Continuum Hypothesis, Axiom of Choice, and Non-Cantorian Theory

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Abstract We prove that the Continuum Hypothesis is equivalent to the Axiom of Choice. Thus, the Hypothesis-Negation is equivalent to the Axiom of No-Choice.

The Non-Cantorian Axioms impose a Non-Cantorian definition of cardinality, that is different from Cantor's cardinality imposed by the Cantorian Axioms.

The Non-Cantorian Theory is the Zermelo-Fraenkel Theory with the No-Choice Axiom, or the Hypothesis-Negation.

This Theory has distinct infinities.

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Introduction

The Continuum Hypothesis says that there is no set X with cardinality that is strictly between $CardN$, and $CardR = 2^{CardN}$.

Thus, the Hypothesis statement assumes that

$$CardN < 2^{CardN}.$$

In [1], we proved that in Cantor's Theory,

$$2^{CardN} = CardN .$$

Therefore, Cantor's claim that $CardN < CardR$ is disproved, but the Hypothesis statement is trivially satisfied.

Consequently, Cantor's theory offers precisely one unique infinity, defying its purpose to supply us with many distinct infinities.

To obtain distinct infinities we need to develop the Non-Cantorian Theory.

1. The Continuum Hypothesis, and Cardinality

In [3] we proved that the Hypothesis is equivalent to

$$(CardN)^2 = CardN .$$

Here, we show that it is equivalent to each of the following Axioms,

● **Continuum Hypothesis**

A. There is no set X so that $CardN < CardX < 2^{CardN}$

● **Countability Axiom**

B. $CardN \times CardN = CardN$

Cantor believed that the Countability Axiom was a Theorem, and "proved" it by his "Zig-Zag proof".

But the Countability Axiom cannot be proved. It is equivalent to the Hypothesis, and it holds under Cantorian Cardinality.

The Cantorian Cardinality is established by the Effective Countability Axiom, that is too equivalent to the Hypothesis.

● **Generalized Countability Axiom**

C. For any $n = 1,2,3,...$ $(CardN)^n = (CardN)^{n+1}$

● **Diagonal Axiom**

D. $2^{CardN} = CardN$

Cantor believed that 2^{CardN} is greater than $CardN$, and "proved" the inequality, which is the ***Non-Diagonal Axiom***, as a Theorem in his Theory.

Actually, the ***Non-Diagonal Axiom*** belongs to Non-Cantorian Theory.

Cantor's "proof" is known as "the Diagonal Argument".

The Cantorian Diagonal Axiom allows only one infinity in Cantor's Theory.

Thus, raising the need for the Non-Cantorian Theory.

● **Generalized Diagonal Axiom**

E. For any $n = 1,2,3,...$ $2^{CardN} = (CardN)^n$

● **Effective Countability Axiom**

F. $Card\{a_1, a_2, a_3,..\} = CardN$, for any $\{a_1, a_2, a_3,..\}$

Any infinite sequence of distinct numbers has $CardN$.

This Axiom establishes Cantorian Cardinality.

The Effective Countability Axiom guarantees that sequencing is sufficient to establish equal Cantorian cardinalities. All sequences have the same cardinality as the sequence of the natural numbers.

Since the Effective Countability is equivalent to the Hypothesis, the Cantorian Cardinality characterizes the Hypothesis exclusively.

Thus, the Effective-Countability Axiom is the key to the Cantorian Theory.

Proof

$\underline{A \Rightarrow B}$

We prove $Negation\ B \Rightarrow Negation\ A$.

By [2, p.155], *For any cardinals* $m_1,\ n_1,,\ m_2,\ \text{and}\ n_2$,

$$m_1 < n_1, \textit{ and } m_2 < n_2 \Rightarrow m_1 \times m_2 < n_1 \times n_2 .$$

Assuming that

$$CardN < 2^{CardN},$$

we have

$$CardN \times CardN < 2^{CardN} \times 2^{CardN}.$$

But

$$2^{CardN} \times 2^{CardN} = 2^{CardN+CardN} = 2^{CardN}.$$

Hence,

$$(CardN)^2 < 2^{CardN}.$$

Therefore,

$$CardN < (CardN)^2 \Rightarrow CardN < (CardN)^2 < 2^{CardN}.$$

Namely, if $CardN < (CardN)^2$, the rationals serve as a set X which cardinality is between $CardN$, and $CardR$, and the Continuum Hypothesis does not hold. That is,

$$Continuum\ Hypothesis \Rightarrow (CardN)^2 = cardN . \square$$

$\underline{B \Rightarrow C}$ is clear.

$\underline{C \Rightarrow D}$

$$2^{CardN} \leq (CardN)^{CardN}$$

$$\leq CardN + (CardN)^2 + (CardN)^3 + ...$$

Now, C implies that for any $n = 1,2,3....$

$$CardN + (CardN)^2 + ... + (CardN)^n \leq cardN.$$

Tarski ([4], or [2, p.174]) proved that

If

$$m_1, m_2, ..m_n \quad and\ m$$

are any cardinal numbers so that for any $n = 1,2,3....$,

$$m_1 + m_2 + ... + m_n \leq m,$$

then,

$$m_1 + m_2 + ... + m_n + ... \leq m.$$

By Tarski result

$$CardN + (CardN)^2 + (CardN)^3 + ... \leq CardN,$$

$$\leq 2^{CardN}.$$

Therefore,

$$(CardN)^n = (CardN)^{n+1} \Rightarrow 2^{CardN} = CardN. \square$$

<u>$D \Rightarrow A$</u> is clear.

<u>$C \ \& \ D \Leftrightarrow E$</u>

Therefore,

$$A \Leftrightarrow B \Leftrightarrow C \Leftrightarrow D \Leftrightarrow E.$$

$\underline{D \Rightarrow F}$

For any $\{a_1, a_2, a_3, ...\}$,

$$CardN \leq Card\{a_1, a_2, a_3, ...\} \leq 2^{CardN} = CardN . \square$$

$\underline{F \Rightarrow B}$ is clear, since $N \times N$ may be sequenced.

Therefore,

$$A \Leftrightarrow B \Leftrightarrow C \Leftrightarrow D \Leftrightarrow E \Leftrightarrow F . \square$$

It follows that Cantor's Cardinality, that does not distinguish between sequences of integers, and sequences of rationals, does not distinguish between cardinalities of Natural, and Real Numbers.

Cantor's Cardinality, is too coarse to distinguish between infinities.

Will a Non-Cantorian Cardinality distinguish between infinities?

In [3] we showed that a Non-Cantorian Theory does not exist under Cantor's Cardinality

But the Non-Cantorian Axioms establish a Non-Cantorian Cardinality.

Cantor's Cardinality ignores the property that makes the rationals <u>seem</u> larger than the natural numbers. Namely, that between any two rationals there is another rational.

Non-Cantorian Cardinality, may count that.

The Non-Cantorian Cardinality is established by the Non-Effective-Countability Axiom.

2. The Non-Cantorian Cardinality

The Non-Cantorian cardinality is established by the following Axioms, that are all equivalent to the Hypothesis-Negation.

■ ***Hypothesis-Negation***

a) There is a set X so that $CardN < CardX < CardR$

■ ***Non-Countability Axiom***

b) For any $n = 1,2,3,...$ $CardN \times CardN > CardN$

The Rationals Non-Cantorian Cardinality is greater than $CardN$.

The Non-Countability Axiom Cannot be proved. It is equivalent to the Hypothesis-Negation, and it holds under Non-Cantorian Cardinality.

The Non-Cantorian Cardinality is established by the Non-Effective-Countability Axiom, that is too equivalent to the Hypothesis-Negation.

■ ***Generalized Non-Countability Axiom***

c) For any $n = 1,2,3,...$ $(CardN)^{n+1} > (CardN)^n$

This Axiom guarantees infinitely many Non-Cantorian distinct infinities, between the Non-Cantorian cardinalities of the natural and the real numbers.

■ ***Non-Diagonal Axiom***

d) $2^{CardN} > CardN$

Cantor "proved" this Non-Cantorian Axiom as a Theorem in his Theory, by his " Diagonal Argument".

In fact, in Cantorian Theory we have the Diagonal Axiom $2^{CardN} = CardN$.

■ ***Generalized Non-Diagonal Axiom***

e) For any $n = 1,2,3,...$ $2^{CardN} > (CardN)^n$

This Axiom too guarantees many Non-Cantorian distinct infinities

■ ***Non-Effective-Countability Axiom***

f) $Card\{a_1, a_2, a_3,..\} > CardN$, *for some* $\{a_1, a_2, a_3,...\}$

There are infinite sequences of distinct numbers with Non-Cantorian cardinalities greater than $CardN$. For instance, the rational numbers, and the real numbers.

This Axiom establishes Non-Cantorian cardinality.

The Non-Effective-Countability Axiom guarantees that sequencing is not sufficient to establish equal Non-Cantorian cardinalities.

Not all sequences have the same cardinality as the sequence of the Natural Numbers.

There are sequences with Non-Cantorian cardinality strictly greater than that of the Natural Numbers.

Since the Non-Effective-Countability is equivalent to the

Hypothesis-Negation, the Non-Cantorian cardinality characterizes the Hypothesis-Negation exclusively.

3. The smallest Non-Cantorian Cardinality

The smallest Non-Cantorian cardinality is

$$CardN = Card\{1,2,3,...\}.$$

If $\{a_1, a_2, a_3,....\}$ is an infinite set with distinct elements that are sequenced, so that between none of two consecutive elements there are no other elements of the sequence, then we say that

$$Card\{a_1,a_2,a_3,....\} = CardN .$$

There are many sets with this cardinality.

The Odd natural numbers

$$\{1,3,5,7,.....\}$$

The Even natural numbers, which are the multiples of the number 2,

$$\{2,4,6,8,....\}$$

The multiples of the number 3,

$$\{3, 6, 9, 12, 15, ...\}$$

The powers of the number 2,

$$\{2, 2^2, 2^3, 2^4, ...\}$$

The powers of the number 3,

$$\{3, 3^2, 3^3, 3^4, ...\}.$$

Theorem $CardN + CardN = CardN$

Proof

Given two infinite disjoint sets,

$$\{a_1, a_2, a_3, ...\},$$

and

$$\{b_1, b_2, b_3, ...\},$$

each with cardinality $CardN$, form the union

$$\{a_1, b_1, a_2, b_2, a_3, b_3, ...\}.$$

Since between any two consecutive elements of the union, there are no elements of the union, we have

$$Card\{a_1, b_1, a_2, b_2, a_3, b_3, ...\} = CardN.$$

We conclude that

$$CardN + CardN = CardN . \square$$

Similarly, for any natural number $n = 1,2,3,...$,

$$\underbrace{CardN + CardN + ... + CardN}_{n\ times} = CardN .$$

If the infinite set is such that between some consecutive elements there is another element, we may expect the Non-Cantorian cardinality of that sequence to be greater than $CardN$.

4. No-Choice Axiom

The **Choice Theorem** says that if for each $n = 1,2,3,...$ there is a non-empty set of numbers A_n, then we can choose from each A_n one number a_n, and obtain a collection of numbers that has a representative from each A_n.

If we replace the index numbers $n = 1,2,3,...$ with an infinite set of numbers I, this choice may not be guaranteed.

There may be an infinite set of numbers I, so that for each index i in it, there is a non-empty set of numbers A_i, with no collection of numbers, that has a representative a_i from each A_i.

The **Axiom of Choice** is the guess that the choice is guaranteed for any infinite set I, and any family of non-empty sets indexed by I.

The ***No-Choice Axiom*** says that there is an infinite index set, and a family of non-empty sets A_i indexed by it, with no collection of numbers, that has a representative a_i from each A_i.

5. No-Well-Ordering Axiom

The Axiom of Choice is equivalent to the Well-Ordering Axiom.

By the **Well-Ordering Theorem**, the Natural Numbers are ordered in such a way that every subset of them has a first element.

The **Well-Ordering Axiom** is the guess that every infinite set of numbers can be well-ordered like the Natural Numbers.

The ***No-Well-Ordering Axiom*** says that there is a set that cannot be well-ordered.

A candidate for such set are the Real Numbers.

In 1963, Cohen claimed that it is not possible to prove that the real numbers can be well-ordered.

6. No-Transfinite-Induction Axiom

The Axiom of Choice is equivalent to the Transfinite Induction Axiom.

The **Induction Theorem** says that

If a property depends on each number $n = 1,2,3,..$, so that

1) The property holds for the first natural number $n = 1$.
2) If the property holds for the natural number k, we can deduct that it holds for the next number $k+1$.

Then, the property holds for any $n = 1,2,3,...$.

The **Transfinite-Induction Axiom** guesses that the same holds for any infinite index set I.

It says that if I is any well-ordered infinite set of numbers, and if there is any property that depends on each index i from I, so that

1) The property holds for the first element of I,
2) If the property holds for all the $k's$ that precede the index j, we can conclude that the property holds for j,

Then, the property holds for any index i in I.

The ***No-Transfinite-Induction Axiom*** says that

There is a well-ordered infinite set of numbers I, and there is a property that depends on each index i from I, so that

1) The property holds for the first element of I,
2) If the property holds for all the $k's$ that precede the index j, we can conclude that the property holds for j,

But the property does not hold for some index i_0 in I.

7. Continuum Hypothesis $\equiv$ Axiom of Choice

We use the Non-Countability Axiom to link the Continuum Hypothesis with the Axiom of Choice, and prove equivalence between them.

We will show here that the Hypothesis-Negation is equivalent to the No-Choice Axiom.

We use a result that Tarski obtained in 1924. [6, p. 165, #1.17(a)]

Tarski proved that the Axiom of Choice is equivalent to the Axiom

- For any infinite cardinals α, and β,

$$\alpha + \beta = \alpha \times \beta .$$

That is, according to Tarski, the No-Choice Axiom is equivalent to the Axiom

- There are infinite cardinals α, and β, so that

$$\alpha + \beta \neq \alpha \times \beta .$$

If we take

$$\alpha = \beta = CardN .$$

Then,

$$\alpha + \beta = CardN + CardN$$

$$= CardN$$

$$< CardN \times CardN$$

$$= \alpha \times \beta$$

That is,

$$\alpha + \beta \neq \alpha \times \beta$$

Thus, the Non-Countability Axiom

$$CardN < CardN \times CardN$$

is equivalent to the No-Choice Axiom.

On the other hand, the Non-Countability Axiom is equivalent to the Hypothesis-Negation.

Therefore, the No-Choice Axiom, and the Hypothesis-Negation are equivalent.

Thus, The Axiom of Choice, and the Hypothesis are equivalent. □

The Continuum Hypothesis is not a stand alone Axiom, independent of the Commonly accepted Axioms of Set Theory.

The Non-Cantorian Theory is based on the Axiom of No-Choice.

8. The Meaning of Godel's Consistency

The failure to identify the Continuum Hypothesis with any of the Axioms of set theory, led Godel in 1938 to confirm the consistency of the Hypothesis with the other Axioms of set theory, and led Cohen in 1963 to confirm the consistency of the Hypothesis-Negation.

Since the Continuum Hypothesis is equivalent to the Axiom of Choice, Godel's Consistency result is self-evident.

The Continuum Hypothesis is consistent with the Axioms of set theory, because it is one of them.

The Continuum Hypothesis is just another statement of the Axiom of Choice.

Therefore, Godel's work amounts to the following:

If the commonly accepted Axioms of Set Theory are consistent, then adding one of them to all of them will cause no inconsistency.

While Godel established a trivial result, his methods enabled Cohen, to establish the Continuum Hypothesis as an independent Axiom of Set Theory.

That stopped work on the Continuum Hypothesis for a long time.

9. Cohen's Consistency Error

Cohen claimed that the addition of the Hypothesis-Negation to the commonly accepted Axioms of set theory, will cause no inconsistency.

But the Hypothesis-Negation is just another statement of the Axiom of No-Choice

Therefore, the addition of the Hypothesis-Negation to the axioms of set theory, means the addition of the Axiom of No-Choice, to the Axiom of Choice.

The mixing of the Axiom of Choice with its Negation, must lead to inconsistency.

Cohen's erroneous consistency result, established the Hypothesis as an independent Axiom of Set Theory.

In fact, the Continuum Hypothesis is equivalent to the Axiom of Choice.

The Hypothesis is one of the commonly accepted Axioms of Set

Theory.

Thus, Non-Cantorian Theory is based on the Axiom of No-Choice.

The Non-Cantorian Theory is the No-Choice Theory of Zermelo and Fraenkel.

10. Non-Cantorian Cardinals

(CardN)×(CardN)

Since the rationals can be listed in an infinite matrix,

$$Card(Rationls) = CardN \times CardN \ .$$

According to the Non-Countability Axiom,

$$CardN < CardN \times CardN \, .$$

That is, the Non-Cantorian cardinality of the rational numbers, is greater than $CardN$.

Since for any $n = 1,2,3,...$,

$$nCardN = CardN \, ,$$

there is a no Non-Cantorian cardinality between the integers, and the rationals.

The Rationals have the smallest Non-Cantorian cardinality that is strictly greater than the cardinality of the Natural numbers.

(CardN)^3

The cardinality of the roots of quadratic polynomials in integer coefficients in R is

$$CardN \times CardN \times CardN .$$

By the Generalized Non-Countability Axiom,

$$CardN \times CardN < CardN \times CardN \times CardN .$$

Since for any $n = 1,2,3,...,$

$$nCardN \times CardN = CardN \times CardN ,$$

there is a no Non-Cantorian cardinality between the Rationals, and the roots of the Quadratic Polynomials in integer coefficients.

The Roots of the Quadratic Polynomials in integer coefficients, have the smallest Non-Cantorian cardinality that is strictly greater than the cardinality of the rationals.

(CardN)^n

For any $n = 1,2,3,\ldots,(CardN)^n$ is the cardinality of all the roots of all the polynomials in integer coefficients of degree n.

By the Generalized Non-Countability Axiom, for any $n = 1,2,3,.$

$$(CardN)^n < (CardN)^{n+1}.$$

Furthermore, there is no Non-Cantorian cardinality between the two.

$$(CardN)^{n+1} \quad is\ the\ smallest\ Non\text{-}Cantorian\ cardinality$$

$$that\ is\ strictly\ greater\ than\ \ (CardN)^n.$$

2^{CardN}

The cardinality of the real numbers is

$$CardR = 2^{CardN}.$$

By the Generalized Non-Diagonal Axiom,

$$2^{CardN} > (CardN)^n,\ for\ any\ n = 1,2,3,\ldots.$$

Theorem $\underline{(CardN)^{CardN} = 2^{CardN}}$

Proof

$(\geq)$ is clear.

$(\leq)$

$$(cardN)^{CardN} \leq CardN + (CardN) \times (CardN)$$

$$+(CardN) \times (CardN) \times (CardN) + \ldots,$$

$$= CardN + (CardN)^2 + (CardN)^3 + \ldots$$

Now, for $n = 1,2,\ldots$

$$CardN + (CardN)^2 + \ldots + (CardN)^n \leq 2^{CardN}.$$

Therefore, by Tarski, [4]

$$CardN + (CardN)^2 + (CardN)^3 + \ldots \leq 2^{CardN}$$

Hence,

$$(cardN)^{CardN} \leq 2^{CardN}.$$

and we conclude

$$(CardN)^{CardN} = 2^{CardN}. \square$$

Theorem for any $n \geq 2$,

$$\underline{(\,cardN\,)^{CardN} \,=\, 2^{CardN} \,=\, n^{CardN}}$$

Proof

$$n^{CardN} \le (\,cardN\,)^{CardN} = 2^{CardN} \le n^{CardN}.\square$$

Theorem $\underline{2^{CardN} \times 2^{CardN} = 2^{CardN}}$

Proof

$$2^{CardN} \times 2^{CardN} = 2^{CardN+CardN} = 2^{CardN}.\square$$

Theorem $\underline{(\,CardN\,)^{n} \uparrow 2^{CardN}}.$

Proof:

$$(\,CardN\,)^{n} \uparrow (\,CardN\,)^{CardN} \ge 2^{CardN} = (\,CardN\,)^{CardN}.\square$$

Algebraic Numbers

For algebraic numbers,

$$(\,CardN\,)^{CardN} = CardN \times CardN \times CardN \times \ldots$$

$$\le CardN + CardN \times CardN$$

$$+CardN \times CardN \times CardN + \ldots$$

$$= CardN + (CardN)^2 + (CardN)^3 + \ldots$$

$$= Card(Algebraic\ Numbers)$$

$$\leq CardR$$

$$= 2^{CardN}$$

$$= (cardN)^{CardN}$$

Hence,

$$\underline{Card(Algebraic\ Numbers) = 2^{CardN}}.\square$$

Transcendental Numbers

By [5],

If a is non-zero, real algebraic number, then e^a is a transcendental number. The mapping

$$a \to e^a$$

is an injection from the algebraic numbers into the transcendental numbers.

Therefore,

$$2^{CardN} = CardR \geq Card(Transcendental\ \ Numbers)$$

$$\geq Card(Algebraic\ \ Numbers) = 2^{CardN}.$$

Thus,

$$\underline{Card(Transcendental\ \ Numbers) = 2^{CardN}}.$$

Irrational Numbers

$$2^{CardN} = CardR \geq Card(Irrational\ \ Numbers)$$

$$\geq Card(Transcendental\ \ Numbers)$$

$$= 2^{CardN}.$$

Hence,

$$\underline{Card(Irrational\ \ Numbers) = 2^{CardN}}.$$

$2^{(CardN)\times(CardN)}$

Theorem $\underline{2^{CardN} < 2^{(CardN)^2}}$.

Proof

If not, then

$$2^{CardN} = 2^{(CardN)^2} = \left(2^{CardN}\right)^{CardN}$$

$$= \left(2^{(CardN)^2}\right)^{CardN} = 2^{(CardN)^3} = ...$$

Therefore,

$$2^{CardN} = \left(2^{CardN}\right)^{CardN}$$

$$= 2^{CardN} \times 2^{CardN} \times 2^{CardN} \times ...$$

$$= 2^{CardN} \times 2^{(CardN)^2} \times 2^{(CardN)^3} \times ...$$

$$= 2^{CardN + (CardN)^2 + (CardN)^3 + ...}$$

$$= 2^{2^{CardN}}$$

According to [2, p.152, #7], we can prove without the Axiom of Choice that

✚ *There is no cardinal number* m *so that* $2^m = 2^{2^m}$

Thus,

$$2^{2^{CardN}} \neq 2^{CardN},$$

and we conclude that

$$2^{CardN} < 2^{(CardN)^2}.\square$$

$2^{(CardN)^n}$

Theorem $\underline{2^{(CardN)^n} < 2^{(CardN)^{n+1}} \ for\ any\ n = 1,2,3,...}$

Proof

If we assume that $2^{(CardN)^n} = 2^{(CardN)^{n+1}}$, then by an argument similar to the one used for $2^{CardN} < 2^{(CardN)^2}$, we will have

$$2^{2^{(CardN)^n}} = 2^{(CardN)^n}.$$

Contradicting ✚.□

Theorem $\underline{2^{(CardN)^n} \uparrow 2^{2^{CardN}}}.$

Proof:

$$2^{(CardN)^n} \uparrow 2^{(CardN)^{CardN}} = 2^{2^{CardN}}.\square$$

11. Cantorian Theory, and Cardinals

In Cantor's theory, any set may be sequenced [1], and there is only one infinity. By the Countability, and Diagonal Axioms,

$$CardN = CardN \times CardN = 2^{CardN}.$$

By the Generalized Countability, and Diagonal Axioms,

$$\begin{aligned} CardN = (CardN)^2 = (CardN)^3 = ... \\ = 2^{CardN} = (CardN)^{CardN} \\ = 2^{(CardN)^2} = 2^{(CardN)^3} = ... \\ = 2^{(CardN)^{CardN}} = 2^{2^{CardN}} = ... \end{aligned}$$

The Algebraic Numbers have cardinality

$$\begin{aligned} CardN + (CardN)^2 + (CardN)^3 + ... \\ = CardN + CardN + CardN + ... \\ = CardN \times CardN = CardN. \end{aligned}$$

The Transcendental, and the Irrational Numbers have cardinality

$$2^{CardN} = CardN.$$

Cantor's theory attempts to prove Axioms, as if they were Theorems, and borrows from the Non-Cantorian Theory Axioms that do not hold in Cantor's Theory.

For instance, it confiscates the Non-Cantorian, Non-Diagonal Axiom

$$CardN < 2^{CardN},$$

and expects it to be compatible with the Cantorian Countability

$$CardN = (CardN)^2,$$

disregarding the Cantorian Generalized Countability, and Diagonal Axioms that guarantee

$$CardN = (CardN)^2 = ... = (CardN)^{CardN} = 2^{CardN}.$$

Cantor's Theory is obtained by augmenting the Zermelo-Fraenkel Theory with the Axiom of Choice, which is equivalent to the Well-Ordering Axiom, and to the Continuum Hypothesis.

Cantor's Theory adds as Axioms, statements that cannot be proved in the Zermelo-Fraenkel Theory.

For instance, according to [6, p. 123], Cohen proved in 1963 that

In the Zermelo-Fraenkel Set Theory, one cannot prove that the set of all real numbers can be well-ordered

This suggests that,

The set of real numbers may not be well-ordered.

But in defiance, Cantorian Theory adds the Well-Ordering Axiom.

At the end, in-spite of all the patching with added Axioms, Cantor's Theory delivers only one infinity.

12. Non-Cantorian Theory, and Cardinals

The Non-Cantorian Theory is the Zermelo-Fraenkel Theory with the No-Choice Axiom, and the equivalent Hypothesis-Negation, No-Well-Ordering, and No-Transfinite-Induction Axioms.

The Non-Cantorian Cardinality is established with the Non-Effective-Countability Axiom.

To obtain distinct infinities, we have to limit the lowest cardinality to sequences that are like the Natural Numbers, and unlike the Rationals. By the Non-Countability Axiom

$$cardN < CardN \times CardN .$$

By the Generalized Non-Countability Axiom,

$$CardN < (CardN)^2 < (CardN)^3 < (CardN)^4 < \ldots.$$

$$< (CardN\,)^{CardN}$$

$$= 2^{CardN} < 2^{(\,CardN\,)^2} < 2^{(\,CardN\,)^3} < ...$$

$$< 2^{2^{CardN}} < ...$$

The non-Cantorian cardinality of the Algebraic Numbers is

$$CardN + (CardN\,)^2 + (CardN\,)^3 + ... = 2^{CardN}.$$

The Non-Cantorian cardinalities of the Transcendental, and the Irrational Numbers are 2^{CardN}.

The first Non-Cantorian infinities in ascending order are:

$CardN$

$CardN \times CardN$

$CardN \times CardN \times CardN$

$(CardN\,)^4$

$(CardN\,)^5$

…………………………………………….

$2^{CardN} = (CardN\,)^{CardN}$

$2^{CardN \times CardN}$

$2^{(CardN)^3}$

$2^{(CardN)^4}$

$2^{(CardN)^5}$

……………………………………………..

$2^{2^{CardN}}$

$2^{2^{(CardN)^2}}$

$2^{2^{(CardN)^3}}$

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