Infinitesimal Complex Calculus

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Abstract We develop here the Infinitesimal Complex Calculus to obtain results that are beyond the reach of the Complex Calculus of Limits.

1) In the Calculus of Limits, Cauchy's Theorem that any loop integral of a Complex $f(z)$ on a Simply-Connected domain, vanishes, requires only Continuity of $f(z)$.

Then, the derivation of the Cauchy Formula requires only continuity.

And since Cauchy Formula guarantees differentiability, it follows that Continuity implies Differentiability.

But the continuous $f(z) = |z|$ is not differentiable.

Thus, the derivation of Cauchy Formula in the Calculus of Limits leads to a falsehood, and must be flawed.

In contrast, the derivation of Cauchy Formula in

Infinitesimal Complex Calculus requires Differentiability of $f(z)$, and avoids the contradiction.

2) Infinitesimal Complex Calculus supplies us with a discontinuous complex function that has a derivative.
No such result exists in the Calculus of Limits.

3) The Cauchy Integral Formula holds for Hyper-Complex Function analytic in an infinitesimal disk in the Hyper-Complex Domain. No infinitesimal disk exists in the Complex Plane, and no such result can exist in the Calculus of Limits.

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Introduction

0.1 The Cauchy Integral Formula

For z in the interior of γ, Cauchy Integral Formula gives an analytic $f(z)$ as the convolution of f with $\frac{1}{2\pi i}\frac{1}{\zeta}$.

$$f(z) = \frac{1}{2\pi i}\oint_{\gamma}\frac{f(\zeta)}{\zeta - z}d\zeta$$

$$= \oint_{\gamma} f(\zeta)\frac{1}{2\pi i}\frac{1}{\zeta - z}d\zeta$$

Thus, the Cauchy Integral Formula recovers the value of a complex function $f(\zeta)$ at the point z in the interior of a loop γ, by sifting through the values of $f(\zeta)$ on γ.

In the Calculus of Limits, the derivation of the Cauchy Integral Formula raises two difficulties:

0.2 The Problem with taking $\varepsilon \downarrow 0$

The Calculus of Limits entertains the notion that the singularity at $\zeta = z$ can be bypassed by tracing a circular path around z, even when the radius of the circle, $\zeta - z$, vanishes.

But in the Calculus of limits, $h(\zeta) = \dfrac{1}{\zeta - z}$, is defined only out of a disk of radius ε about $\zeta = z$, and a vanishing radius, requires $\varepsilon \downarrow 0$, and $\dfrac{1}{\varepsilon} \to \infty$.

To see the flaw in the Calculus of Limits evaluation of

$$\lim_{\varepsilon \to 0} \oint_{|\zeta - z| = \varepsilon} \frac{1}{\zeta - z} d\zeta,$$

put

$$\zeta - z = \varepsilon e^{i\phi}$$

$$d\zeta = i\varepsilon e^{i\phi} d\phi.$$

Then,

$$\oint_{|\zeta - z| = \varepsilon} \frac{1}{\zeta - z} d\zeta = \int_{\phi=0}^{\phi=2\pi} \frac{1}{\varepsilon e^{i\phi}} i\varepsilon e^{i\phi} d\phi$$

$$= \frac{1}{\varepsilon} \varepsilon i \int_{\phi=0}^{\phi=2\pi} d\phi$$

$$= 2\pi i \frac{1}{\varepsilon} \varepsilon.$$

Whenever $\varepsilon > 0$, we have

$$\frac{1}{\varepsilon}\varepsilon = 1, \text{ and } \oint_{|\zeta - z| = \varepsilon} \frac{1}{\zeta - z} d\zeta = 2\pi i.$$

But for $\varepsilon \downarrow 0$, we have $\lim_{\varepsilon \to 0} \varepsilon = 0$, and

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{\varepsilon}=\frac{\lim\limits_{\varepsilon\to 0}\varepsilon}{\lim\limits_{\varepsilon\to 0}\varepsilon}=\frac{0}{0},$$

which is undefined.

Therefore, $\lim\limits_{\varepsilon\downarrow 0}\oint\limits_{|\zeta|=\varepsilon}\frac{1}{\zeta}d\zeta$ is undefined.

In the Calculus of Limits,

$$\lim_{\varepsilon\to 0}\varepsilon=0,$$

and the limit process $\varepsilon\downarrow 0$, drives ε to 0, without stopping at some positive value, so that ε may be cancelled out.

On the real line, there is no such ε that can decrease to zero, and have a nonzero limit.

ε alludes to the hyper-real infinitesimals. But infinitesimals do not exist on the real line, or in the complex plane, and cannot be used in the Calculus of Limits.

Thus, <u>to derive the Cauchy Integral Formula, we need the Complex Infinitesimals</u>.

0.3 Problem of Continuity implying Differentiability

The derivation of the Cauchy Formula, uses Cauchy's Theorem by which any loop integral of a complex $f(z)$ on a simply connected domain, vanishes.

Cauchy's Theorem has a proof that seems to require only Continuity of $f(z)$ on the domain.

And the flawed proof of Cauchy Integral Formula in the Calculus of Limits, requires only continuity.

Since by Cauchy Formula, $f(z)$ is analytic, it seems that Continuity can imply Differentiability, which is impossible: The continuous $|z|$ is not differentiable.

In contrast, the proof of the Cauchy Integral Formula in Infinitesimal Complex Calculus, requires differentiability.

We develop here the Infinitesimal Hyper-Complex Calculus.

In particular, we show that the Hyper-complex step function has an infinite Hyper-Complex valued derivative at its discontinuity.

We derive the Cauchy Integral Formula for Hyper-Complex Function analytic in an infinitesimal disk in the Hyper-Complex Domain. This result cannot be obtained in the Complex Calculus of Limits.

Finally, we derive the Cauchy Integral Formula requiring the differentiability of $f(z)$ in a simply connected hyper-complex domain.

1.

Hyper-Complex Plane

The Hyper-Complex Plane is the cross product of a Hyper-real line, with a hyper-real line which elements are multiplied by $i = \sqrt{-1}$.

Each complex number $\alpha + i\beta$ can be represented by a Cauchy sequence of rational complex numbers, $\left\langle r_1 + is_1, r_2 + is_2, r_3 + is_3 ... \right\rangle$ so that $r_n + is_n \to \alpha + i\beta$.

The constant sequence $(\alpha + i\beta, \alpha + i\beta, \alpha + i\beta, ...)$ is a Constant Hyper-Complex Number.

Following [Dan2] we claim that,

1. Any set of sequences $(\iota_1 + io_1, \iota_2 + io_2, \iota_3 + io_3, ...)$, where $(\iota_1, \iota_2, \iota_3, ...)$ belongs to one family of infinitesimal hyper reals, and $(o_1, o_2, o_3, ...)$ belongs to another family of infinitesimal hyper-reals, constitutes a family of infinitesimal hyper-complex numbers.

2. Each hyper-complex infinitesimal has a polar representation $dz = (dr)e^{i\phi} = o_* e^{i\phi}$, where $dr = o_*$ is an infinitesimal, and $\phi = \arg(dz)$.
3. The infinitesimal hyper-complex numbers are smaller in length, than any complex number, yet strictly greater than zero.
4. Their reciprocals $\left(\frac{1}{\iota_1 + io_1}, \frac{1}{\iota_2 + io_2}, \frac{1}{\iota_3 + io_3}, ...\right)$ are the infinite hyper-complex numbers.
5. The infinite hyper-complex numbers are greater in length than any complex number, yet strictly smaller than infinity.
6. The sum of a complex number with an infinitesimal hyper-complex is a non-constant hyper-complex.
7. The Hyper-Complex Numbers are the totality of constant hyper-complex numbers, a family of hyper-complex infinitesimals, a family of infinite hyper-complex, and non-constant hyper-complex.
8. The <u>Hyper-Complex Plane</u> is the direct product of a Hyper-Real Line by an imaginary Hyper-Real Line.

9. In Cartesian Coordinates, the Hyper-Real Line serves as an x coordinate line, and the imaginary as an iy coordinate line.
10. In Polar Coordinates, the Hyper-Real Line serves as a Range r line, and the imaginary as an $i\theta$ coordinate. Radial symmetry leads to Polar Coordinates.
11. The Hyper-Complex Plane includes the complex numbers separated by the non-constant hyper-complex numbers. Each complex number is the center of a disk of hyper-complex numbers, that includes no other complex number.
12. In particular, zero is separated from any complex number by a disk of complex infinitesimals.
13. Zero is not a complex infinitesimal, because the length of zero is not strictly greater than zero.
14. We do not add infinity to the hyper-complex plane.
15. The hyper-complex plane is embedded in $\mathbb{C}^{\infty}$, and is not homeomorphic to the Complex Plane $\mathbb{C}$. There is

no bi-continuous one-one mapping from the hyper-complex Plane onto the Complex Plane.

16. In particular, there are no points in the Complex Plane that can be assigned uniquely to the hyper-complex infinitesimals, or to the infinite hyper-complex numbers, or to the non-constant hyper-complex numbers.

17. No neighbourhood of a hyper-complex number is homeomorphic to a $\mathbb{C}^n$ ball. Therefore, the Hyper-Complex Plane is not a manifold.

18. The Hyper-Complex Plane is not spanned by two elements, and is not two-dimensional.

2.

Hyper-Complex Function

2.1 Definition of a hyper-complex function

$f(z)$ is a hyper-complex function, iff it is from the hyper-complex numbers into the hyper-complex numbers.

This means that any number in the domain, or in the range of a hyper-complex $f(x)$ is either one of the following

complex

complex + infinitesimal

infinitesimal

infinite hyper-complex

2.2 *Every function from complex numbers into complex numbers is a hyper-complex function.*

2.3 $\dfrac{\sin(dz)}{dz}$ *has the constant hyper-complex value* 1

Proof: $\sin(dz) = dz - \frac{(dz)^3}{3!} + \frac{(dz)^5}{5!} - ...$

$$\frac{\sin(dz)}{dz} = 1 - \frac{(dz)^2}{3!} + \frac{(dz)^4}{5!} - ...$$

2.4 $\cos(dz)$ ***has the constant hyper-complex value*** 1

Proof: $\cos(dz) = 1 - \frac{(dz)^2}{2!} + \frac{(dz)^4}{4!} - ...$

2.5 e^{dz} ***has the constant hyper-complex value*** 1

Proof: $e^{dz} = 1 + dz + \frac{(dz)^2}{2!} + \frac{(dz)^3}{3!} + \frac{(dz)^4}{4!} + ...$

2.6 $e^{\frac{1}{dz}}$ **is an infinite hyper-complex, and** $\left|e^{\frac{1}{dz}}\right| = e^{\frac{1}{dr}\cos\phi}$.

Proof: $\left|e^{\frac{1}{dz}}\right| = e^{\frac{1}{dr}\mathrm{Re}[e^{-i\phi}]} = e^{\frac{1}{dr}\cos\phi}$.

2.7 $\log(dz)$ **is an infinite hyper-complex, and** $\left|\log(dz)\right| > \frac{1}{dr}$

Proof: $\left|\log(dz)\right| = \sqrt{[\log(dr)]^2 + \phi^2} > \log(dr) > \frac{1}{dr}$

3.

Hyper-Complex Continuity

3.1 Hyper-Complex Continuity Definition

$f(z)$ *is continuous at* z_0 <u>iff</u> *for any* $dz = (dr)e^{i\theta}$,

$$f(z_0 + (dr)e^{i\theta}) - f(z_0) = \text{infinitesimal}.$$

3.2 $f(z) = z^2$ *is Continuous at* $z = 1$

Proof: $f(1 + (dr)e^{i\theta}) - f(1) = (1 + (dr)e^{i\theta})^2 - 1^2$

$$= 2(dr)e^{i\theta} + (dr)^2 e^{2i\theta}$$

$$= \text{infinitesimal}.\square$$

3.3 $h(z) = \begin{cases} 0, & |z| \le 1 \\ 1, & |z| > 1 \end{cases}$ *is discontinuous on* $z = e^{i\phi}$.

Proof: $h(e^{i\phi} + (dr)e^{i\theta}) - h(e^{i\phi}) = 1 - 0.\square$

3.4 $f(z) = |z|$ *is continuous at any* z_0

Proof: $\left|z_0 + (dr)e^{i\theta}\right| - \left|z_0\right| \leq \left|(dr)e^{i\theta}\right| = dr.\square$

3.5 $g(\overline{z}) = \overline{z}$ ***is discontinuous at any*** z_0

Proof: $\overline{z_0 + (dr)e^{i\theta}} - z_0 = \overline{z}_0 - z_0 + (dr)e^{-i\theta}$

$$\neq \text{infinitesimal}.\square$$

4.

$\frac{1}{z}$, $z \neq 0$

In the Calculus of Limits, the function

$$f(z) = \frac{1}{z} \quad \text{is defined for all } z \neq 0.$$

We avoid $z = 0$, because the oscillation of $f(z) = \frac{1}{z}$ over a disk that includes $z = 0$, is infinite.

However,

$$z \to 0 \Rightarrow \frac{1}{z} \to \infty.$$

Therefore, $f(z) = \frac{1}{z}$ has to avoid a <u>disk</u> of radius ε, that includes $z = 0$. Namely,

4.1 *In the Calculus of limits,* $f(z) = \frac{1}{z}$ *, is defined only out of a disk of radius* ε *about* $z = 0$.

In Infinitesimal Calculus, if $dz = \left\langle \frac{1}{n} \right\rangle$, then $\frac{1}{dz} = \left\langle n \right\rangle < \infty$, and we have,

4.2 *In Infinitesimal Calculus, the Hyper-Complex function*

$f(z) = \frac{1}{z}$, *is defined, for any* $z \neq 0$.

4.3 $f(z) = \frac{1}{z}$ *is discontinuous at* $(d\rho)e^{i\phi}$.

because

$$\left| f((d\rho)e^{i\phi} + (dr)e^{i\theta}) - f((d\rho)e^{i\phi}) \right| =$$

$$= \left| \frac{1}{(d\rho)e^{i\phi} + (dr)e^{i\theta}} - \frac{1}{(d\rho)e^{i\phi}} \right|$$

$$= \left| \frac{(dr)e^{i\theta}}{(d\rho)^2 e^{2i\phi} + (d\rho)(dr)e^{i(\theta+\phi)}} \right|$$

$$= \frac{dr}{\left| (d\rho)^2 e^{i\phi} + (d\rho)(dr)e^{i\theta} \right|}$$

$$\sim \frac{1}{d\rho}. \square$$

5.

$\mathrm{Log}(z)$, $z \neq 0$

In the Calculus of Limits, the function

$$f(z) = \mathrm{Log}(z) = \log\left|z\right| + i\theta \quad \text{is defined for all } z \neq 0.$$

We avoid $z = 0$, because the oscillation of $\log\left|z\right|$ over a disk that includes $z = 0$, is infinite.

However, for $\varepsilon > 0$,

$$-\frac{1}{2}\log\varepsilon = \frac{1-\varepsilon}{1+\varepsilon} + \frac{1}{3}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^3 + \frac{1}{5}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^5 + ...$$

To first order $\dfrac{1}{1+\varepsilon} \approx 1-\varepsilon$, and we have,

$$-\frac{1}{2}\log\varepsilon \approx 1 - 2\varepsilon + \left(\frac{1}{3} - \varepsilon\right) + \left(\frac{1}{5} - \varepsilon\right) + ...$$

Therefore,

$$\varepsilon \to 0 \Rightarrow \log\varepsilon \to -\infty.$$

Consequently, the domain of $f(z) = \mathrm{Log}(z)$ has to avoid a disk of radius ε about $z = 0$. Namely,

5.1 *In the Calculus of limits,* $f(z) = \mathrm{Log}(z)$, *is defined only out of a disk of radius* ε *about* $z = 0$.

In Infinitesimal Calculus, if $dz = \left\langle \frac{1}{n} \right\rangle$, then

$$\log(dz) = \left\langle \log \tfrac{1}{n} \right\rangle = \left\langle -\log n \right\rangle > \left\langle -n \right\rangle > -\infty$$

Consequently, we have,

5.2 *In Infinitesimal Calculus, the Hyper-Complex function* $f(z) = \mathrm{Log}(z)$*, is defined for any* $z \neq 0$.

6.

Complex Derivative

6.1 Complex Derivative Definition

$f(z)$ defined at z_0, has a Complex Derivative at z_0, $f'(z_0)$, iff for any complex infinitesimal dz,

$$\frac{f(z_0 + dz) - f(z_0)}{dz}$$

equals a unique hyper-complex number.

If that number is an infinite hyper-complex number, then it is the complex derivative $f'(z_0)$.

If that number is a finite Non-Constant Hyper-complex, then it is the sum of a constant hyper-complex and a complex infinitesimal. Then, the constant Hyper-Complex part is the Complex Derivative $f'(z_0)$.

6.2 Derivative of $f(z) = z^3$ at $z = 1$

For any dz,

$$\frac{(1 + dz)^3 - (1)^3}{dz} = 3 + 3dz + (dz)^2.$$

Therefore, $f(z) = z^3$ has derivative $f'(1) = 3$.□

6.3 $f(z) = |z|$ *has no derivative at* $z = 0$

For $dz = (dr)e^{2\pi i} = dr$,

$$\frac{f(dz) - f(0)}{dz} = \frac{|dz| - |0|}{dr} = \frac{dr}{dr} = 1.$$

For $dz = (dr)e^{\pi i} = -dr$,

$$\frac{f(dz) - f(0)}{dz} = \frac{|dz| - |0|}{-dr} = \frac{dr}{-dr} = -1.$$

Thus, the derivative of $f(z) = |z|$ at $z = 0$, does not exist.□

6.4 $g(\overline{z}) = \overline{z}$ *has no derivative with respect to* z *at any* z_0

Proof: $$\frac{d\overline{z}}{dz} = a \Rightarrow \frac{dx - idy}{dx + idy} = a \Rightarrow \begin{matrix} dx = adx \\ -idy = iady \end{matrix} \Rightarrow \begin{matrix} a = 1 \\ a = -1 \end{matrix}$$

7.

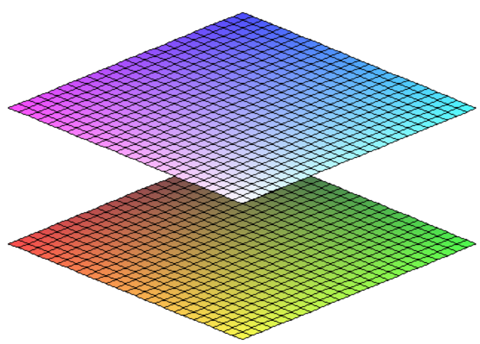
Step Functions

7.1 the Step-Up Function Definition

we define $h(z) = \begin{cases} 0, & |z| = 0 \\ 1, & |z| > 0 \end{cases}$.

$$\text{plot3d}\left(\left[0, \begin{cases} 0 & \sqrt{x^2+y^2} = 0 \\ 1 & \sqrt{x^2+y^2} > 0 \end{cases}\right], x = -2..2, y = -2..2\right)$$

gives its plot on the plane $Z = 0$ in Maple.



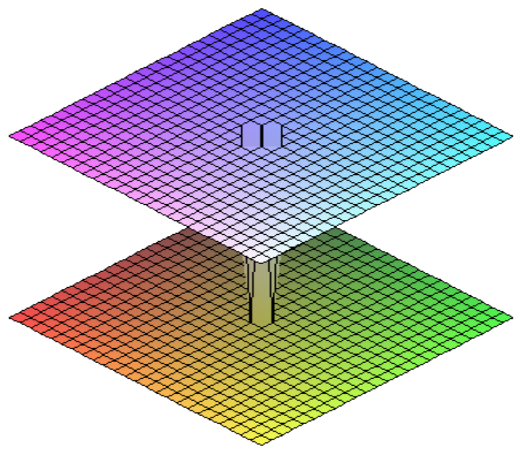
7.2 the Step-Down function definition

We define the step-down function as $\begin{cases} 1, & |z| = 0 \\ 0, & |z| > 0 \end{cases}$,

7.3 *The Step Function is discontinuous at* $z = 0$

The discontinuity jump of the step-up function, is seen with

$$\text{plot3d}\left(\left[0, \begin{cases} 0 & \sqrt{x^2+y^2} < 0.2 \\ 1 & \sqrt{x^2+y^2} > 0.2 \end{cases}\right], x=-2..2, y=-2..2\right)$$



7.4 $$\frac{d}{dz}h(z) = \frac{1}{dz}\chi_{\{|z| \le |dz|\}}(z) = \begin{cases} \frac{1}{dz}, |z| \le |dz| \\ 0, otherwise \end{cases}$$

Proof: For any dz, $\dfrac{h(dz) - h(0)}{dz} = \dfrac{1-0}{dz} = \dfrac{1}{dz}$.□

7.5 *The step-up function is differentiable at its discontinuity at* $z = 0$. *Its derivative is the infinite hyper-complex* $\frac{1}{dz}$.

8.

Cauchy-Riemann Equations

8.1 *If* $f(z) = u(x,y) + iv(x,y)$ *has derivative at* $z_0 = x_0 + iy_0$

Then, $\boxed{\begin{matrix} u,_x = v,_y \\ u,_y = -v,_x \end{matrix}}$ *at* (x_0, y_0)

Proof: $\partial_x \left(u + iv\right) = \partial_{iy}(u + iv) \Rightarrow \begin{cases} \partial_x u = \partial_y v \\ \partial_y u = -\partial_x v \end{cases}$

8.2 $f(z) = z$ *satisfies Cauchy-Riemann equations at any* z

Proof: $z = x + iy \;\Rightarrow\; u = x\,,\; v = y$

$$\begin{matrix} \partial_x u = 1 = \partial_y v \\ \partial_y u = 0 = -\partial_x v \end{matrix} \Rightarrow \text{Cauchy Riemann equations hold.}$$

8.3 $f(z) = \left|z\right|$ *has no derivative with respect to* z *at any* z_0

Proof: By 6.3, $f(z) = \left|z\right|$ has no derivative at $z = 0$,

At $z \neq 0$, $\left|z\right| = \sqrt{x^2 + y^2} \;\Rightarrow\; u = \sqrt{x^2 + y^2}\,,\; v = 0$

$$\partial_x u = \frac{x}{\sqrt{x^2 + y^2}} \qquad \partial_y v = 0$$
$$\partial_y u = \frac{y}{\sqrt{x^2 + y^2}} \qquad \partial_x v = 0$$

$\Rightarrow$ Cauchy Riemann equations do not hold, at $z \neq 0$ and by 8.1 there is no derivative.□

8.4 $g(\overline{z}) = \overline{z}$ *has no derivative with respect to* z *at any* z_0

Proof: $\overline{z} = x - iy \;\Rightarrow\; u = x\,,\; v = -y$

$\partial_x u = 1 \neq -1 = \partial_y v \;\Rightarrow$ Cauchy Riemann equations do not hold, and by 8.1 there is no derivative

8.5 $h(z) = \begin{cases} 0, & |z| = 0 \\ 1, & |z| > 0 \end{cases}$ *satisfies the Cauchy Riemann equations at any* z.

Proof: $u = \begin{cases} 0, & r = 0 \\ 1, & r > 0 \end{cases}, \quad v = 0\,.$

9.

Hyper-Complex Path Integral

Following the definition of the Hyper-real Integral in [Dan3], the Hyper-Complex Integral of $f(z)$ over a path $z(t)$, $t \in [\alpha, \beta]$, in its domain, is the sum of the areas $f(z)z'(t)dt$ of the rectangles with base $z'(t)dt$, and height $f(z)$.

9.1 Hyper-Complex Path Integral Definition

Let $f(z)$ be hyper-complex function, defined on a domain in the Hyper-Complex Plane. The domain may not be bounded.

$f(z)$ may take infinite hyper-complex values, and need not be bounded.

Let $z(t)$, $t \in [\alpha, \beta]$, be a path, $\gamma(a, b)$, so that $dz = z'(t)dt$, and $z'(t)$ is continuous.

For each t, there is a hyper-complex rectangle with base $[z(t) - \frac{dz}{2}, z(t) + \frac{dz}{2}]$, height $f(z)$, and area $f(z(t))z'(t)dt$.

We form the **Integration Sum** of all the areas that start at $z(\alpha) = a$, and end at $z(\beta) = b$,

$$\sum_{t\in[\alpha,\beta]} f(z(t))z'(t)dt.$$

If for any infinitesimal $dz = z'(t)dt$, the Integration Sum equals the same hyper-complex number, then $f(z)$ is Hyper-Complex Integrable over the path $\gamma(a,b)$.

Then, we call the Integration Sum the Hyper-Complex Integral of $f(z)$ over the $\gamma(a,b)$, and denote it by $\int\limits_{\gamma(a,b)} f(z)dz$.

If the hyper-complex number is an infinite hyper-complex, then it equals $\int\limits_{\gamma(a,b)} f(z)dz$.

If the hyper-complex number is finite, then its constant part equals $\int\limits_{\gamma(a,b)} f(z)dz$.□

The Integration Sum may take infinite hyper-complex values, such as $\frac{1}{dz}$, but may not equal to ∞.

The Hyper-Complex Integral of the function $f(z) = \dfrac{1}{|z|}$ over a path that goes through $z = 0$ diverges.

9.2 The Countability of the Integration Sum

In [Dan1], we established the equality of all positive infinities:

We proved that the number of the Natural Numbers, $Card\mathbb{N}$, equals the number of Real Numbers, $Card\mathbb{R} = 2^{Card\mathbb{N}}$, and we have

$$Card\mathbb{N} = (Card\mathbb{N})^2 = ... = 2^{Card\mathbb{N}} = 2^{2^{Card\mathbb{N}}} = ... \equiv \infty.$$

In particular, we demonstrated that the real numbers may be well-ordered.

Consequently, there are countably many real numbers in the interval $[\alpha, \beta]$, and the Integration Sum has countably many terms.

While we do not sequence the real numbers in the interval, the summation takes place over countably many $f(z)dz$.

9.3 Continuous $f(z)$ is Path-Integrable

Hyper-Complex $f(z)$ *Continuous on* D *is Path-Integrable on* D

Proof:

Let $z(t)$, $t \in [\alpha, \beta]$, be a path, $\gamma(a, b)$, so that $dz = z'(t)dt$, and $z'(t)$ is continuous. Then,

$$f(z(t))z'(t) = \left(u(x(t),y(t)) + iv(x(t),y(t))\right)\left(x'(t) + iy'(t)\right)$$

$$= \underbrace{\left[u(x(t),y(t))x'(t) - v(x(t),y(t))y'(t)\right]}_{U(t)} +$$

$$+i\underbrace{\left[u(x(t),y(t))y'(t) + v(x(t),y(t))x'(t)\right]}_{V(t)}$$

$$= U(t) + iV(t),$$

where $U(t)$, and $V(t)$ are Hyper-Real Continuous on $[\alpha,\beta]$.

Therefore, by [Dan3, 12.4], $U(t)$, and $V(t)$ are integrable on $[\alpha,\beta]$.

Hence, $f(z(t))z'(t)$ is integrable on $[\alpha,\beta]$.

Since

$$\int_{t=\alpha}^{t=\beta} f(z(t))z'(t)dt = \int_{\gamma(a,b)} f(z)dz,$$

$f(z)$ is Path-Integrable on $\gamma(a,b)$. $\square$

10.

The Fundamental Theorem of Path Integration

The Fundamental Theorem of Path Integration guarantees that Integration and Differentiation are well defined inverse operations, that when applied consecutively yield the original function.

The Fundamental Theorem requires Hyper-Complex Integrability of the Hyper-Complex Function.

10.1 The Fundamental Theorem

Let $f(z(t))$ *be Hyper-Complex Integrable on* $\gamma[a,b]$

Then, *for any* $z \in \gamma[a,b]$,

$$\frac{d}{dz(t)} \int_{u(\tau_0)=a(\alpha)}^{u(\tau)=z(t)} f(u(\tau))du(\tau) = f(z(t))$$

Proof:

$$\frac{d}{dz(t)} \int_{u(\tau_0)=a(\alpha)}^{u(\tau)=z(t)} f(u(\tau))du(\tau) =$$

$$= \frac{\int\limits_{\zeta(\tau)=a(\alpha)}^{\zeta(\tau)=z(t)+\frac{1}{2}dz(t)} f(\zeta(\tau))d\zeta(\tau) - \int\limits_{\zeta(\tau)=a(\alpha)}^{\zeta(\tau)=z(t)-\frac{1}{2}dz(t)} f(\zeta(\tau))d\zeta(\tau)}{dz(t)}$$

$$= \frac{\sum\limits_{\tau\in[\alpha,t+\frac{dt}{2}]} f(\zeta(\tau))\zeta'(\tau)d\tau - \sum\limits_{\tau\in[\alpha,t-\frac{dt}{2}]} f(\zeta(\tau))\zeta'(\tau)d\tau}{z'(t)dt}$$

$$= \frac{f(z(t))z'(t)dt}{z'(t)dt}$$

$$= f(z(t)).\square$$

11.

Path Independence, and Loop Integrals

The Fundamental Theorem of Path Integration implies Path Independence. we have,

11.1 *If the Hyper-Complex* $f(z)$ *is Path-Integrable on a Hyper-Complex Domain.*

Then, $\int\limits_{\gamma(a,b)} f(z)dz$ *is Path-independent*

Proof:

By 10.1, the Principal Value Derivative of $\int\limits_{\gamma(a,z)} f(\zeta)d\zeta$ with respect to z is $f(z)$, for any path $\gamma(a,z)$.

Therefore, $\int\limits_{\gamma(a,b)} f(z)dz$ does not depend on the path $\gamma(a,b)$.

Only on the endpoints, a, and b.$\square$

Path independence is equivalent to the vanishing of the Circulation of $f(z)$.

11.2 *Let the Hyper-Complex* $f(z)$ *be defined on a Hyper-Complex Domain. Then the following are equivalent*

A. $\int\limits_{\gamma(a,b)} f(z)dz$ *is Path-independent*

B. *For any loop* γ *with interior in the domain*,

$$\oint\limits_{\gamma} f(z)dz = 0.$$

12.

Cauchy Integral Theorem

By Cauchy Integral Theorem any loop integral of a Differentiable $f(z)$ on a Simply-Connected domain, vanishes.

It seems that a Continuous $f(z)$ on its Domain may suffice.

The argument is as follows

By 9.3, The Continuity of $f(z)$ with respect to z, on the Domain D, guarantees that $f(z)$ is Path-Integrable on D.

By 11.1, for any path $\gamma(a,b)$ in D,

$$\int_{\gamma(a,b)} f(z)dz \text{ is Path-independent.}$$

By 11.2,

For any loop γ with interior in the domain, $\oint_{\gamma} f(z)dz = 0$.

However, Cauchy Integral Theorem leads to the Cauchy Integral Formula for $f(z)$, and to the conclusion that $f(z)$ is differentiable. But the continuous function $f(z) = |z|$ is not differentiable.

Consequently, the Cauchy Integral Theorem requires differentiability of $f(z)$, and we present a proof that requires differentiability:

12.1 Cauchy Integral Theorem

If the Hyper-Complex $f(z)$ *is Differentiable on a Hyper-Complex Simply Connected Domain* D

Then, for any loop γ *with interior in the domain*,

$$\oint_{\gamma} f(z)dz = 0.$$

Proof:

$$\oint_{\gamma} f(z)dz = \oint_{\gamma} \left(u + iv\right)(dx + idy)$$

$$= \oint_{\gamma} udx - vdy + i\oint_{\gamma} vdx + udy$$

Simple-Connectedness allows the use of Green's Theorem,

$$= \iint_{\operatorname{int}\gamma} \underbrace{\begin{vmatrix} \partial_x & \partial_y \\ u & -v \end{vmatrix}}_{-(u_y + v_x)} dxdy + i\iint_{\operatorname{int}\gamma} \underbrace{\begin{vmatrix} \partial_x & \partial_y \\ v & u \end{vmatrix}}_{u_x - v_y} dxdy,$$

which vanishes by Cauchy Riemann equations.□

13.

Cauchy Integral Formula in an Infinitesimal Disk

13.1 $$\oint_{|\zeta - z_0| = dr} \frac{1}{\zeta - z_0} d\zeta = 2\pi i .$$

Proof: Put

$$\zeta - z_0 = (dr)e^{i\phi}$$

$$d\zeta = i(dr)e^{i\phi} d\phi .$$

Then,

$$\oint_{|\zeta - z_0| = dr} \frac{1}{\zeta - z_0} d\zeta = \int_{\phi=0}^{\phi=2\pi} \frac{1}{(dr)e^{i\phi}} i(dr)e^{i\phi} d\phi = i \int_{\phi=0}^{\phi=2\pi} d\phi = 2\pi i$$

because $(dr)e^{i\phi} \neq 0$, for any infinitesimal dr, and any ϕ.□

The precision of 13.1, enables us to obtain the Cauchy Integral Formula in an infinitesimal disk: A result that cannot be obtained in the Complex calculus of Limits.

13.2 Cauchy Integral Formula in $\left|\zeta - z_0\right| \le dr$

If $f(z)$ *is Hyper-Complex function Differentiable at* $z = z_0$

Then, $$f(z_0) = \frac{1}{2\pi i} \oint_{\left|\zeta - z_0\right| = dr} \frac{f(\zeta)}{\zeta - z_0} d\zeta,$$

Proof:

Since f is differentiable at z_0, then, on the circle $\zeta - z_0 = (dr)e^{i\theta}$,

$$f(z_0 + (dr)e^{i\phi}) = f(z_0) + f'(z_0)(dr)e^{i\phi},$$

Therefore,

$$\oint_{\left|\zeta - z_0\right| = dr} \frac{f(\zeta)}{\zeta - z_0} d\zeta = \oint_{\left|\zeta - z_0\right| = dr} \frac{f(z_0) + f'(z_0)(dr)e^{i\phi}}{\zeta - z_0} d\zeta,$$

$$= f(z_0) \underbrace{\oint_{\left|\zeta - z_0\right| = dr} \frac{1}{\zeta - z_0} d\zeta}_{2\pi i} + f'(z_0)(dr) \oint_{\left|\zeta - z_0\right| = dr} \frac{e^{i\phi}}{\zeta - z_0} d\zeta$$

Substitute

$$\zeta - z_0 = (dr)e^{i\phi}$$

$$d\zeta = i(dr)e^{i\phi} d\phi.$$

Then,

$$= 2\pi i f(z_0) + f'(z_0)(dr) \int_{\phi=0}^{\phi=2\pi} e^{i\phi} \frac{1}{(dr)e^{i\phi}} i(dr)e^{i\phi} d\phi$$

$$= 2\pi i f(z_0) + f'(z_0)(dr)\underbrace{\int_{\phi=0}^{\phi=2\pi} e^{i\phi}d\phi}_{=0}$$

$$= 2\pi i f(z_0). \square$$

Since the Formula can be differentiated at $z = z_0$ with respect to z, to any order, we conclude

13.2 *a Hyper-Complex function, Differentiable at* $z = z_0$*, is differentiable to any order at* $z = z_0$.

14.

Cauchy Integral Formula

14.1 Cauchy Integral Formula

If $f(z)$ *is Hyper-Complex Differentiable function on a Hyper-Complex Simply-Connected Domain* D.

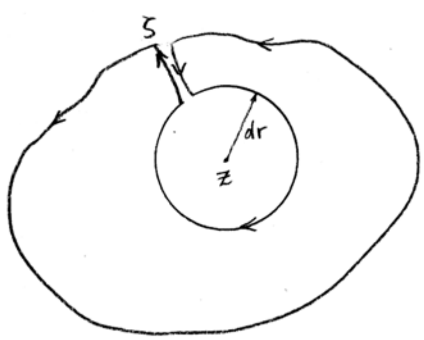
Then,

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta \,,$$

for any loop γ, *and any point* z *in its interior.*

Proof:

The Hyper-Complex function $\dfrac{f(\zeta)}{\zeta - z}$ is Differentiable on the Hyper-Complex Simply-Connected domain D, and on a path that includes γ and an infinitesimal circle about z.



Then, the integral over the infinitesimal circle has a an opposite sign because its direction is opposite to the direction on γ.

By Cauchy Integral Theorem, we have

$$\oint_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta - \underbrace{\oint_{|\zeta - z| = dr} \frac{f(\zeta)}{\zeta - z} d\zeta}_{2\pi i f(z)} = 0 . \square$$

Since the Formula can be differentiated with respect to z,to any order, we conclude

14.2 *A Hyper-Complex* $f(z)$ *Differentiable on a Hyper-Complex Simply-Connected Domain is differentiable to any order.*

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