi \\ 0 \end{pmatrix} = \partial_{\theta\phi}\vec{x}$$

$$\vec{N}\cdot\partial_{\phi\theta}\vec{x} = \begin{pmatrix} -\sin\theta\cos\phi \\ -\sin\theta\sin\phi \\ -\cos\theta \end{pmatrix}\cdot\begin{pmatrix} -R\cos\theta\sin\phi \\ R\cos\theta\cos\phi \\ 0 \end{pmatrix} = 0$$

$$\partial_{\phi\phi}\vec{x} = \partial_{\phi}\begin{pmatrix} -R\sin\theta\sin\phi \\ R\sin\theta\cos\phi \\ 0 \end{pmatrix} = \begin{pmatrix} -R\sin\theta\cos\phi \\ -R\sin\theta\sin\phi \\ 0 \end{pmatrix}$$

$$\vec{N}\cdot\partial_{\phi\phi}\vec{x} = \begin{pmatrix} -\sin\theta\cos\phi \\ -\sin\theta\sin\phi \\ -\cos\theta \end{pmatrix}\cdot\begin{pmatrix} -R\sin\theta\cos\phi \\ -R\sin\theta\sin\phi \\ 0 \end{pmatrix} = R\sin^2\theta$$

The matrix for the Second Fundamental Form is

$$\vec{N}\cdot\partial_{ij}\vec{x} = \begin{pmatrix} R & 0 \\ 0 & R\sin^2\theta \end{pmatrix}$$

The Second Fundamental Form is

$$\det(\vec{N}\cdot\partial_{ij}\vec{x}) = R^2\sin^2\theta$$

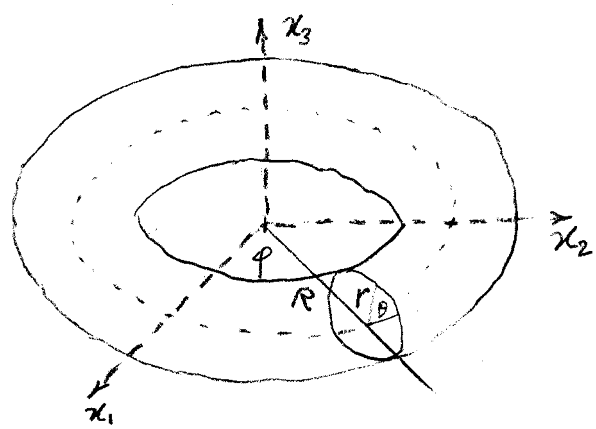
The Curvature of the 2-sphere is

$$\frac{\det(\vec{N}\cdot\partial_{ij}\vec{x})}{\det(g_{ij})} = \frac{R^2\sin^2\theta}{R^4\sin^2\theta} = \frac{1}{R^2}.\square$$

3.

2-Torus in E^3

A point $\vec{x}(\phi,\theta)$ on a 2-Torus with radius R, Azimuth angle ϕ, tube radius r, and elevation angle in the tube θ,



is represented by

$$\vec{x}(\phi,\theta) = \begin{bmatrix} (R + r\sin\theta)\cos\phi \\ (R + r\sin\theta)\sin\phi \\ r\cos\theta \end{bmatrix}.$$

The Tangent vectors at $\vec{x}(\phi,\theta)$ are

$$\partial_\phi \vec{x} = \partial_\phi \begin{bmatrix} (R + r\sin\theta)\cos\phi \\ (R + r\sin\theta)\sin\phi \\ r\cos\theta \end{bmatrix} = \begin{bmatrix} -(R + r\sin\theta)\sin\phi \\ (R + r\sin\theta)\cos\phi \\ 0 \end{bmatrix},$$

and

$$\partial_\theta \vec{x} = \partial_\theta \begin{bmatrix} (R + r\sin\theta)\cos\phi \\ (R + r\sin\theta)\sin\phi \\ r\cos\theta \end{bmatrix} = \begin{bmatrix} r\cos\theta\cos\phi \\ r\cos\theta\sin\phi \\ -r\sin\theta \end{bmatrix}.$$

The Metric Tensor is

$$g_{ij} = \begin{bmatrix} (\partial_\phi \vec{x}) \cdot (\partial_\phi \vec{x}) & (\partial_\phi \vec{x}) \cdot (\partial_\theta \vec{x}) \\ (\partial_\theta \vec{x}) \cdot (\partial_\phi \vec{x}) & (\partial_\theta \vec{x}) \cdot (\partial_\theta \vec{x}) \end{bmatrix} = \begin{bmatrix} (R + r\sin\theta)^2 & 0 \\ 0 & r^2 \end{bmatrix}.$$

The First Fundamental Form is

$$\det(g_{ij}) = (R + r\sin\theta)^2 r^2$$

The Unit Normal at $\vec{x}$ lies along $-\vec{x}$

$$\vec{N} = \begin{bmatrix} -\cos\phi\sin\theta \\ -\sin\phi\sin\theta \\ -\cos\theta \end{bmatrix}$$

$\vec{N}$ is perpendicular to the tangent vectors:

$$\vec{N} \cdot \partial_\theta \vec{x} = \begin{bmatrix} -\cos\phi\sin\theta \\ -\sin\phi\sin\theta \\ -\cos\theta \end{bmatrix} \cdot \begin{bmatrix} r\cos\theta\cos\phi \\ r\cos\theta\sin\phi \\ -r\sin\theta \end{bmatrix} = 0$$

and

$$\vec{N} \cdot \partial_\phi \vec{x} = \begin{bmatrix} -\cos\phi\sin\theta \\ -\sin\phi\sin\theta \\ -\cos\theta \end{bmatrix} \cdot \begin{bmatrix} -(R + r\sin\theta)\sin\phi \\ (R + r\sin\theta)\cos\phi \\ 0 \end{bmatrix} = 0\,.$$

For the Matrix of the Second Fundamental Form,

$$\partial_{\phi\phi} \vec{x} = \partial_\phi \begin{bmatrix} -(R + r\sin\theta)\sin\phi \\ (R + r\sin\theta)\cos\phi \\ 0 \end{bmatrix} = \begin{bmatrix} -(R + r\sin\theta)\cos\phi \\ -(R + r\sin\theta)\sin\phi \\ 0 \end{bmatrix}$$

$$\vec{N} \cdot \partial_{\phi\phi} \vec{x} = \begin{bmatrix} -\cos\phi\sin\theta \\ -\sin\phi\sin\theta \\ -\cos\theta \end{bmatrix} \cdot \begin{bmatrix} -(R + r\sin\theta)\cos\phi \\ -(R + r\sin\theta)\sin\phi \\ 0 \end{bmatrix} = (R + r\sin\theta)\sin\theta$$

$$\partial_{\phi\theta}\vec{x} = \partial_\theta \begin{bmatrix} -(R + r\sin\theta)\sin\phi \\ (R + r\sin\theta)\cos\phi \\ 0 \end{bmatrix} = \begin{bmatrix} -r\cos\theta\sin\phi \\ r\cos\theta\cos\phi \\ 0 \end{bmatrix}$$

$$\vec{N} \cdot \partial_{\phi\theta}\vec{x} = \begin{bmatrix} -\cos\phi\sin\theta \\ -\sin\phi\sin\theta \\ -\cos\theta \end{bmatrix} \cdot \begin{bmatrix} -r\cos\theta\sin\phi \\ r\cos\theta\cos\phi \\ 0 \end{bmatrix} = 0$$

$$\partial_{\theta\theta}\vec{x} = \partial_\theta \begin{bmatrix} r\cos\theta\cos\phi \\ r\cos\theta\sin\phi \\ -r\sin\theta \end{bmatrix} = \begin{bmatrix} -r\sin\theta\cos\phi \\ -r\sin\theta\sin\phi \\ -r\cos\theta \end{bmatrix}$$

$$\vec{N} \cdot \partial_{\theta\theta}\vec{x} = \begin{bmatrix} -\cos\phi\sin\theta \\ -\sin\phi\sin\theta \\ -\cos\theta \end{bmatrix} \cdot \begin{bmatrix} -r\sin\theta\cos\phi \\ -r\sin\theta\sin\phi \\ -r\cos\theta \end{bmatrix} = r$$

The Second Fundamental Form is

$$\det(\vec{N} \cdot \partial_{ij}\vec{x}) = \begin{vmatrix} (R + r\sin\theta)\sin\theta & 0 \\ 0 & r \end{vmatrix} = r(R + r\sin\theta)\sin\theta$$

The Curvature of the 2-torus is

$$\kappa = \frac{\det(\vec{N} \cdot \partial_{ij}\vec{x})}{\det(g_{ij})} = \frac{r(R + r\sin\theta)\sin\theta}{(R + r\sin\theta)^2 r^2} = \boxed{\frac{\sin\theta}{(R + r\sin\theta)r}}$$

4.

3-Sphere in E^4

The curvature of a 3-sphere of Radius R in E^4 is the product of the curvatures of three perpendicularly intersecting great circles, each with curvature $\kappa = \frac{1}{R}$.

$$\boxed{\frac{1}{R}\cdot\frac{1}{R}\cdot\frac{1}{R} = \frac{1}{R^3}}.$$

Parametrically, For the 3-sphere in E^4, there is a constant R, and three angles ψ, θ, and ϕ so that

$$\vec{x}(\psi,\theta,\phi) = \begin{pmatrix} R\cos\psi \\ R\sin\psi\cos\theta \\ R\sin\psi\sin\theta\cos\phi \\ R\sin\psi\sin\theta\sin\phi \end{pmatrix}$$

The three Tangent vectors at $\vec{x}$ are

$$\partial_\psi \vec{x} = \begin{pmatrix} -R\sin\psi \\ R\cos\psi\cos\theta \\ R\cos\psi\sin\theta\cos\phi \\ R\cos\psi\sin\theta\sin\phi \end{pmatrix},$$

$$\partial_\theta \vec{x} = \begin{pmatrix} 0 \\ -R\sin\psi\sin\theta \\ R\sin\psi\cos\theta\cos\phi \\ R\sin\psi\cos\theta\sin\phi \end{pmatrix},$$

and

$$\partial_{\phi}\vec{x} = \begin{pmatrix} 0 \\ 0 \\ -R\sin\psi\sin\theta\sin\phi \\ R\sin\psi\sin\theta\cos\phi \end{pmatrix}$$

The Metric Tensor is

$$g_{ij} = (\partial_{i}\vec{x})\cdot(\partial_{j}\vec{x}) = \begin{bmatrix} R^2 & 0 & 0 \\ 0 & R^2\sin^2\psi & 0 \\ 0 & 0 & R^2\sin^2\psi\sin^2\theta \end{bmatrix}$$

The First Fundamental Form is

$$\det(g_{ij}) = R^6\sin^4\psi\sin^2\theta$$

The Unit Normal at $\vec{x}$ lies along $-\vec{x}$

$$\vec{N} = \begin{pmatrix} -\cos\psi \\ -\sin\psi\cos\theta \\ -\sin\psi\sin\theta\cos\phi \\ -\sin\psi\sin\theta\sin\phi \end{pmatrix}$$

$\vec{N}$ is perpendicular to the Tangent Vectors:

$$\vec{N}\cdot\partial_{\psi}\vec{x} = \begin{pmatrix} -\cos\psi \\ -\sin\psi\cos\theta \\ -\sin\psi\sin\theta\cos\phi \\ -\sin\psi\sin\theta\sin\phi \end{pmatrix}\cdot\begin{pmatrix} -R\sin\psi \\ R\cos\psi\cos\theta \\ R\cos\psi\sin\theta\cos\phi \\ R\cos\psi\sin\theta\sin\phi \end{pmatrix} = 0$$

$$\vec{N}\cdot\partial_{\theta}\vec{x} = \begin{pmatrix} -\cos\psi \\ -\sin\psi\cos\theta \\ -\sin\psi\sin\theta\cos\phi \\ -\sin\psi\sin\theta\sin\phi \end{pmatrix}\cdot\begin{pmatrix} 0 \\ -R\sin\psi\sin\theta \\ R\sin\psi\cos\theta\cos\phi \\ R\sin\psi\cos\theta\sin\phi \end{pmatrix} = 0$$

$$\vec{N}\cdot\partial_{\phi}\vec{x}=\begin{pmatrix}-\cos\psi\\ -\sin\psi\cos\theta\\ -\sin\psi\sin\theta\cos\phi\\ -\sin\psi\sin\theta\sin\phi\end{pmatrix}\cdot\begin{pmatrix}0\\ 0\\ -R\sin\psi\sin\theta\sin\phi\\ R\sin\psi\sin\theta\cos\phi\end{pmatrix}=0$$

For the matrix of the Second Fundamental Form:

$$\partial_{\psi\psi}\vec{x}=\partial_{\psi}\begin{pmatrix}-R\sin\psi\\ R\cos\psi\cos\theta\\ R\cos\psi\sin\theta\cos\phi\\ R\cos\psi\sin\theta\sin\phi\end{pmatrix}=\begin{pmatrix}-R\cos\psi\\ -R\sin\psi\cos\theta\\ -R\sin\psi\sin\theta\cos\phi\\ -R\sin\psi\sin\theta\sin\phi\end{pmatrix},$$

$$\vec{N}\cdot\partial_{\psi\psi}\vec{x}=\begin{pmatrix}-\cos\psi\\ -\sin\psi\cos\theta\\ -\sin\psi\sin\theta\cos\phi\\ -\sin\psi\sin\theta\sin\phi\end{pmatrix}\cdot\begin{pmatrix}-R\cos\psi\\ -R\sin\psi\cos\theta\\ -R\sin\psi\sin\theta\cos\phi\\ -R\sin\psi\sin\theta\sin\phi\end{pmatrix}=R$$

$$\partial_{\theta\psi}\vec{x}=\partial_{\theta}\begin{pmatrix}-R\sin\psi\\ R\cos\psi\cos\theta\\ R\cos\psi\sin\theta\cos\phi\\ R\cos\psi\sin\theta\sin\phi\end{pmatrix}=\begin{pmatrix}0\\ -R\cos\psi\sin\theta\\ R\cos\psi\cos\theta\cos\phi\\ R\sin\psi\cos\theta\sin\phi\end{pmatrix}=\partial_{\psi\theta}\vec{x}$$

$$\vec{N}\cdot\partial_{\theta\psi}\vec{x}=\begin{pmatrix}-\cos\psi\\ -\sin\psi\cos\theta\\ -\sin\psi\sin\theta\cos\phi\\ -\sin\psi\sin\theta\sin\phi\end{pmatrix}\cdot\begin{pmatrix}0\\ -R\cos\psi\sin\theta\\ R\cos\psi\cos\theta\cos\phi\\ R\sin\psi\cos\theta\sin\phi\end{pmatrix}=0$$

$$\partial_{\phi\psi}\vec{x}=\partial_{\phi}\begin{pmatrix}-R\sin\psi\\ R\cos\psi\cos\theta\\ R\cos\psi\sin\theta\cos\phi\\ R\cos\psi\sin\theta\sin\phi\end{pmatrix}=\begin{pmatrix}0\\ 0\\ -R\cos\psi\sin\theta\sin\phi\\ R\cos\psi\sin\theta\cos\phi\end{pmatrix}=\partial_{\psi\phi}\vec{x}$$

$$\vec{N}\cdot\partial_{\phi\psi}\vec{x}=\begin{pmatrix}-\cos\psi\\-\sin\psi\cos\theta\\-\sin\psi\sin\theta\cos\phi\\-\sin\psi\sin\theta\sin\phi\end{pmatrix}\cdot\begin{pmatrix}0\\0\\-R\cos\psi\sin\theta\sin\phi\\R\cos\psi\sin\theta\cos\phi\end{pmatrix}=0$$

$$\partial_{\theta\theta}\vec{x}=\partial_{\theta}\begin{pmatrix}0\\-R\sin\psi\sin\theta\\R\sin\psi\cos\theta\cos\phi\\R\sin\psi\cos\theta\sin\phi\end{pmatrix}=\begin{pmatrix}0\\-R\sin\psi\cos\theta\\-R\sin\psi\sin\theta\cos\phi\\-R\sin\psi\sin\theta\sin\phi\end{pmatrix}$$

$$\vec{N}\cdot\partial_{\theta\theta}\vec{x}=\begin{pmatrix}-\cos\psi\\-\sin\psi\cos\theta\\-\sin\psi\sin\theta\cos\phi\\-\sin\psi\sin\theta\sin\phi\end{pmatrix}\cdot\begin{pmatrix}0\\-R\sin\psi\cos\theta\\-R\sin\psi\sin\theta\cos\phi\\-R\sin\psi\sin\theta\sin\phi\end{pmatrix}=R\sin^2\psi$$

$$\partial_{\phi\theta}\vec{x}=\partial_{\phi}\begin{pmatrix}0\\-R\sin\psi\sin\theta\\R\sin\psi\cos\theta\cos\phi\\R\sin\psi\cos\theta\sin\phi\end{pmatrix}=\begin{pmatrix}0\\0\\-R\sin\psi\sin\theta\sin\phi\\R\sin\psi\sin\theta\cos\phi\end{pmatrix}=\partial_{\theta\phi}\vec{x}$$

$$\vec{N}\cdot\partial_{\phi\theta}\vec{x}=\begin{pmatrix}-\cos\psi\\-\sin\psi\cos\theta\\-\sin\psi\sin\theta\cos\phi\\-\sin\psi\sin\theta\sin\phi\end{pmatrix}\cdot\begin{pmatrix}0\\0\\-R\sin\psi\sin\theta\sin\phi\\R\sin\psi\sin\theta\cos\phi\end{pmatrix}=0$$

$$\partial_{\phi\phi}\vec{x}=\partial_{\phi}\begin{pmatrix}0\\0\\-R\sin\psi\sin\theta\sin\phi\\R\sin\psi\sin\theta\cos\phi\end{pmatrix}=\begin{pmatrix}0\\0\\-R\sin\psi\sin\theta\cos\phi\\-R\sin\psi\sin\theta\sin\phi\end{pmatrix}$$

$$\vec{N}\cdot\partial_{\phi\phi}\vec{x} = \begin{pmatrix} -\cos\psi \\ -\sin\psi\cos\theta \\ -\sin\psi\sin\theta\cos\phi \\ -\sin\psi\sin\theta\sin\phi \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 0 \\ -R\sin\psi\sin\theta\cos\phi \\ -R\sin\psi\sin\theta\sin\phi \end{pmatrix} = R\sin^2\psi\sin^2\theta$$

The matrix of the Second Fundamental Form is

$$\vec{N}\cdot\partial_{ij}\vec{x} = \begin{pmatrix} R & 0 & 0 \\ 0 & R\sin^2\psi & 0 \\ 0 & 0 & R\sin^2\psi\sin^2\theta \end{pmatrix}.$$

The Second Fundamental Form is

$$\det(\vec{N}\cdot\partial_{ij}\vec{x}) = -R^3\sin^4\psi\sin^2\theta\,.$$

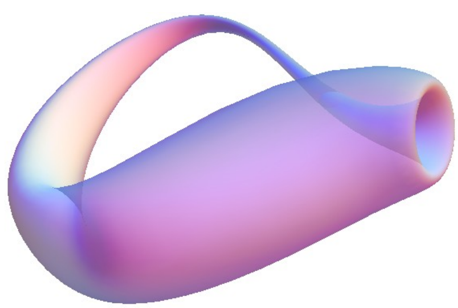
The Curvature of the 3-sphere is

$$\frac{\det(\vec{N}\cdot\partial_{ij}\vec{x})}{\det(g_{ij})} = \frac{R^3\sin^4\psi\sin^2\theta}{R^6\sin^4\psi\sin^2\theta} = \frac{1}{R^3}.\square$$

5.

Klein Bottle in E^4

A Klein Bottle is two-parameter surface that to not intersect itself needs a four dimensional Euclidean space. To visualize it in three dimensional space, we allow it to intersect itself as if it is a Surface imbedded in E^3. From Wikipedia[1]



In four dimensions, there are positive constants R, and r, and two angles $0 \le \theta < 2\pi$, and $0 \le \phi < 2\pi$ so that

$$\vec{x}(\theta,\phi) = \begin{bmatrix} (R + r\cos\theta)\cos\phi \\ (R + r\cos\theta)\sin\phi \\ r\sin\theta\cos\frac{1}{2}\phi \\ r\sin\theta\sin\frac{1}{2}\phi \end{bmatrix}.$$

[1] https://en.wikipedia.org/wiki/Klein_bottle#

The Tangent vectors are

$$\partial_\theta \vec{x} = \partial_\theta \begin{bmatrix} (R + r\cos\theta)\cos\phi \\ (R + r\cos\theta)\sin\phi \\ r\sin\theta\cos\frac{1}{2}\phi \\ r\sin\theta\sin\frac{1}{2}\phi \end{bmatrix} = \begin{bmatrix} -r\sin\theta\cos\phi \\ -r\sin\theta\sin\phi \\ r\cos\theta\cos\frac{1}{2}\phi \\ r\cos\theta\sin\frac{1}{2}\phi \end{bmatrix},$$

and

$$\partial_\phi \vec{x} = \partial_\phi \begin{bmatrix} (R + r\cos\theta)\cos\phi \\ (R + r\cos\theta)\sin\phi \\ r\sin\theta\cos\frac{1}{2}\phi \\ r\sin\theta\sin\frac{1}{2}\phi \end{bmatrix} = \begin{bmatrix} -(R + r\cos\theta)\sin\phi \\ (R + r\cos\theta)\cos\phi \\ -\frac{1}{2}r\sin\theta\sin\frac{1}{2}\phi \\ \frac{1}{2}r\sin\theta\cos\frac{1}{2}\phi \end{bmatrix}.$$

$$g_{\theta\theta} = (\partial_\theta \vec{x}) \cdot (\partial_\theta \vec{x}) = r^2$$

$$g_{\phi\phi} = (\partial_\phi \vec{x}) \cdot (\partial_\phi \vec{x}) = (R + r\cos\theta)^2 + \tfrac{1}{4}r^2\sin^2\theta$$

$$g_{\theta\phi} = (\partial_\theta \vec{x}) \cdot (\partial_\phi \vec{x}) = \begin{bmatrix} -r\sin\theta\cos\phi \\ -r\sin\theta\sin\phi \\ r\cos\theta\cos\frac{1}{2}\phi \\ r\cos\theta\sin\frac{1}{2}\phi \end{bmatrix} \cdot \begin{bmatrix} -(R + r\cos\theta)\sin\phi \\ (R + r\cos\theta)\cos\phi \\ -\frac{1}{2}r\sin\theta\sin\frac{1}{2}\phi \\ \frac{1}{2}r\sin\theta\cos\frac{1}{2}\phi \end{bmatrix} = 0.$$

The Metric Tensor is

$$g_{ij} = (\partial_i \vec{x}) \cdot (\partial_j \vec{x}) = \begin{bmatrix} r^2 & 0 \\ 0 & g(R + r\cos\theta)^2 + \frac{1}{4}r^2\sin^2\theta \end{bmatrix}$$

The First Fundamental Form is

$$\det(g_{ij}) = r^2\Big[(R + r\cos\theta)^2 + \tfrac{1}{4}r^2\sin^2\theta\Big].$$

$$\vec{N} = \begin{bmatrix} -r\cos\theta\cos\phi \\ -r\cos\theta\sin\phi \\ -r\sin\theta\cos\frac{1}{2}\phi \\ -r\sin\theta\sin\frac{1}{2}\phi \end{bmatrix} \text{ is the Unit Normal at } \vec{x}$$

because it is perpendicular to the two Tangent Vectors

$$\vec{N}\cdot\partial_{\theta}\vec{x}=\begin{bmatrix}-r\cos\theta\cos\phi\\-r\cos\theta\sin\phi\\-r\sin\theta\cos\frac{1}{2}\phi\\-r\sin\theta\sin\frac{1}{2}\phi\end{bmatrix}\cdot\begin{bmatrix}-r\sin\theta\cos\phi\\-r\sin\theta\sin\phi\\r\cos\theta\cos\frac{1}{2}\phi\\r\cos\theta\sin\frac{1}{2}\phi\end{bmatrix}=0\,,$$

$$\vec{N}\cdot\partial_{\phi}\vec{x}=\begin{bmatrix}-r\cos\theta\cos\phi\\-r\cos\theta\sin\phi\\-r\sin\theta\cos\frac{1}{2}\phi\\-r\sin\theta\sin\frac{1}{2}\phi\end{bmatrix}\cdot\begin{bmatrix}-(R+r\cos\theta)\sin\phi\\(R+r\cos\theta)\cos\phi\\-\frac{1}{2}r\sin\theta\sin\frac{1}{2}\phi\\\frac{1}{2}r\sin\theta\cos\frac{1}{2}\phi\end{bmatrix}=0\,.$$

For the matrix of the Second Fundamental Form,

$$\partial_{\theta\theta}\vec{x}=\partial_{\theta}\begin{bmatrix}-(R+r\cos\theta)\sin\phi\\(R+r\cos\theta)\cos\phi\\-\frac{1}{2}r\sin\theta\sin\frac{1}{2}\phi\\\frac{1}{2}r\sin\theta\cos\frac{1}{2}\phi\end{bmatrix}=\begin{bmatrix}r\sin\theta\sin\phi\\-r\sin\theta\cos\phi\\-\frac{1}{2}r\cos\theta\sin\frac{1}{2}\phi\\\frac{1}{2}r\cos\theta\cos\frac{1}{2}\phi\end{bmatrix}$$

$$\vec{N}\cdot\partial_{\theta\theta}\vec{x}=\begin{bmatrix}-r\cos\theta\cos\phi\\-r\cos\theta\sin\phi\\-r\sin\theta\cos\frac{1}{2}\phi\\-r\sin\theta\sin\frac{1}{2}\phi\end{bmatrix}\cdot\begin{bmatrix}r\sin\theta\sin\phi\\-r\sin\theta\cos\phi\\-\frac{1}{2}r\cos\theta\sin\frac{1}{2}\phi\\\frac{1}{2}r\cos\theta\cos\frac{1}{2}\phi\end{bmatrix}=0$$

$$\partial_{\phi\theta}\vec{x}=\partial_{\phi}\begin{bmatrix}-(R+r\cos\theta)\sin\phi\\(R+r\cos\theta)\cos\phi\\-\frac{1}{2}r\sin\theta\sin\frac{1}{2}\phi\\\frac{1}{2}r\sin\theta\cos\frac{1}{2}\phi\end{bmatrix}=\begin{bmatrix}-(R+r\cos\theta)\cos\phi\\-(R+r\cos\theta)\sin\phi\\-\frac{1}{4}r\sin\theta\cos\frac{1}{2}\phi\\-\frac{1}{4}r\sin\theta\sin\frac{1}{2}\phi\end{bmatrix}=\partial_{\theta\phi}\vec{x}$$

$$\vec{N}\cdot\partial_{\phi\theta}\vec{x}=\begin{bmatrix}-r\cos\theta\cos\phi\\-r\cos\theta\sin\phi\\-r\sin\theta\cos\frac{1}{2}\phi\\-r\sin\theta\sin\frac{1}{2}\phi\end{bmatrix}\cdot\begin{bmatrix}-(R+r\cos\theta)\cos\phi\\-(R+r\cos\theta)\sin\phi\\-\frac{1}{4}r\sin\theta\cos\frac{1}{2}\phi\\-\frac{1}{4}r\sin\theta\sin\frac{1}{2}\phi\end{bmatrix}$$

$$=r\cos\theta(R+r\cos\theta)+\tfrac{1}{4}r^{2}\sin^{2}\theta$$

$$\partial_{\phi\phi}\vec{x} = \partial_{\phi}\begin{bmatrix} -(R + r\cos\theta)\sin\phi \\ (R + r\cos\theta)\cos\phi \\ -\frac{1}{2}r\sin\theta\sin\frac{1}{2}\phi \\ \frac{1}{2}r\sin\theta\cos\frac{1}{2}\phi \end{bmatrix} = \begin{bmatrix} -(R + r\cos\theta)\cos\phi \\ -(R + r\cos\theta)\sin\phi \\ -\frac{1}{4}r\sin\theta\cos\frac{1}{2}\phi \\ -\frac{1}{4}r\sin\theta\sin\frac{1}{2}\phi \end{bmatrix}$$

$$\vec{N}\cdot\partial_{\phi\phi}\vec{x} = \begin{bmatrix} -r\cos\theta\cos\phi \\ -r\cos\theta\sin\phi \\ -r\sin\theta\cos\frac{1}{2}\phi \\ -r\sin\theta\sin\frac{1}{2}\phi \end{bmatrix}\cdot\begin{bmatrix} -(R + r\cos\theta)\cos\phi \\ -(R + r\cos\theta)\sin\phi \\ -\frac{1}{4}r\sin\theta\cos\frac{1}{2}\phi \\ -\frac{1}{4}r\sin\theta\sin\frac{1}{2}\phi \end{bmatrix}$$

$$= r\cos\theta(R + r\cos\theta) + \tfrac{1}{4}r^2\sin^2\theta$$

The matrix for the Second Fundamental Form is

$$\vec{N}\cdot\partial_{ij}\vec{x} = \begin{pmatrix} 0 & r\cos\theta(R + r\cos\theta) + \frac{1}{4}r^2\sin^2\theta \\ r\cos\theta(R + r\cos\theta) + \frac{1}{4}r^2\sin^2\theta & r\cos\theta(R + r\cos\theta) + \frac{1}{4}r^2\sin^2\theta \end{pmatrix}$$

The Second Fundamental Form is

$$\det(\vec{N}\cdot\partial_{ij}\vec{x}) = -\Big[r\cos\theta(R + r\cos\theta) + \tfrac{1}{4}r^2\sin^2\theta\Big]^2$$

The Curvature of the four dimensional Klein Bottle is

$$\frac{\det(\vec{N}\cdot\partial_{ij}\vec{x})}{\det(g_{ij})} = \frac{-\Big[r\cos\theta(R + r\cos\theta) + \frac{1}{4}r^2\sin^2\theta\Big]^2}{r^2\Big[(R + r\cos\theta)^2 + \frac{1}{4}r^2\sin^2\theta\Big]}$$

$$= \boxed{-\frac{\cos\theta(R + r\cos\theta) + \frac{1}{4}r\sin^2\theta}{r}}$$

References

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