# The Curvature of the Universe

H. Vic Dannon
vic0user@gmail.com
September 2025

**Abstract** Riemann's inaugural talk is about Geometry at astronomical distances, and the Shape of the Universe.

The possibilities are an infinite hyperbolic universe with negative curvature, or an infinite flat universe with zero curvature, or a finite elliptic universe with positive curvature.

Riemann suggested that the Universe is an unbounded but finite 3-sphere with positive curvature, embedded in a 4-dimensional Euclidean space. And expected empirical results to determine the Curvature, and shape of the universe.

Recent measurements confirm that with 93% confidence, the Curvature of the Universe is positive.

The measured curvatures range between $-0.0012$, and $0.0026$, average $0.0007$, and are distributed normally, on a Gaussian bell shaped curve. Statistics yields that

with 93% likelihood the curvature of the universe is positive

Consequently, Minkowski Space-Time that has a definitely negative curvature does not model the Universe.

Neither does Einstein Space-Time.

That is, the universe is Elliptic with 93% certainty.

Therefore, the universe may be Three Dimensional Sphere or a Three dimensional Torus, embedded in a Four Dimensional Euclidean Space.

The 3-Sphere is a simpler model.

We show that the Curvature of a 3-sphere of radius R is 1/R^3.

It is highly likely that Riemann's universe is a 3-Sphere that bends with curvature 1/R^3 in the 4 dimensional Euclidean flat space.

**Keywords:** Riemannian Geometry, Curvature, elliptic, Hyperbolic, Flat, 3-Surface, Universe, 3-Sphere, 3-Torus, Space-Time, Normal Distribution,

# Contents

# 1.

# Riemann's Universe is a 3-Sphere with Radius $R$

Riemann said[1]

> *"When we extend constructions in space to the immeasurably large, a distinction has to be made between the* ***unlimited*** *and the* ***infinite****;....*
>
> *That space is an unlimited tree-dimensional Manifold is a hypothesis that is constantly being verified in its applications.*
>
> *The property of the unboundedness of space posses therefore a greater empirical certainty than any other fact established by observation.*
>
> *However, the infiniteness of space does not in any way follow from this.*
>
> *On the contrary, if one assumes that the size of solid bodies does not depend on their position, and consequently ascribes to space a constant curvature, then space is necessarily finite, whenever this constant is positive, no matter how small.*
>
> *If we were to extend the infinitesimal line element in each initial direction in an element of surface into lines*

[1] Collected Papers, Bernhard Riemann, Translated from the 1892 edition by Roger Baker, Charles Christenson, and Henry Orde. pp.269-270, Kendrick Press,2004

*segments along paths of shortest lengths, the lines would lie in an unbounded surface of constant curvature.*

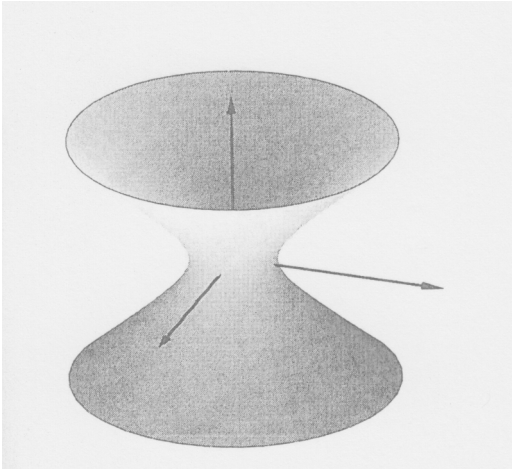
*This surface in a flat three-dimensional manifold would take the form of a sphere, so that it would be finite.*.....

*At any rate, the reciprocal of the curvature would correspond to a surface with a radius compared with which the range of our present telescopes would be negligibly small."*

# 2.

# Space-Time Does Not Model the Universe

A two-dimensional Hyperbolic surface is



By the "shape of the universe"[2], Minkowski Space-Time is an unbounded hyperbolic surface in three dimensions.

2018 measurements of the fluctuations of the temperature of the Cosmic Microwave Background Radiation suggest that the Universe curvature is almost zero, in conflict with the definitely negative curvature of the Minkowski space-time

In Einstein Space-Time, the binding gravitational forces curve the space. The curvature is extremely high around black holes. But this is irrelevant to the determination of the mean curvature of the universe.

[2] https://en.wikipedia.org/wiki/Shape_of_the_universe

# 3.

# The Universe Curves Positively with 93% Certainty

By the "shape of the universe"[3], Fluctuations in the Temperature of the Cosmic Background Radiation observed by satellites like NASA's WMAP, and the ESA Planck indicate that the curvature of the universe is very close to zero.

By the Planck mission results from 2018, the curvature is between

$$-0.0012, \quad \text{and} \quad 0.0026.$$

The mean curvature is the positive

$$0.0007$$

And the measurement error is

$$\pm 0.0019.$$

If many measurements of the curvature were made, the curvature possible values are distributed normally. That is, the probability density of the curvatures distribution is the bell shaped, Gaussian curve.

That curve peaks at the mean

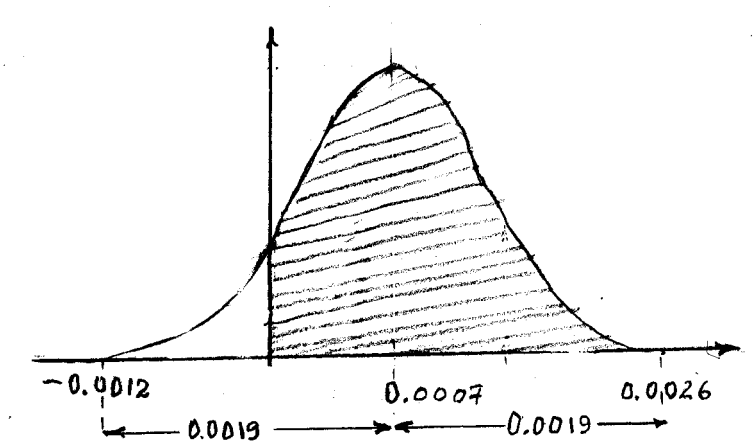
$$0.0007$$

and ranges

$$\text{from } 0.0007 - 0.0019 = -0.012$$

$$\text{to } \quad 0.0007 + 0.0019 = 0.0026$$

[3] https://en.wikipedia.org/wiki/Shape_of_the_universe



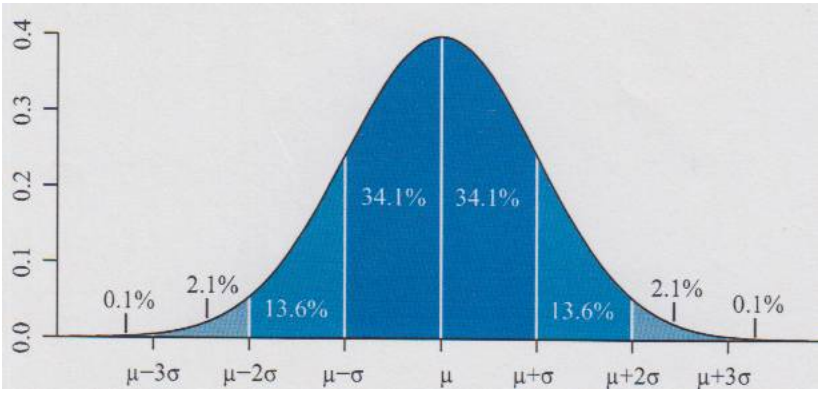

The chance for the curvature to be positive, is the area under the Gaussian curve from

$$x = 0$$

till the end of the positive range

$$\text{at } x = 0.0026 .$$

That area is tabulated for a Gaussian curve in normalized form.




In normalized form, with mean $\mu = 0$, and standard deviation $\sigma = 1$, the normal distribution density is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} .$$

$x$ ranges from $-\infty$ to $\infty$,

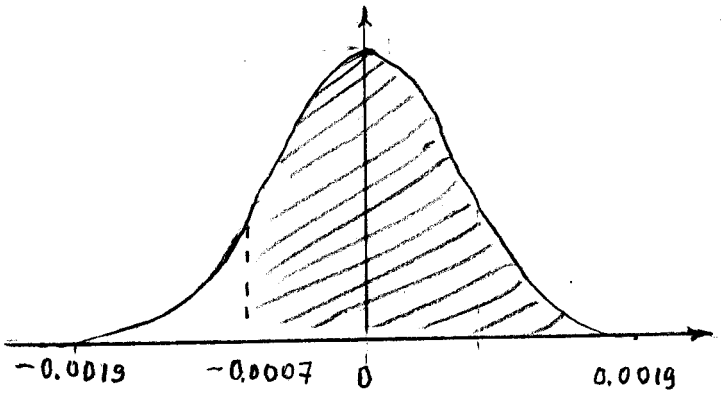
But the curve vanishes out of the interval $-4\sigma = -4$ to $4\sigma = 4$.

Adjusting to the standard normal curve, we seek the area from

$$-0.0007 \text{ to } 0.0019$$

with standard deviation

$$\frac{0.0019}{4} = 0.000475\,.$$

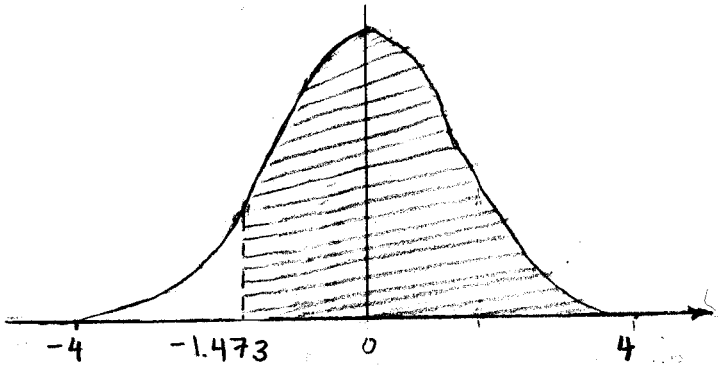



On the normal curve,

$$-\frac{0.0007}{0.000475} = -1.473$$

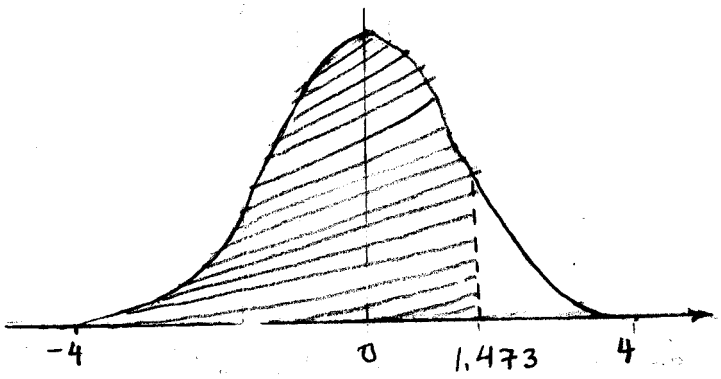
That is, the chance for positive curvature is the area under the normalized curve from

$$x = -1.473 \text{ till } x = 4\,.$$




Due to the symmetry of the normal curve with respect to $x = 0$, the area equals the area under the normal curve from

$$x = -\infty \text{ to } \quad x = 1.473$$




The area is

$$\operatorname{erf}(1.473) = \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{x=1.473} e^{-\frac{1}{2}x^2} dx = 0.9292 .^{4}$$

Therefore, with about 93% confidence, the Curvature of the universe is positive.

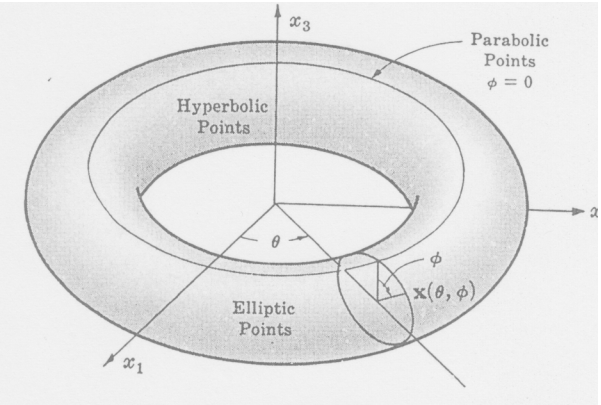
Thus, the universe has to be elliptic, such as a three dimensional sphere, or a three dimensional Torus.

[4] Murray Spiegel: "Mathematical Handbook", Schaum's 1968, p. 257

# 4.

# 3-Torus Universe

A two-dimensional Torus is




It is given by

$$\begin{pmatrix} x(\theta,\phi) \\ y(\theta,\phi) \\ z(\theta,\phi) \end{pmatrix} = \begin{pmatrix} (b + a\sin\phi)\cos\theta \\ (b + a\sin\phi)\sin\theta \\ a\cos\phi \end{pmatrix}$$

In the "shape of the universe"[5], observations of the fluctuations of the Temperature of the Cosmic Microwave Background Radiation have shown a striking amount of missing wavelengths on scales that are beyond the size of the universe.

That spectrum fits better a curvature of a universe which is a three-dimensional Torus.

But without a reasoning for such universe, we will not pursue this any further.

[5] https://en.wikipedia.org/wiki/Shape_of_the_universe

# 5.

# The Curvature of a 3-Sphere with Radius $R$ is $\frac{1}{R^3}$

A circle of radius $R$ is $\vec{x}(\phi)=\begin{pmatrix} R\cos\phi \\ R\sin\phi \end{pmatrix}$.

The Tangent at $\vec{x}$ is $\partial_\phi \vec{x}(\phi)=\begin{pmatrix} -R\sin\phi \\ R\cos\phi \end{pmatrix}$

The Metric Tensor at $\vec{x}$ is $g_{\phi\phi}=\partial_\phi \vec{x}\cdot\partial_\phi \vec{x}=R^2$

The First Fundamental Form at $\vec{x}$ is $\det(g_{\phi\phi})=R^2$

The Unit Normal lies along $-\vec{x}$, $\vec{N}=\begin{bmatrix} -\cos\phi \\ -\sin\phi \end{bmatrix}$

$\vec{N}$ is normal to the Tangent, $\vec{N}\cdot\vec{T}=\begin{bmatrix} -\cos\phi \\ -\sin\phi \end{bmatrix}\cdot\begin{pmatrix} -R\sin\phi \\ R\cos\phi \end{pmatrix}=0$

For the Second Fundamental Form,

$$\partial_{\phi\phi}\vec{x}(\phi)=\begin{pmatrix} x"(\phi) \\ y"(\phi) \end{pmatrix}=\begin{pmatrix} -R\cos\phi \\ -R\sin\phi \end{pmatrix}$$

The Second Fundamental Form at $\vec{x}$ is

$$\det(\vec{N}\cdot\partial_{\phi\phi}\vec{x}(\phi))=R$$

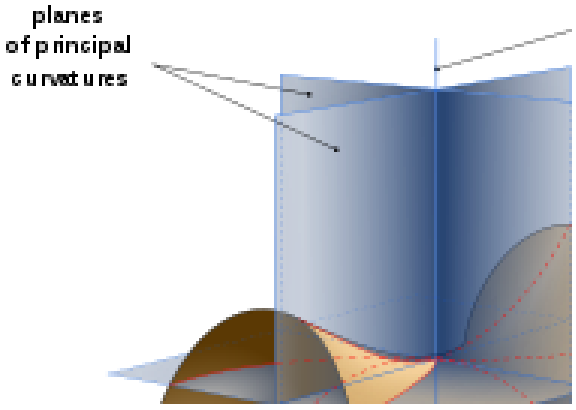
Then, the circle's curvature is

$$\kappa=\frac{\text{Second Fundamental Form}}{\text{First Fundamental Form}}=\frac{\det(\vec{N}\cdot\partial_{\phi\phi}\vec{x}(\phi))}{\det(g_{\phi\phi})}=\frac{R}{R^2}=\frac{1}{R}.$$

At each point of a differentiable two-dimensional surface, there is a normal to the surface, that belongs to a family of such planes. Each plane cuts the surface in a curve.

$\kappa_{\min} \equiv \kappa_1$, and $\kappa_{\max} \equiv \kappa_2$ are the principal curvatures.

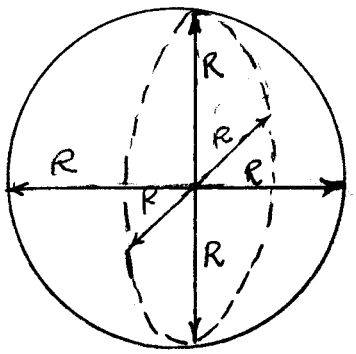
They belong to perpendicularly intersecting curves on the surface.




The Gaussian curvature of a two-dimensional surface is

$$K = \kappa_1 \kappa_2 .$$

On a 2-sphere with radius $R$, the great circles are geodesics, with curvature $\frac{1}{R}$. Any two perpendicularly intersecting great circles have the principal curvatures $\kappa_{\min} = \kappa_{\max} = \frac{1}{R}$.




The Gaussian curvature of a 2-sphere of Radius $R$ is

$$\frac{1}{R}\frac{1}{R}=\frac{1}{R^2}.$$

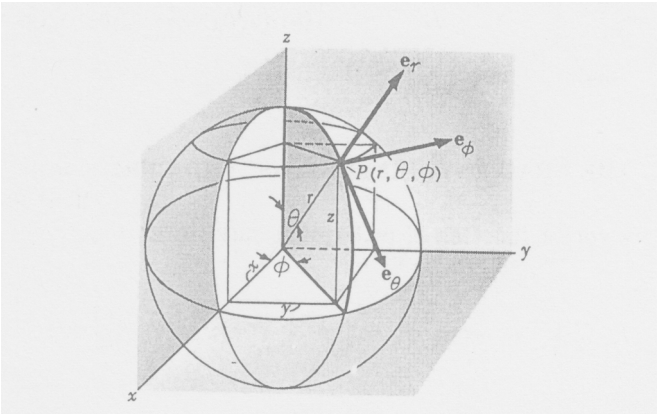
For a three parameters sphere of Radius $R$, the Curvature is the product of the curvatures of three perpendicularly intersecting great circles, each with curvature $\kappa=\frac{1}{R}$.

Therefore, the Curvature of a 3-sphere of Radius $R$ is

$$\boxed{\frac{1}{R}\frac{1}{R}\frac{1}{R}=\frac{1}{R^3}}.$$

This may be derived from the surfaces' representations.

A 2-sphere of Radius $R$, with Azimuth angle $\phi$, and polar angle $\theta$




is represented by

$$\vec{x}(\theta,\phi)=\begin{pmatrix}R\sin\theta\cos\phi\\R\sin\theta\sin\phi\\R\cos\theta\end{pmatrix}$$

A 3-sphere with radius $R$ in a 4-Euclidean space is

$$\vec{x}(\psi,\theta,\phi)=\begin{pmatrix}R\cos\psi\\R\sin\psi\cos\theta\\R\sin\psi\sin\theta\cos\phi\\R\sin\psi\sin\theta\sin\phi\end{pmatrix}$$

At each point on the 3-sphere, the Tangent space is spanned by

$$\partial_{\psi}\vec{x} = \begin{pmatrix} -R\sin\psi \\ R\cos\psi\cos\theta \\ R\cos\psi\sin\theta\cos\phi \\ R\cos\psi\sin\theta\sin\phi \end{pmatrix},$$

$$\partial_{\theta}\vec{x} = \begin{pmatrix} 0 \\ -R\sin\psi\sin\theta \\ R\sin\psi\cos\theta\cos\phi \\ R\sin\psi\cos\theta\sin\phi \end{pmatrix},$$

$$\partial_{\phi}\vec{x} = \begin{pmatrix} 0 \\ 0 \\ -R\sin\psi\sin\theta\sin\phi \\ R\sin\psi\sin\theta\cos\phi \end{pmatrix}$$

The Metric Tensor is

$$g_{ij} = (\partial_i\vec{x})\cdot(\partial_j\vec{x}) = \begin{bmatrix} R^2 & 0 & 0 \\ 0 & R^2\sin^2\psi & 0 \\ 0 & 0 & R^2\sin^2\psi\sin^2\theta \end{bmatrix}$$

The First Fundamental Form is

$$\det(g_{ij}) = R^6\sin^4\psi\sin^2\theta$$

The Unit Normal lies along $-\vec{x}$,

$$\vec{N} = \begin{pmatrix} -\cos\psi \\ -\sin\psi\cos\theta \\ -\sin\psi\sin\theta\cos\phi \\ -\sin\psi\sin\theta\sin\phi \end{pmatrix}.$$

$\vec{N}$ is perpendicular to the Tangent Vectors,

$$\vec{N}\cdot\partial_{\psi}\vec{x}=\begin{pmatrix}-\cos\psi\\-\sin\psi\cos\theta\\-\sin\psi\sin\theta\cos\phi\\-\sin\psi\sin\theta\sin\phi\end{pmatrix}\cdot\begin{pmatrix}-R\sin\psi\\R\cos\psi\cos\theta\\R\cos\psi\sin\theta\cos\phi\\R\cos\psi\sin\theta\sin\phi\end{pmatrix}=0\,,$$

$$\vec{N}\cdot\partial_{\theta}\vec{x}=\begin{pmatrix}-\cos\psi\\-\sin\psi\cos\theta\\-\sin\psi\sin\theta\cos\phi\\-\sin\psi\sin\theta\sin\phi\end{pmatrix}\cdot\begin{pmatrix}0\\-R\sin\psi\sin\theta\\R\sin\psi\cos\theta\cos\phi\\R\sin\psi\cos\theta\sin\phi\end{pmatrix}=0\,,$$

$$\vec{N}\cdot\partial_{\phi}\vec{x}=\begin{pmatrix}-\cos\psi\\-\sin\psi\cos\theta\\-\sin\psi\sin\theta\cos\phi\\-\sin\psi\sin\theta\sin\phi\end{pmatrix}\cdot\begin{pmatrix}0\\0\\-R\sin\psi\sin\theta\sin\phi\\R\sin\psi\sin\theta\cos\phi\end{pmatrix}=0\,.$$

For the matrix of the Second Fundamental Form:

$$\partial_{\psi\psi}\vec{x}=\partial_{\psi}\begin{pmatrix}-R\sin\psi\\R\cos\psi\cos\theta\\R\cos\psi\sin\theta\cos\phi\\R\cos\psi\sin\theta\sin\phi\end{pmatrix}=\begin{pmatrix}-R\cos\psi\\-R\sin\psi\cos\theta\\-R\sin\psi\sin\theta\cos\phi\\-R\sin\psi\sin\theta\sin\phi\end{pmatrix},$$

$$\vec{N}\cdot\partial_{\psi\psi}\vec{x}=R$$

$$\partial_{\theta\psi}\vec{x}=\partial_{\theta}\begin{pmatrix}-R\sin\psi\\R\cos\psi\cos\theta\\R\cos\psi\sin\theta\cos\phi\\R\cos\psi\sin\theta\sin\phi\end{pmatrix}=\begin{pmatrix}0\\-R\cos\psi\sin\theta\\R\cos\psi\cos\theta\cos\phi\\R\sin\psi\cos\theta\sin\phi\end{pmatrix}=\partial_{\psi\theta}\vec{x}$$

$$\vec{N}\cdot\partial_{\theta\psi}\vec{x}=0$$

$$\partial_{\phi\psi}\vec{x} = \partial_{\phi}\begin{pmatrix} -R\sin\psi \\ R\cos\psi\cos\theta \\ R\cos\psi\sin\theta\cos\phi \\ R\cos\psi\sin\theta\sin\phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -R\cos\psi\sin\theta\sin\phi \\ R\cos\psi\sin\theta\cos\phi \end{pmatrix} = \partial_{\psi\phi}\vec{x}$$

$$\vec{N}\cdot\partial_{\phi\psi}\vec{x} = 0$$

$$\partial_{\theta\theta}\vec{x} = \partial_{\theta}\begin{pmatrix} 0 \\ -R\sin\psi\sin\theta \\ R\sin\psi\cos\theta\cos\phi \\ R\sin\psi\cos\theta\sin\phi \end{pmatrix} = \begin{pmatrix} 0 \\ -R\sin\psi\cos\theta \\ -R\sin\psi\sin\theta\cos\phi \\ -R\sin\psi\sin\theta\sin\phi \end{pmatrix}$$

$$\vec{N}\cdot\partial_{\theta\theta}\vec{x} = R\sin^2\psi$$

$$\partial_{\phi\theta}\vec{x} = \partial_{\phi}\begin{pmatrix} 0 \\ -R\sin\psi\sin\theta \\ R\sin\psi\cos\theta\cos\phi \\ R\sin\psi\cos\theta\sin\phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -R\sin\psi\sin\theta\sin\phi \\ R\sin\psi\sin\theta\cos\phi \end{pmatrix} = \partial_{\theta\phi}\vec{x}$$

$$\vec{N}\cdot\partial_{\phi\theta}\vec{x} = 0$$

$$\partial_{\phi\phi}\vec{x} = \partial_{\phi}\begin{pmatrix} 0 \\ 0 \\ -R\sin\psi\sin\theta\sin\phi \\ R\sin\psi\sin\theta\cos\phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -R\sin\psi\sin\theta\cos\phi \\ -R\sin\psi\sin\theta\sin\phi \end{pmatrix}$$

$$\vec{N}\cdot\partial_{\phi\phi}\vec{x} = R\sin^2\psi\sin^2\theta$$

The matrix for the Second Fundamental Form is

$$\vec{N}\cdot\partial_{ij}\vec{x} = \begin{pmatrix} R & 0 & 0 \\ 0 & R\sin^2\psi & 0 \\ 0 & 0 & R\sin^2\psi\sin^2\theta \end{pmatrix}$$

The Second Fundamental Form is

$$\det(\vec{N}\cdot\partial_{ij}\vec{x}) = R^3\sin^4\psi\sin^2\theta$$

The Curvature of the 3-sphere is

$$\boxed{\frac{\det(\vec{N}\cdot\partial_{ij}\vec{x})}{\det(g_{ij})} = \frac{R^3\sin^4\psi\sin^2\theta}{R^6\sin^4\psi\sin^2\theta} = \frac{1}{R^3}}$$

## *References*

Roger Baker Translation "The Hypothesis on which Geometry is based", Collected Papers, Bernhard Riemann, Translated from the 1892 edition by Roger Baker, Charles Christenson, and Henry Orde. p. 257, Kendrick Press, 2004.

William Kingdom Clifford Translation "On the Hypothesis which lie at the basis of Geometry", Mathematical papers, William Kingdom Clifford, p. 56, Chelsea, 1882.

Henry S. White Translation "On the Hypothesis which lie at the foundation of Geometry", A source Book in Mathematics, by David Eugene Smith, p.411, McGraw Hill, 1929.

Michael Spivak Translation "On the Hypothesis which Lie at the Foundations of Geometry" A Comprehensive Introduction to Differential Geometry, Volume Two, p.151, Publish or Perish, 1970.

[Banchoff, Lovett], Thomas Banchoff, and Stephen Lovett, "Differential Geometry of Curves, and Surfaces" A.K. Peters, 2010.

[McCleary] John McCleary, "Geometry From a Differentiable Viewpoint", Cambridge, 1994.

[McInerney] Andrew McInerney "First Steps In Differential Geometry", Springer, 2013.

[Lipschutz], Martin Lipschutz, "Differential Geometry" Schaum's Outlines, 1969.

https://en.wikipedia.org/wiki/Shape_of_the_universe

https://en.wikipedia.org/wiki/Planck_(spacecraft)

https://en.wikipedia.org/wiki/3-torus

https://en.wikipedia.org/wiki/3-sphere

https://en.wikipedia.org/wiki/Metric_tensor

https://en.wikipedia.org/wiki/Curvature_of_Riemannian_manifolds

https://en.wikipedia.org/wiki/Riemann_curvature_tensor

https://en.wikipedia.org/wiki/Curvature

https://en.wikipedia.org/wiki/Parametric_surface

https://en.wikipedia.org/wiki/Gaussian_curvature

https://en.wikipedia.org/wiki/Principal_curvature

https://en.wikipedia.org/wiki/Sectional_curvature