The Universe is Riemannian 3-Sphere with 93% Certainty

H. Vic Dannon
November 2022

Abstract Riemann's inaugural talk is about Geometry at its extremes. the Geometry of the infinite, and the Geometry of the infinitesimal.

The generations that tried to unveil the mathematical machinery behind Riemann's statements got lost in that machinery, and lost sight of these two themes.

Like Gauss who was guided in his theory of surfaces by observations of the earth, Riemann's infinity is the Astronomical realm, and Riemann's infinitesimal is the Atomic realm.

In Astronomical Space, Riemann suggested that the Universe is an unbounded but finite 3-sphere with almost zero, positive curvature, in a 3-dimensional zero-curvature space.

<u>We show that the Curvature of a 3-sphere of radius R is 1/R^3</u>.

It is beyond us to see bending in 4 dimensional flat space. But Riemann's universe does bend with curvature 1/R^3.

This must have been known to Riemann. But he did not state it. And it never concerned anyone after Riemann. Generations after Riemann sought instead the curvatures of the 4-dimensional matrix representing the 4x4x4x4 Riemann Curvature Tensor.

Riemann was expecting empirical results to determine the curvature of the universe.

Recent measurements confirm that the curvature of the Universe is almost zero.

Consequently, Minkowski Space-Time that has a definitely negative curvature does not model the Universe.

Neither does Einstein Space-Time. The high curvature around black holes is irrelevant to the mean curvature of the universe.

The 2018 measured curvature ranges between -0.0012, and 0.0026, and averages 0.0007.

To the Wikipedia, this seemed like an undetermined curvature, equally allowing an infinite hyperbolic universe with negative curvature, or an infinite flat universe with zero curvature, or a finite elliptic, Riemannian universe with positive curvature.

But since many measurements were made, the curvature is distributed normally, on the Gaussian bell shaped curve. And statistics yields that

the curvature of the universe is positive with 93% likelihood

That is, the universe is Riemannian with 93% certainty.

In Atomic Space, Riemann suspected that the electromagnetic forces curve the space, and the curvature of the space changes from one point to the other.

To this date, the Geometry of Atomic Space did not progress beyond Riemann's time.

Contents

1.

Riemann's Universe is a 3-Sphere with Radius R

Riemann said[1]

> *"When we extend constructions in space to the immeasurably large, a distinction has to be made between the* ***unlimited*** *and the* ***infinite****;....*
>
> *That space is an unlimited tree-dimensional Manifold is a hypothesis that is constantly being verified in its applications.*
>
> *The property of the unboundedness of space posses therefore a greater empirical certainty than any other fact established by observation.*
>
> *However, the infiniteness of space does not in any way follow from this.*
>
> *On the contrary, if one assumes that the size of solid bodies does not depend on their position, and consequently ascribes to space a constant curvature, then space is necessarily finite, whenever this constant is positive, no matter how small.*

---

[1] Collected Papers, Bernhard Riemann, Translated from the 1892 edition by Roger Baker, Charles Christenson, and Henry Orde. pp.269-270, Kendrick Press,2004

If we were to extend the infinitesimal line element in each initial direction in an element of surface into lines segments along paths of shortest lengths, the lines would lie in an unbounded surface of constant curvature.
This surface in a flat three-dimensional manifold would take the form of a sphere, so that it would be finite......
At any rate, the reciprocal of the curvature would correspond to a surface with a radius compared with which the range of our present telescopes would be negligibly small."

2.

The Curvature of Riemann's 3-Sphere Universe is $\frac{1}{R^3}$

A circle of radius R is given parametrically by

$$\vec{x}(\phi) = \begin{pmatrix} x(\phi) \\ y(\phi) \end{pmatrix} = \begin{pmatrix} R\cos\phi \\ R\sin\phi \end{pmatrix}.$$

The Tangent is

$$\partial_{\phi}\vec{x}(\phi) = \begin{pmatrix} x'(\phi) \\ y'(\phi) \end{pmatrix} = \begin{pmatrix} -R\sin\phi \\ R\cos\phi \end{pmatrix}$$

The Normal is

$$\partial_{\phi\phi}\vec{x}(\phi) = \begin{pmatrix} x''(\phi) \\ y''(\phi) \end{pmatrix} = \begin{pmatrix} -R\cos\phi \\ -R\sin\phi \end{pmatrix}$$

The curvature is

$$\kappa = \frac{\begin{vmatrix} x'(\phi) & y'(\phi) \\ x''(\phi) & y''(\phi) \end{vmatrix}}{\left([x'(\phi)]^2 + [y'(\phi)]^2\right)^{\frac{3}{2}}}$$

$$= \frac{\begin{vmatrix} -R\sin\phi & R\cos\phi \\ -R\cos\phi & -R\sin\phi \end{vmatrix}}{\left([-R\sin\phi]^2 + [R\cos\phi]^2\right)^{\frac{3}{2}}}$$

$$= \frac{R^2}{R^3} = \frac{1}{R}$$

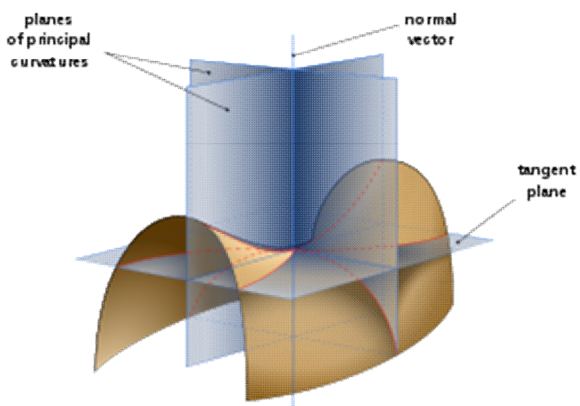
The Curvature of the circle is $\kappa = \dfrac{1}{R}$.

At each point of a differentiable two-dimensional surface, there is a normal to the surface, that is contained in a family of planes normal to the surface. Each plane cuts the surface in a plane curve that has a curvature at the point.

$$\kappa_{\min} \equiv \kappa_1 \text{, and } \kappa_{\max} \equiv \kappa_2$$

are the principal values of the curvature.

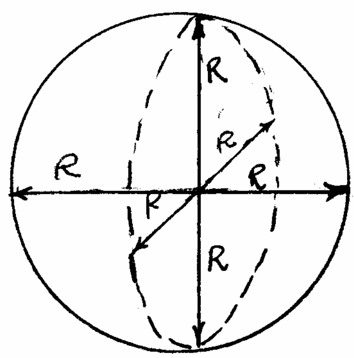
They belong to perpendicularly intersecting curves on the surface.



The Gaussian curvature of a two-dimensional surface is

$$K = \kappa_1 \kappa_2 .$$

On a 2-sphere with radius R, the great circles are geodesics, with curvature $\dfrac{1}{R}$. Any two perpendicularly intersecting great circles have the principal curvatures $\kappa_{\min} = \kappa_{\max} = \dfrac{1}{R}$.



The Gaussian curvature of a 2-sphere of Radius R is

$$\frac{1}{R}\frac{1}{R}=\frac{1}{R^2}.$$

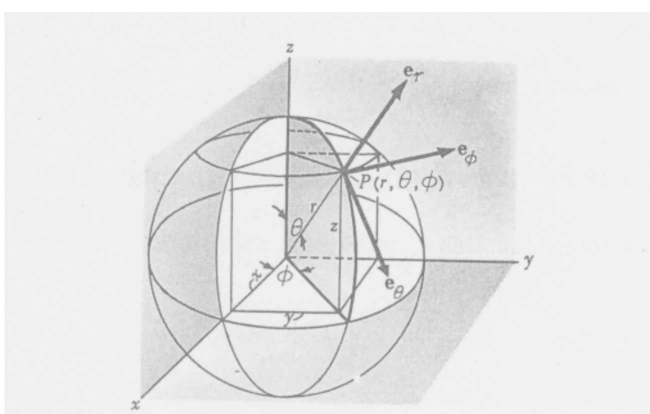
For a 3-sphere of Radius R, the Riemann Curvature is the product of the curvatures of three perpendicularly intersecting great circles, each with curvature $\kappa=\dfrac{1}{R}$.

Therefore, the Riemann Curvature of a 3-sphere of Radius R is

$$\boxed{\frac{1}{R}\frac{1}{R}\frac{1}{R}=\frac{1}{R^3}}.$$

This may be derived from the surfaces' parametric representations

A 2-sphere of Radius R, with Azimuth angle ϕ, and polar angle θ



is represented parametrically by

$$\vec{x}(\theta,\phi) = \begin{pmatrix} x(\theta,\phi) \\ y(\theta,\phi) \\ z(\theta,\phi) \end{pmatrix} = \begin{pmatrix} R\sin\theta\cos\phi \\ R\sin\theta\sin\phi \\ R\cos\theta \end{pmatrix}$$

Denote

$$\frac{\partial}{\partial\theta} \equiv \partial_1, \qquad \frac{\partial}{\partial\phi} \equiv \partial_2 .$$

At each point on the 2-sphere, the Tangent space is spanned by

$$\partial_\theta \vec{x} = \partial_\theta \begin{pmatrix} R\sin\theta\cos\phi \\ R\sin\theta\sin\phi \\ R\cos\theta \end{pmatrix} = \begin{pmatrix} R\cos\theta\cos\phi \\ R\cos\theta\sin\phi \\ -R\sin\theta \end{pmatrix}$$

$$\partial_\phi \vec{x} = \partial_\phi \begin{pmatrix} R\sin\theta\cos\phi \\ R\sin\theta\sin\phi \\ R\cos\theta \end{pmatrix} = \begin{pmatrix} -R\sin\theta\sin\phi \\ R\sin\theta\cos\phi \\ 0 \end{pmatrix}$$

The Metric Tensor is

$$g_{ij} = (\partial_i \vec{x}) \cdot (\partial_j \vec{x}) = \begin{bmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{bmatrix}$$

The First Fundamental Form is

$$\det(g_{ij}) = R^4 \sin^2\theta$$

The Unit Normal at $\vec{x}$ is

$$\vec{N} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

$$\partial_{\theta\theta} \vec{x} = \partial_\theta \begin{pmatrix} R\cos\theta\cos\phi \\ R\cos\theta\sin\phi \\ -R\sin\theta \end{pmatrix} = \begin{pmatrix} -R\sin\theta\cos\phi \\ -R\sin\theta\sin\phi \\ -R\cos\theta \end{pmatrix}$$

$$\vec{N} \cdot \partial_{\theta\theta}\vec{x} = -R$$

$$\partial_{\phi\theta}\vec{x} = \partial_{\phi}\begin{pmatrix} R\cos\theta\cos\phi \\ R\cos\theta\sin\phi \\ -R\sin\theta \end{pmatrix} = \begin{pmatrix} -R\cos\theta\sin\phi \\ R\cos\theta\cos\phi \\ 0 \end{pmatrix} = \partial_{\theta\phi}\vec{x}$$

$$\vec{N} \cdot \partial_{\phi\theta}\vec{x} = 0$$

$$\partial_{\phi\phi}\vec{x} = \partial_{\phi}\begin{pmatrix} -R\sin\theta\sin\phi \\ R\sin\theta\cos\phi \\ 0 \end{pmatrix} = \begin{pmatrix} -R\sin\theta\cos\phi \\ -R\sin\theta\sin\phi \\ 0 \end{pmatrix}$$

$$\vec{N} \cdot \partial_{\phi\phi}\vec{x} = -R\sin^2\theta$$

The matrix for the Second Fundamental Form is

$$\vec{N} \cdot \partial_{ij}\vec{x} = \begin{pmatrix} -R & 0 \\ 0 & -R\sin^2\theta \end{pmatrix}$$

The Second Fundamental Form is

$$\det(\vec{N} \cdot \partial_{ij}\vec{x}) = R^2\sin^2\theta$$

The Curvature of the 2-sphere is

$$\frac{\det(\vec{N} \cdot \partial_{ij}\vec{x})}{\det(g_{ij})} = \frac{R^2\sin^2\theta}{R^4\sin^2\theta} = \frac{1}{R^2}$$

The parameterized 3-sphere in 4-Euclidean space is

$$\begin{pmatrix} x^1(R,\psi,\theta,\phi) \\ x^2(R,\psi,\theta,\phi) \\ x^3(R,\psi,\theta,\phi) \\ x^4(R,\psi,\theta,\phi) \end{pmatrix} = \begin{pmatrix} R\cos\psi \\ R\sin\psi\cos\theta \\ R\sin\psi\sin\theta\cos\phi \\ R\sin\psi\sin\theta\sin\phi \end{pmatrix}$$

Denote

$$\frac{\partial}{\partial\psi} \equiv \partial_1, \qquad \frac{\partial}{\partial\theta} \equiv \partial_2, \qquad \frac{\partial}{\partial\phi} \equiv \partial_3.$$

At each point on the 3-sphere, the Tangent space is spanned by

$$\partial_\psi \vec{x} = \begin{pmatrix} -R\sin\psi \\ R\cos\psi\cos\theta \\ R\cos\psi\sin\theta\cos\phi \\ R\cos\psi\sin\theta\sin\phi \end{pmatrix} \equiv \vec{e}_1,$$

$$\partial_\theta \vec{x} = \begin{pmatrix} 0 \\ -R\sin\psi\sin\theta \\ R\sin\psi\cos\theta\cos\phi \\ R\sin\psi\cos\theta\sin\phi \end{pmatrix} \equiv \vec{e}_2,$$

$$\partial_\phi \vec{x} = \begin{pmatrix} 0 \\ 0 \\ -R\sin\psi\sin\theta\sin\phi \\ R\sin\psi\sin\theta\cos\phi \end{pmatrix} \equiv \vec{e}_3$$

The Metric Tensor is

$$g_{ij} = (\partial_i \vec{x}) \cdot (\partial_j \vec{x}) = \begin{bmatrix} R^2 & 0 & 0 \\ 0 & R^2\sin^2\psi & 0 \\ 0 & 0 & R^2\sin^2\psi\sin^2\theta \end{bmatrix}$$

The First Fundamental Form is

$$\det(g_{ij}) = R^6\sin^4\psi\sin^2\theta$$

The Unit Normal at $\vec{x}$ is

$$\vec{N} = \begin{pmatrix} -\cos\psi \\ -\sin\psi\cos\theta \\ -\sin\psi\sin\theta\cos\phi \\ -\sin\psi\sin\theta\sin\phi \end{pmatrix}$$

$$\partial_{\psi\psi}\vec{x} = \partial_{\psi}\begin{pmatrix} -R\sin\psi \\ R\cos\psi\cos\theta \\ R\cos\psi\sin\theta\cos\phi \\ R\cos\psi\sin\theta\sin\phi \end{pmatrix} = \begin{pmatrix} -R\cos\psi \\ -R\sin\psi\cos\theta \\ -R\sin\psi\sin\theta\cos\phi \\ -R\sin\psi\sin\theta\sin\phi \end{pmatrix},$$

$$\vec{N}\cdot\partial_{\psi\psi}\vec{x} = R$$

$$\partial_{\theta\psi}\vec{x} = \partial_{\theta}\begin{pmatrix} -R\sin\psi \\ R\cos\psi\cos\theta \\ R\cos\psi\sin\theta\cos\phi \\ R\cos\psi\sin\theta\sin\phi \end{pmatrix} = \begin{pmatrix} 0 \\ -R\cos\psi\sin\theta \\ R\cos\psi\cos\theta\cos\phi \\ R\sin\psi\cos\theta\sin\phi \end{pmatrix} = \partial_{\psi\theta}\vec{x}$$

$$\vec{N}\cdot\partial_{\theta\psi}\vec{x} = 0$$

$$\partial_{\phi\psi}\vec{x} = \partial_{\phi}\begin{pmatrix} -R\sin\psi \\ R\cos\psi\cos\theta \\ R\cos\psi\sin\theta\cos\phi \\ R\cos\psi\sin\theta\sin\phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -R\cos\psi\sin\theta\sin\phi \\ R\cos\psi\sin\theta\cos\phi \end{pmatrix} = \partial_{\psi\phi}\vec{x}$$

$$\vec{N}\cdot\partial_{\phi\psi}\vec{x} = 0$$

$$\partial_{\theta\theta}\vec{x} = \partial_{\theta}\begin{pmatrix} 0 \\ -R\sin\psi\sin\theta \\ R\sin\psi\cos\theta\cos\phi \\ R\sin\psi\cos\theta\sin\phi \end{pmatrix} = \begin{pmatrix} 0 \\ -R\sin\psi\cos\theta \\ -R\sin\psi\sin\theta\cos\phi \\ -R\sin\psi\sin\theta\sin\phi \end{pmatrix}$$

$$\vec{N}\cdot\partial_{\theta\theta}\vec{x} = R\sin^2\psi$$

$$\partial_{\phi\theta}\vec{x} = \partial_{\phi}\begin{pmatrix} 0 \\ -R\sin\psi\sin\theta \\ R\sin\psi\cos\theta\cos\phi \\ R\sin\psi\cos\theta\sin\phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -R\sin\psi\sin\theta\sin\phi \\ R\sin\psi\sin\theta\cos\phi \end{pmatrix} = \partial_{\theta\phi}\vec{x}$$

$$\vec{N}\cdot\partial_{\phi\theta}\vec{x} = 0$$

$$\partial_{\phi\phi}\vec{x} = \partial_{\phi}\begin{pmatrix} 0 \\ 0 \\ -R\sin\psi\sin\theta\sin\phi \\ R\sin\psi\sin\theta\cos\phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -R\sin\psi\sin\theta\cos\phi \\ -R\sin\psi\sin\theta\sin\phi \end{pmatrix}$$

$$\vec{N}\cdot\partial_{\phi\phi}\vec{x} = R\sin^2\psi\sin^2\theta$$

The matrix for the Second Fundamental Form is

$$\vec{N}\cdot\partial_{ij}\vec{x} = \begin{pmatrix} R & 0 & 0 \\ 0 & R\sin^2\psi & 0 \\ 0 & 0 & R\sin^2\psi\sin^2\theta \end{pmatrix}$$

The Second Fundamental Form is

$$\det(\vec{N}\cdot\partial_{ij}\vec{x}) = R^3\sin^4\psi\sin^2\theta$$

The Curvature of the 3-sphere is

$$\boxed{\frac{\det(\vec{N}\cdot\partial_{ij}\vec{x})}{\det(g_{ij})} = \frac{R^3\sin^4\psi\sin^2\theta}{R^6\sin^4\psi\sin^2\theta} = \frac{1}{R^3}}$$

Riemann's Curvature of his 3-Sphere universe is

$$\frac{1}{R^3}$$

3.

The Curvature of the Universe is Positive with 93% Certainty

By the "shape of the universe"[2], Fluctuations in the Temperature of the Cosmic Background Radiation observed by satellites like NASA's WMAP, and the ESA Planck indicate that the curvature of the universe is very close to zero.

By the Planck mission results from 2018, the curvature is between

$$-0.0012\,, \quad \text{and} \quad 0.0026\,.$$

The mean curvature is the positive

$$0.0007$$

And the measurement error is

$$\pm 0.0019\,.$$

If many measurements of the curvature were made, the curvature possible values are distributed normally. That is, the probability density of the curvatures distribution is the bell shaped, Gaussian curve.

That curve peaks at the mean

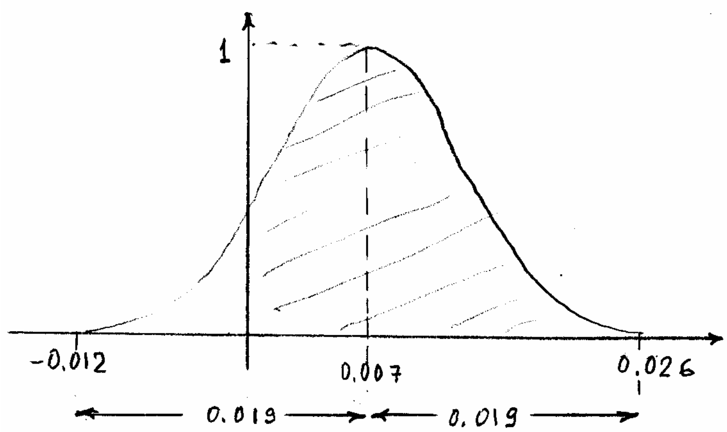
0.0007

and ranges

$$\text{from } 0.0007 - 0.0019 = -0.012$$

[2] https://en.wikipedia.org/wiki/Shape_of_the_universe

to $\quad 0.0007 + 0.0019 = 0.0026$



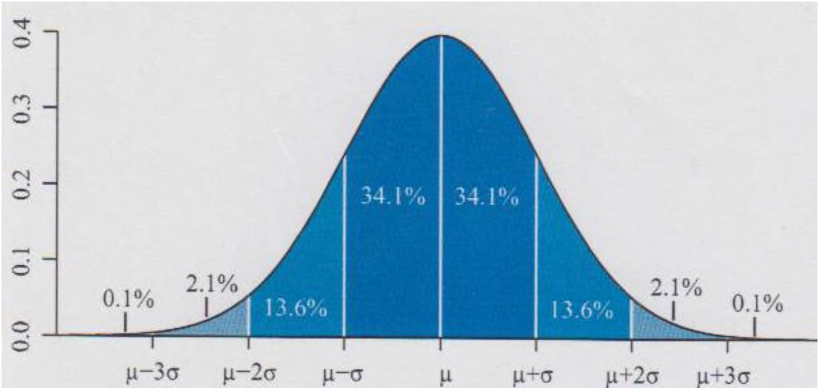
The chance for the curvature to be positive, is the area under the Gaussian curve from

$$x = 0$$

till the end of the positive range

$$\text{at } x = 0.0026\,.$$

That area is tabulated for a Gaussian curve in normalized form.



In normalized form, with mean $\mu = 0$, and standard deviation $\sigma = 1$, the normal distribution density is

$$f(x) = \tfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,.$$

x ranges from $-\infty$ to ∞,

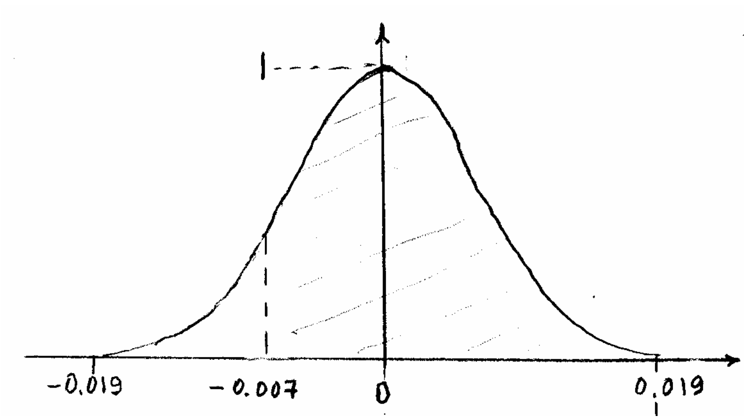
But the curve vanishes out of the interval $-4\sigma = -4$ to $4\sigma = 4$.

Adjusting to the standard normal curve, we seek the area from

$$-0.0007 \text{ to } 0.0019$$

with standard deviation

$$\frac{0.0019}{4} = 0.000475\,.$$

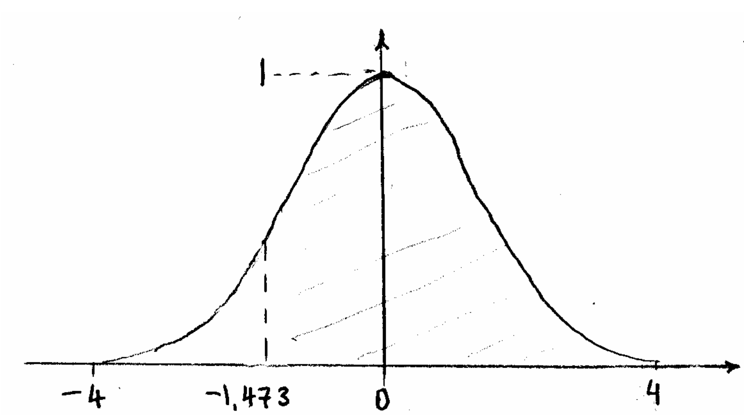


On the normal curve,

$$-\frac{0.0007}{0.00475} = -1.473$$

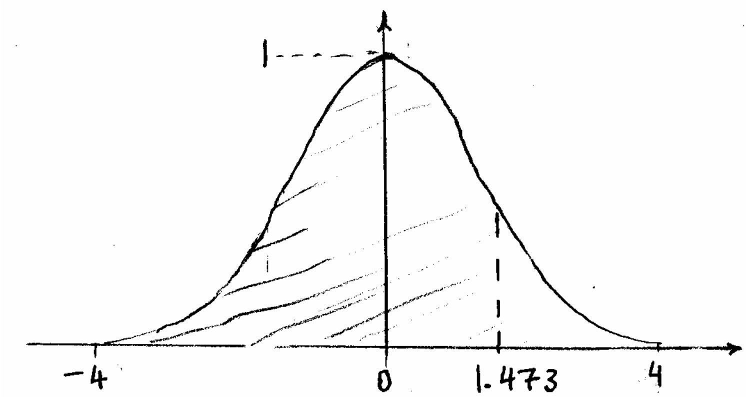
That is, the chance for positive curvature is the area under the normalized curve from

$$x = -1.473 \text{ till } x = 4\,.$$



Due to the symmetry of the normal curve with respect to $x = 0$, the area equals the area under the normal curve from

$x = -\infty$ to $x = 1.473$



The area is

$$\operatorname{erf}(1.473) = \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{x=1.473} e^{-\frac{1}{2}x^2} dx = 0.9292 .$$ [3]

Therefore, with about 93% confidence, the Curvature of the universe is positive.

Riemann expected empirical results to determine the curvature of the universe.

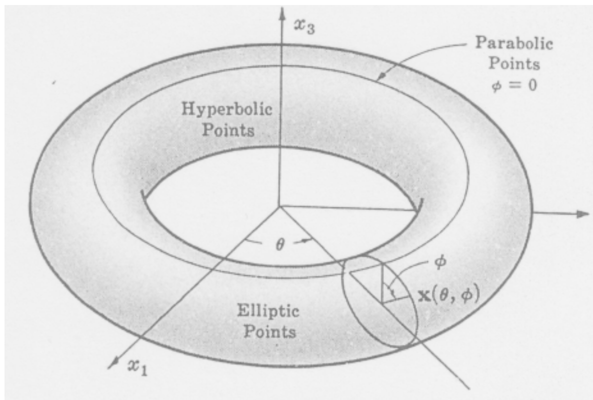
Recent empirical results determine with 93% confidence that the curvature is positive, and therefore, the Universe is a 3-Sphere.

[3] Murray Spiegel: "Mathematical Handbook", Schaum's 1968, p. 257

4.

3-Torus Universe

A two-dimensional Torus is



It is given parametrically by

$$\begin{pmatrix} x(\theta,\phi) \\ y(\theta,\phi) \\ z(\theta,\phi) \end{pmatrix} = \begin{pmatrix} (b + a\sin\phi)\cos\theta \\ (b + a\sin\phi)\sin\theta \\ a\cos\phi \end{pmatrix}$$

In the "shape of the universe"[4], observations of the fluctuations of the Temperature of the Cosmic Microwave Background Radiation have shown a striking amount of missing wavelengths on scales that are beyond the size of the universe.

That spectrum fits better a curvature of a universe which is a three-dimensional Torus.

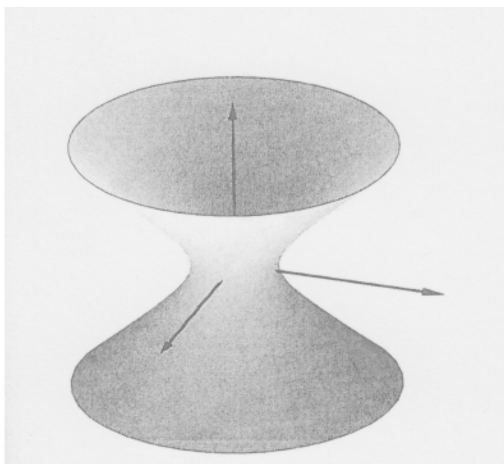
But without a reasoning for such universe, we will not pursue this any further.

[4] https://en.wikipedia.org/wiki/Shape_of_the_universe

5.

Space-Time Does Not Model the Universe

By the "shape of the universe"[5], Minkowski Space-Time is an unbounded hyperbolic space in three dimensions.
A two-dimensional Hyperbolic surface is



The Minkowski space-time has a definitely negative curvature that conflicts with the almost zero curvature of the universe.
2018 measurements of the fluctuations of the temperature of the Cosmic Microwave Background Radiation suggest that the Universe curvature is almost zero. They do not determine whether it is positive, negative, or zero. But they conflict with a definitely negative curvature.
Einstein Space-Time is irrelevant to the curvature of the universe.

[5] https://en.wikipedia.org/wiki/Shape_of_the_universe

Einstein applied Riemann suggestion for Atomic space to Astronomical space.

In Atomic Space, Riemann suspected that the binding electromagnetic forces curve the space, and the curvature of the space changes from one point to the other.

In Einstein Space-Time, the binding gravitational forces curve the space. The curvature is extremely high around black holes, and very low around hydrogen clouds.

But this observation is irrelevant to the determination of the mean curvature of the universe.

6.

Riemann's Atomic Space

Riemann said,

> *If we don't assume the independence of the size of bodies from their position, then the curvature in three dimensions could have arbitrary values close to zero at every point in space.*
>
> *Even more complicated relations can occur if the line element can not be represented by quadratic differential form* $(ds)^2 = \sum (dx)^2$.
>
> *Then, the empirical concepts on which space measurements are based, the concept of a rigid body, and the concept of a light ray, become meaningless in infinitesimally small space.*
>
> *Accordingly, in infinitesimally small space metric relations may not conform to the postulates of continuous manifolds.*
>
> *If Infinitesimally small space is a continuous manifold, then its metric relations depend on the binding forces that act upon it.*

Riemann might have suspected that in Atomic distances, the binding electromagnetic forces curve the space, and the curvature of the space changes from one point to the other.

To this date, the Geometry of Atomic Space did not progress beyond Riemann's time.

References

Roger Baker Translation "The Hypothesis on which Geometry is based", Collected Papers, Bernhard Riemann, Translated from the 1892 edition by Roger Baker, Charles Christenson, and Henry Orde. pp.269-270, Kendrick Press, 2004.

William Kingdom Clifford Translation "On the Hypothesis which lie at the basis of Geometry", Mathematical papers, William Kingdom Clifford, p. 56, Chelsea, 1882.

Henry S. White Translation "On the Hypothesis which lie at the foundation of Geometry", A source Book in Mathematics, by David Eugene Smith, p.411, McGraw Hill, 1929.

Michael Spivak Translation "On the Hypothesis which Lie at the Foundations of Geometry" A Comprehensive Introduction to Differential Geometry, Volume Two, p.151, Publish or Perish, 1970.

[Lipschutz], Martin Lipschutz, "Differential Geometry" Schaum's Outlines, 1969.

[Banchoff, Lovett], Thomas Banchoff, and Stephen Lovett, Differential Geometry of Curves, and Surfaces" A.K. Peters, 2010.

https://en.wikipedia.org/wiki/Shape_of_the_universe

https://en.wikipedia.org/wiki/Planck_(spacecraft)

https://en.wikipedia.org/wiki/3-torus

https://en.wikipedia.org/wiki/3-sphere

https://en.wikipedia.org/wiki/Metric_tensor

https://en.wikipedia.org/wiki/Curvature_of_Riemannian_manifolds

https://en.wikipedia.org/wiki/Riemann_curvature_tensor

https://en.wikipedia.org/wiki/Curvature

https://en.wikipedia.org/wiki/Parametric_surface

https://en.wikipedia.org/wiki/Gaussian_curvature

https://en.wikipedia.org/wiki/Principal_curvature

https://en.wikipedia.org/wiki/Sectional_curvature