Fermat Last Theorem Complete Statement

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Abstract: Following Diaphanous, Fermat stated

"*It is impossible for a cube to be written as a sum of two cubes, or for a fourth power to be written as the sum of two fourth powers, or, in general, for any number which is a power greater than the second to be written as a sum of a two like powers*".

It was self-evident to Diaphanous that the sides are measured in natural numbers. That is, for $n = 3, 4, 5, \dots$,

For natural numbers $1 < k < l < m$

the equality $k^n + l^n = m^n$ leads to a contradiction

Fermat proved that the equality

$k^4 + l^4 = m^4$ leads to a contradiction.

But the Fermat proof cannot be extended even to $k^3 + l^3 = m^3$

Consequently, we present the case $n = 4$ in an Appendix.

In terms of rationals, and irrationals, The Fermat Conjecture can

be stated as

For any rationals p and q in (0,1) there are no rational points (p,q) on the superellipse $x^n + y^n = 1$

Or,

$$\boxed{q = \text{rational in } (0,1) \Rightarrow \sqrt[n]{1 - q^n} = irrational}$$

For any rational $0 < r < 1$,

$$q = \frac{2r}{1+r^2}$$

$$p = \frac{1-r^2}{1+r^2}.$$

is a rational point on the unit circle.

We observe that then

$$\frac{1+q}{1-q} = \underbrace{\left(\frac{1+r}{1-r}\right)}_{s}^{2}$$

That is, for the rational number $s = \dfrac{1+r}{1-r} > 1$

$$1 + q = s^2(1-q).$$

Hence, the rational points on the circle satisfy two equations

$$(1+q)(1-q) = p^2,$$

and

$$\frac{1+q}{1-q} = s^2.$$

Observing that

$$1-q^n = (1-q)(1+q+q^2+....+q^{n-1})$$

the Fermat Conjecture complete version is

For $n = 3,4,5,6,...$

$$\boxed{\begin{array}{c} q = \text{rational in } (0,1) \Rightarrow \text{there is no rational } 0 < p < 1 \\ \text{so that} \quad (1-q)(1+q+q^2+....+q^{n-1}) = p^n \end{array}}$$

and

$$\boxed{\begin{array}{c} q = \text{rational in } (0,1) \Rightarrow \text{there is no rational } s > 1 \\ \text{so that} \quad \dfrac{1+q+q^2+....+q^{n-1}}{1-q} = s^n \end{array}}$$

Contents

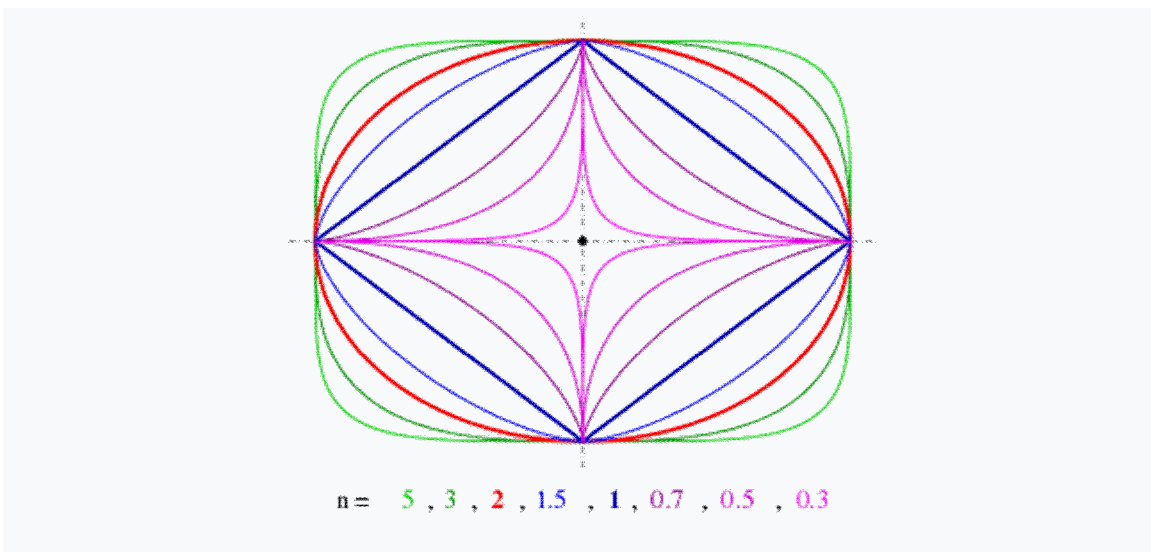
1.

Rational Points on the Super-ellipse $x^n + y^n = 1$

For $n = 1, 2, 3, 4, 5, \ldots$ the superellipse

$$x^n + y^n = 1$$

has an oval shape with rounded corners between a square, and the rhombus



The existence of Pythagorean triplets such as

$$(3, 4, 5),$$

so that

$$3^2 + 4^2 = 5^2,$$

$$\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = 1,$$

means that the unit circle has points with rational coordinates.

In fact, infinitely many such points

$$(k, l, m)$$

with k, l, and m natural numbers so that

$$m > l > k > 1$$

$$k^2 + l^2 = m^2,$$

$$\underbrace{\left(\frac{k}{m}\right)^2}_{p} + \underbrace{\left(\frac{l}{m}\right)^2}_{q} = 1$$

$$p^2 + q^2 = 1$$

$$p^2 = 1 - q^2$$

Then, p depends on q linearly, with slope r, and constant 1.

To keep p rational, r must be rational. That is,

$$p = 1 - rq$$

Then,

$$p^2 = 1 - 2rq + r^2q^2$$

$$\not{1} - 2rq + r^2q^2 = \not{1} - q^2$$

$$-2r + r^2q = -q$$

$$(1 + r^2)q = 2r$$

$$q = \frac{2r}{1+r^2}$$

$$p = 1 - r\frac{2r}{1+r^2} = \frac{1-r^2}{1+r^2}.$$

For $r = \frac{1}{2}$**,**

$$q = \frac{2(\frac{1}{2})}{1+(\frac{1}{2})^2} = \frac{4}{5}, \quad p = \frac{1-(\frac{1}{2})^2}{1+(\frac{1}{2})^2} = \frac{3}{5}$$**, is a rational point on**

$$x^2 + y^2 = 1$$

But

$$q = \frac{1}{2}, \quad \sqrt{1-(\frac{1}{2})^2} = \frac{\sqrt{3}}{2}$$ **is not a rational point on the circle.**

2.

$$\underbrace{(1-q)(1+q+q^2+....+q^{n-1})}_{1-q^n} \neq p^n$$

The first equivalent to the conjecture is

$$q = \text{rational in } (0,1) \Rightarrow \text{there is no rational } 0 < p < 1$$
$$\text{so that} \qquad 1 - q^n = p^n$$

Assuming $p = 1 - rq$ applies to prove for $\boxed{n = 3}$

$$\boxed{q = \text{rational in } (0,1) \Rightarrow \sqrt[3]{1-q^3} = irrational}$$

Proof: Assume that p and q are relatively prime rationals in $(0,1)$ and

$$p^3 = 1 - q^3$$

Then, p depends on q linearly, with slope r, and constant 1.

To keep p rational, r must be rational. That is,

$$p = 1 - rq$$

$$p^3 = \not{1} - 3rq + 3r^2q^2 - r^3q^3 = \not{1} - q^3$$

$$-3r + 3r^2q - r^3q^2 + q^2 = 0$$

$$q^2(1-r^3) + 3r^2q - 3r = 0$$

$$q_{+,-} = -\frac{3r^2}{2(1-r^3)} \pm \frac{\sqrt{9r^4 + 12r(1-r^3)}}{2(1-r^3)}$$

$$= -\frac{3r^2}{2(1-r^3)} \pm \frac{\sqrt{12r - 3r^4}}{2(1-r^3)}$$

For q to be rational,

$$\sqrt{12r - 3r^4} = rational \text{ that depends on } r$$

To annihilate $12r$, choose

$$\sqrt{12r - 3r^4} = 1 + sr$$

$$12r - 3r^4 = (1 + sr)^2$$

$$12r - 3r^4 = 1 + 2sr + s^2r^2$$

$$12r = 2sr \Rightarrow s = 6$$

$$-3r^4 = 1 + 6^2r^2$$

$$3r^4 + 36r^2 + 1 = 0$$

$$r^2 = -\frac{36}{6} \pm \frac{\sqrt{1296 - 12}}{6}$$

$$r = irrational$$

$$\sqrt{12r - 3r^4} = 1 + 6r = irrational$$

$$q = irrational$$

$$\text{contradiction to } q = rational$$

$$p \neq rational$$

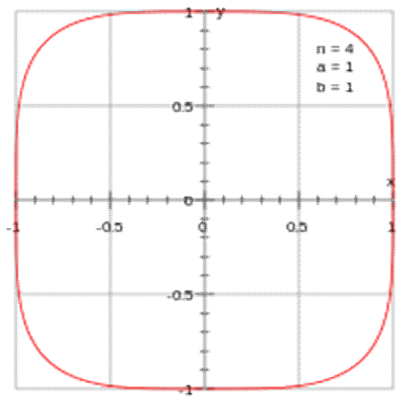
$$\sqrt[3]{1 - q^3} \neq rational$$

$$\sqrt[3]{1 - q^3} = irrational. \square$$

For $\boxed{n=4}$, the superellipse

$$x^4 + y^4 = 1$$

has the shape



If we Assume that p and q are rationals in $(0,1)$ and

$$p^4 = 1 - q^4,$$

and

$$p = 1 - rq,$$

$$p^4 = \not{1} - 4rq + 6r^2q^2 - 4r^3q^3 + r^4q^4 = \not{1} - q^4$$

$$-4r + 6r^2q - 4r^3q^2 + r^4q^3 + q^3 = 0$$

$$(1 + r^4)q^3 - 4r^3q^2 + 6r^2q - 4r = 0$$

For q to be rational, we need to solve the cubic equation

$$q^3 - \frac{4r^3}{1+r^4}q^2 + \frac{6r^2}{1+r^4}q - \frac{4r}{1+r^4} = 0$$

Let[1]

[1] Spiegel, "Mathematical Handbook", 1968, Algebraic equations, p. 32

$$Q = \frac{1}{9}\left(3\frac{6r^2}{1+r^4} - \left[\frac{4r^3}{1+r^4}\right]^2\right) = \frac{2}{9}r^2\frac{9-r^4}{(1+r^4)^2}$$

$$R = \frac{1}{54}\left(9\frac{-4r^3}{1+r^4}\frac{6r^2}{1+r^4} + 27\frac{4r}{1+r^4} + 2\left[\frac{4r^3}{1+r^4}\right]^3\right) = 2\frac{r}{(1+r^4)^3}$$

$$Q^3 = \frac{2^3}{3^6}r^6\frac{(3^2-r^4)^3}{(1+r^4)^6}$$

$$R^2 = \frac{4r^2}{(1+r^4)^6}$$

$$Q^3 + R^2 = \frac{2^3}{3^6}r^6\frac{(3^2-r^4)^3}{(1+r^4)^6} + \frac{4r^2}{(1+r^4)^6}$$

$$S = \sqrt[3]{R + \sqrt{Q^3 + R^2}}$$

$$T = \sqrt[3]{R - \sqrt{Q^3 + R^2}}$$

Then, the solutions are

$$x_1 = S + T + \frac{1}{3}\frac{4r^3}{1+r^4}$$

$$x_2 = -\frac{1}{2}(S+T) + \frac{1}{3}\frac{4r^3}{1+r^4} + i\frac{\sqrt{3}}{2}(S-T)$$

$$x_3 = -\frac{1}{2}(S+T) + \frac{1}{3}\frac{4r^3}{1+r^4} - i\frac{\sqrt{3}}{2}(S-T)$$

That is why Euler did not pursue $n > 3$.

3.

$$\frac{1+q+q^2+....+q^{n-1}}{1-q} \neq s^n$$

For $n = 2$, there are rational points (p,q) on the unit circle so that

$$1-q^2 = (1-q)(1+q) = p^2$$

Then, there is r rational so that

$$q = \frac{2r}{1+r^2}$$

$$1+q = 1+\frac{2r}{1+r^2} = \frac{(1+r)^2}{1+r^2}$$

$$1-q = 1-\frac{2r}{1+r^2} = \frac{(1-r)^2}{1+r^2}$$

$$\frac{1+q}{1-q} = \underbrace{\left(\frac{1+r}{1-r}\right)}_{s}^2 = s^2$$

$$1+q = s^2(1-q)$$

There is a rational point (p,q) on the unit circle $\Leftrightarrow$

$\Leftrightarrow$ there is a rational $s > 1$ so that $1+q = s^2(1-q)$

For $n = 3$,

There is a rational point (p,q) on the superellipse $x^3 + y^3 = 1$ $\Leftrightarrow$

$\Leftrightarrow$ there is a rational $s > 1$ so that $1+q+q^2 = s^3(1-q)$

There are no rational points (p,q) on $x^3+y^3=1 \Leftrightarrow$

$\Leftrightarrow$ for any $0<q<1$, and any positive rational $s>1$,

$$\boxed{1+q+q^2 \neq s^3(1-q)}$$

That is,

$$(1-s^3)+(1+s^3)q+q^2 \neq 0$$

Proof: Assume that

$$(1-s^3)+(1+s^3)q+q^2 = 0$$

$$q_{1,2} = \frac{1+s^3}{2} \pm \frac{1}{2}\sqrt{(1+s^3)^2+4(s^3-1)}$$

$$= \frac{1+s^3}{2} \pm \frac{1}{2}\sqrt{s^6+6s^3-3}$$

To have $q=$ rational, assume

$$\sqrt{s^6+6s^3-3} = 1+ts^3$$

$$s^6+6s^3-3 = 1+2ts^3+t^2s^6$$

To annihilate s^3, set $6s^3 = 2ts^3 \Rightarrow t=3$

$$8s^6 = -4$$

$$s^3 = \frac{1}{\sqrt{2}}i.\square$$

For $n=4$,

There is a rational point (p,q) on the superellipse $x^4+y^4=1 \Leftrightarrow$

$\Leftrightarrow$ there is a rational $s>1$ so that $1+q+q^2+q^4 = s^4(1-q)$

There are no rational points (p,q) on $x^4 + y^4 = 1 \Leftrightarrow$

$\Leftrightarrow$ for any $0 < q < 1$, and any rational $s > 1$,

$$\boxed{1 + q + q^2 + q^3 \neq s^4(1-q)}$$

That is,

$$(1 - s^4) + (1 + s^4)q + q^2 + q^3 \neq 0$$

For $n = 3,4,5,...$

There are no rational points (p,q) on $x^n + y^n = 1 \Leftrightarrow$

$\Leftrightarrow$ for any $0 < q < 1$, and any rational $s > 1$,

$$\boxed{1 + q + q^2 + ... + q^{n-1} \neq s^n(1-q)}$$

That is,

$$\boxed{(1 - s^n) + (1 + s^n)q + q^2 + ... + q^{n-1} \neq 0}$$

Appendix: Fermat Proof for $n = 4$

For $n = 4$, Fermat gave a proof that does not extend to $n \neq 4$.

Assume that there are relatively prime $m > k > l > 0$ so that

$$k^4 + l^4 = m^4$$

$$\underbrace{\left(\frac{k}{m}\right)}_{p}^4 + \underbrace{\left(\frac{l}{m}\right)}_{q}^4 = 1$$

$$(p^2)^2 + (q^2)^2 = 1$$

Then, by $n = 2$, there are relatively prime $m_1 > n_1 > 0$, and a rational $r_1 = \dfrac{n_1}{m_1}$ so that

$$\underbrace{p^2}_{\frac{l^2}{m^2}} = \frac{1 - r_1^2}{1 + r_1^2} = \frac{1 - \left(\frac{n_1}{m_1}\right)^2}{1 + \left(\frac{n_1}{m_1}\right)^2} = \frac{m_1^2 - n_1^2}{m_1^2 + n_1^2} \Rightarrow \begin{cases} l^2 = m_1^2 - n_1^2 \\ m^2 = m_1^2 + n_1^2 \end{cases},$$

$$\underbrace{q^2}_{\frac{k^2}{m^2}} = \frac{2r_1}{1 + r_1^2} = \frac{2\frac{n_1}{m_1}}{1 + \left(\frac{n_1}{m_1}\right)^2} = \frac{2m_1 n_1}{m_1^2 + n_1^2} \Rightarrow \begin{cases} k^2 = 2m_1 n_1 \\ m^2 = m_1^2 + n_1^2 \end{cases},$$

Then,

$$l^2 + n_1^2 = m_1^2$$

$$\left(\frac{l}{m_1}\right)^2 + \left(\frac{n_1}{m_1}\right)^2 = 1$$

Then, by $n = 2$, there are relatively prime $m_2 > n_2 > 0$, and a rational $r_2 = \frac{n_2}{m_2}$ so that

$$\frac{l}{m_1} = \frac{1 - r_2^2}{1 + r_2^2} = \frac{1 - \left(\frac{n_2}{m_2}\right)^2}{1 + \left(\frac{n_2}{m_2}\right)^2} = \frac{m_2^2 - n_2^2}{m_2^2 + n_2^2} \Rightarrow \begin{cases} l = m_2^2 - n_2^2 \\ m_1 = m_2^2 + n_2^2 \end{cases},$$

$$\frac{n_1}{m_1} = \frac{2r_2}{1 + r_2^2} = \frac{2\frac{n_2}{m_2}}{1 + \left(\frac{n_2}{m_2}\right)^2} = \frac{2m_2 n_2}{m_2^2 + n_2^2} \Rightarrow \begin{cases} n_1 = 2m_2 n_2 \\ m_1 = m_2^2 + n_2^2 \end{cases}$$

Therefore,

$$k^2 = 2\underbrace{m_1}_{m_2^2 + n_2^2}\underbrace{n_1}_{2m_2 n_2} = 4m_2 n_2 (m_2^2 + n_2^2)$$

Any prime that divides $m_2 n_2$ divide either m_2, or n_2 but not both, and not $m_2^2 + n_2^2$. Therefore, both $m_2 n_2$, and $m_2^2 + n_2^2$ are squares. And

$$m_2 = k_1^2$$

$$n_2 = l_1^2$$

Hence,

$$k_1^4 + l_1^4 = m_2^2 + n_2^2 = square$$

And

$$k_1^4 + l_1^4 = m_2^2 + n_2^2 = m_1 < m_1^2 + n_1^2 = m^2 < m^4 = k^4 + l^4$$

This produces an impossible infinite sequence of positive k_j, l_j, m_j

so that

$$k_{j+1}^4 + l_{j+1}^4 < k_j^4 + l_j^4 . \square$$

References

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[Euler] Leonard Euler, "Elements of Algebra", 3rd Edition, Longman, 1822.

[Savant] Marilyn vos Savant, "The World's Most Famous Math Problem", St. Martin Press, 1993.

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