Cardinality, Measure, and Category

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Abstract Lebesgue procedure to find the measure of a general set leads to contradictions. In particular, the set of rational numbers does not have measure zero. In fact, by Lebesgue own criteria, the set of rational numbers in $[0,1]$ is not measurable.

Introduction

Lebesgue defined the measure of an interval to be its length. He defined the measure of the union of infinitely many disjoint intervals (a_i, b_i) in $[0,1]$ to be the sum of the intervals' lengths

$$\mathrm{m}[(a_1,b_1)\cup(a_2,b_2)\cup(a_3,b_3)\cup\ldots] = (b_1 - a_1) + (b_2 - a_2) + (b_3 - a_3) + \ldots.$$

For a general set E in the interval $[0,1]$, he wrote [1, p. 182]

Cover E *by finitely many, or countably many intervals of lengths* $l_1, l_2, \ldots$

We wish to have

$$m(E) \leq l_1 + l_2 + \ldots$$

Then,

$$\inf_{\text{all covers of E}} \{l_1 + l_2 + \ldots\}$$

is an upper bound of $m(E)$*, that we denote* $\overline{m(E)}$*,*

and we have

$$m(E) \leq \overline{m(E)}.$$

Similarly, we have

$$m(E^c) \leq \overline{m(E^c)}.$$

We want to have

$$m(E) + m(E^c) = m([0,1]) = 1.$$

Hence, we must have

$$m(E) = 1 - m(E^c)$$

$$\geq 1 - \overline{m(E^c)}$$

In all we need to have

$$1 - \overline{m(E^c)} \leq m(E) \leq \overline{m(E)}$$

When

$$1-\overline{m(E^c)}=\overline{m(E)},$$

then $m(E)$ *is defined, and we say that* E *is measurable.*

Lebesgue applied his procedure to determine the measure of the set of the rational numbers in the interval $[0,1]$ [2, p.35].

He sequenced the rationals

$$\{r_1, r_2, r_3, \ldots\}$$

and covered them by the intervals

$$(r_1-\tfrac{1}{4}\varepsilon, r_1+\tfrac{1}{4}\varepsilon), (r_2-\tfrac{1}{8}\varepsilon, r_2+\tfrac{1}{8}\varepsilon), (r_3-\tfrac{1}{16}\varepsilon, r_3+\tfrac{1}{16}\varepsilon)\ldots$$

of lengths

$$\frac{1}{2}\varepsilon, \frac{1}{2^2}\varepsilon, \frac{1}{2^3}\varepsilon, \ldots$$

Then,

$$\overline{m(E)}\leq\frac{1}{2}\varepsilon+\frac{1}{2^2}\varepsilon+\frac{1}{2^3}\varepsilon+\ldots=\varepsilon.$$

Taking the infimum on $\varepsilon>0$, he effectively set ε to zero, and concluded that $\overline{m(E)}=0.\square$

We aim to show that

- Lebesgue procedure to find the measure of a general set leads to contradictions. In particular, the set of rational numbers does not have measure zero.
- By Lebesgue's own criteria, the set of the rational numbers in $[0,1]$ is not measurable.

We conclude with Lebesgue integration, and Category.

1. First Critique of the Lebesgue procedure

If ε is set to zero, the summation

$$l_1 + l_2 + \ldots$$

is over infinitely many degenerate intervals of length zero, and the sum of their lengths is of the form

$$\frac{1}{2}\varepsilon + \frac{1}{2^2}\varepsilon + \frac{1}{2^3}\varepsilon + \ldots$$

$$= 0 + 0 + 0 + \ldots$$

$$= 0 \bullet \infty,$$

and no one knows what $0 \bullet \infty$ means.

$0 \bullet \infty$ may equal any number a:

$$\underbrace{\frac{a}{n} \times n}_{=a \to a} \to 0 \times \infty$$

In particular, $0 \bullet \infty$ may equal ∞:

$$\underbrace{\frac{1}{n} \times n^2}_{=n \to \infty} \to 0 \times \infty$$

Or, it may equal 0:

$$\underbrace{\frac{1}{n^2} \times n}_{=\frac{1}{n} \to 0} \to 0 \times \infty$$

Lebesgue perceived a countable set in terms of the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...$$

that has in $[0,1]$ a simple distribution compared with the intricate distribution of the rational numbers.

Then, the complement of the sequence in $[0,1]$

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}^c = (\frac{1}{2}, 1) \cup (\frac{1}{3}, \frac{1}{2}) \cup (\frac{1}{4}, \frac{1}{3}) \cup ....$$

has the length

$$(1-\tfrac{1}{2})+(\tfrac{1}{2}-\tfrac{1}{3})+(\tfrac{1}{3}-\tfrac{1}{4})+...=1.$$

Therefore, $\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},....\right\}$ is measurable, and its measure is

$$m\left(\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},....\right\}\right)=m[0,1]-m\left[(\frac{1}{2},1)\bigcup(\frac{1}{3},\frac{1}{2})\bigcup(\frac{1}{4},\frac{1}{3})\cup....\right]=0.$$

The effect of the set-elements' distribution in $[0,1]$, in determining the measure of a set, is evident in the construction of the Cantor set [3].

The Cantor set has the same cardinality as $[0,1]$, but it is constructed in such a way that its complement

$$(\frac{1}{3},\frac{2}{3})\cup(\frac{1}{3^2},\frac{2}{3^2})\cup(\frac{7}{3^2},\frac{8}{3^2})\cup\ldots$$

has length 1.

Therefore, the Cantor set is measurable, and its measure is

$$m(CantorSet)=m[0,1]-m\left[(\frac{1}{3},\frac{2}{3})\cup(\frac{1}{3^2},\frac{2}{3^2})\cup(\frac{7}{3^2},\frac{8}{3^2})\cup\ldots\right]=0.$$

2. Second Critique of the Lebesgue procedure

There are no rational-only intervals, or irrationals-only intervals. In any interval with irrational endpoints, there are infinitely many rational numbers, and in any interval with rational endpoints, there are infinitely many irrational numbers.

The sequencing of the rationals does not alter their dense distribution in the irrationals. We can sequence the rationals, but we cannot squeeze them into any subinterval of $[0,1]$. Not even into a subinterval of size $1-\delta$, for any $\delta > 0$. Similarly, the irrationals are dense in the rationals.

The cardinality of the rationals and irrationals is irrelevant to the density of each set in the other.

Recall Lebesgue's cover of the rationals in $[0,1]$

$$(r_1 - \tfrac{1}{4}\varepsilon, r_1 + \tfrac{1}{4}\varepsilon), (r_2 - \tfrac{1}{8}\varepsilon, r_2 + \tfrac{1}{8}\varepsilon), (r_3 - \tfrac{1}{16}\varepsilon, r_3 + \tfrac{1}{16}\varepsilon) \ldots$$

with length tailored to be $< \varepsilon$.

Its complement in $[0,1]$ is a union of intervals with length $> 1-\varepsilon$.

And according to Lebesgue, there are no rational numbers in those non-degenerate intervals…

Can there be a non-degenerate interval void of rational numbers?

Lebesgue's claim to be able to keep rationals out of infinitely many intervals in $[0,1]$ is not credible.

There is no open cover of the rationals in $[0,1]$ of length $\varepsilon < 1$ that contains all the rational numbers in $[0,1]$.

Thus, the Lebesgue procedure to extend the definition of measure to a general set is based on an impossibility, and it is invalid.

Perhaps, the concept of length in $[0,1]$ can not be extended to sets more general than the union of disjoint open intervals and a real sequence, or a Cantor-like set.

3. Third Critique of the Lebesgue procedure

Actually, Lebesgue's procedure ignores his own characterization of a measurable set. We quote him from [4, p.1051]

"A set E is measurable if and only if

for as small as we wish $\varepsilon > 0$, E *has a cover by* $\alpha(\varepsilon)$ *open intervals, and* E^c *has a cover by* $\beta(\varepsilon)$ *open intervals so that the sum of the lengths of the intervals of intersection of the covers is* $< \varepsilon$*"*

Clearly, this characterization has in mind the simple structures of a real sequence, or a Cantor-like set, where the complement E^c is the union of disjoint open intervals. Then, the open covers may be refined so that their common intersection shrinks and is $< \varepsilon$.

But rational numbers cannot be separated from each other by open intervals of irrational numbers.

The density of the rationals in $[0,1]$ guarantees their presence in any subinterval of any interval in a cover of the irrationals in $[0,1]$.

Therefore, given any $\varepsilon > 0$, there are no refined open covers, so that the sum of the lengths of the intervals that belong to the intersection of the covers is $< \varepsilon$.

That is, by Lebesgue's characterization, both the rationals and the irrationals in $[0,1]$ are non-measurable.

This characterization strengthens the impression that the most general measurable set that is characterized by the Lebesgue criteria, is a union of disjoint open intervals and a real sequence or a Cantor-like set.

4. Integration

Since the set of rational numbers, and the set of irrational numbers in $[0,1]$, are non-measurable, the characteristic functions

$$\chi_{\{Q\bigcap[0,1]\}},$$

and

$$\chi_{\{\{\text{irrationals}\}\subseteq[0,1]\}}$$

are non-measurable, and their Lebesgue integrals do not exist.

Since many general sets may be unmeasurable, the Lebesgue integral may not be the general integral that it purports to be. It may not deliver more than the Riemann integral. Rotating the page for the sake of integration need not resolve essential difficulties.

4. Baire Category

Baire defined a set of numbers to be of *1st category* if the set can be represented as a countable union of nowhere dense sets.

This definition was meant to characterize the set of the rational numbers, and distinguish it from the reals which, according to Cantor's claim, have strictly greater cardinality.

Baire's Theorem [5, p. 2] concludes that

- The complement of any set of first category on the line is dense in the real numbers
- No interval of real numbers is of first category

In a recent paper [6], we proved that in Cantor's set theory, there is only a single cardinality

$$\aleph_0 = 2^{\aleph_0}.$$

Hence,

- Intervals are of first category,

In fact,

- All sets of real numbers are of first category.

Therefore,

- The complement of a set of first category on the line need not be dense in the real numbers.

Consequently,

- Category does not distinguish between sets of real numbers

5. Baire Functions

The continuous functions are Baire class 0 functions [7, p. 137]. Functions that are limits of sequences of continuous functions are Baire class 1 functions. Functions that are limits of sequences of Baire class 1 functions, are Baire class 2 functions.

Thus, Baire class n functions are defined for any $n = 0,1,2,3...$ [7, p. 137]

By [7, p.140, THEOREM 3], the cardinality of the family of Baire functions is $2^{\aleph_0}$.

Therefore, there are countably many Baire functions.

References

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