The Infinite Series $\cos x + \cos 2x + \cos 3x + ...$ is Sumless for any $x \neq 0$

H. Vic Dannon
vic0user@gmail.com
May 2025

Abstract: An infinite series is characterized as convergent if it sums up to a finite number, and as divergent if it sums up to infinity.

The possibility of a sumless infinite series seems to be beyond the grasp of writings on infinite series.

In the introduction to his book “Divergent Series”, Hardy treats infinite series as if they are finite sums in order to sum up series that are sumless.

The non-existing sums became established facts to Hardy. And he applies them in the rest of the book.

For instance, he claims that the sumless series

$$1 + \cos\theta + \cos 2\theta + \cos 3\theta + ...$$

sums up to $\frac{1}{2}$.

And that the sumless series

$$\sin\theta + \sin 2\theta + \sin 3\theta ...$$

sums up to $\dfrac{\cos\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta}$.

These false claims apply later in the book, and make it highly unreliable.

Sumless series are useless, and should not be confused with divergent series like the harmonic series.

Contents

1.

The Series $1-1+1-....$ is Sumless

We see that

$$s_2 = 1-1 = 0,$$

as well as any even numbered partial sum of the series equals 0,

$$s_{2n} = 0.$$

And

$$s_3 = 1-1+1 = 1,$$

as well as any odd numbered partial sum of the series equals 1,

$$s_{2n+1} = 1.$$

The sum of the series may be neither 1, nor 0.

Assuming that the sum is 1, means that the sum is finite. In fact, the summation never ends. That is, after the wishful end we subtract 1. Then, the sum will be 0. But then, we add 1 and the sum will be 1. Therefore, infinite summation means that

The series is sumless.

Hardy missed the meaning of infinite summation

and assumed that the series has sum

$$s = 1-1+1-... = 1-\underbrace{(1-1+1-...)}_{s}$$

$$s = 1-s$$

But this equation cannot be correct.

According to this equation,

if $s=1$ then $1 = 1-1 = 0$,

and

$$\text{if } s = 0 \text{ then } 0 = 1 - 0 = 1.$$

So Hardy did not realize that there is no such s

And proceeded to conclude that

$$s = \frac{1}{2}.$$

But a sum of integers will never equal a quotient.

We see that

$$1 - 1 + 1 - 1 + 1 - .... \text{has no meaning whatsoever}$$

Therefore,

1.2 $1 + x + x^2 + ...$ **is Sumless at** $x = -1$

2.

$$1+x+x^2+....=\frac{1}{1-x} \quad \textbf{for} \quad -1<x<1$$

But the equality does not hold at $\left|x\right|=1$

<u>Proof</u>

$\underline{x=1}|$ $1+1^2+1^3+...=$ infinite hyper real $\neq \frac{1}{1-1}=$ undefined.$\square$

$\underline{x=-1}|$ $-1+(-1)^2+(-1)^3+...\neq \frac{1}{1-(-1)}=\frac{1}{2}.\square$

2.1 **for any** θ**,** $\left|e^{i\theta}\right|=1 \Rightarrow \quad 1+e^{i\theta}+e^{2i\theta}+e^{3i\theta}+...\neq \frac{1}{1-e^{i\theta}}$

3.

$1+e^{i\theta}+e^{2i\theta}+e^{3i\theta}+...$ is Sumless

Hardy argued that the series sums to $\dfrac{1}{1-e^{i\theta}}$ as follows:

$$\text{“}1+x+x^2+...=\frac{1}{1-x} \quad \textit{if} \quad |x|<1.$$

Putting $x=e^{i\theta}$*, where* $0<\theta<2\pi$ *So that* $x\neq 1$*,*

$$1+e^{i\theta}+e^{2i\theta}+...=\frac{1}{1-e^{i\theta}}\text{”}$$

So Hardy did not know that for any θ**,** $\left|e^{i\theta}\right|=1$**.**

3.1
$$\boxed{1+e^{i\theta}+e^{2i\theta}+...+e^{ni\theta}=\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}}$$

Proof
$$s_n=1+e^{i\theta}+e^{2i\theta}+e^{3i\theta}+...+e^{ni\theta}$$

$$s_n=1+e^{i\theta}(1+e^{i\theta}+e^{2i\theta}+...+e^{(n-1)i\theta})$$

$$s_n=1+e^{i\theta}(s_n-e^{in\theta})$$

$$s_n(1-e^{i\theta})=1-e^{i(n+1)\theta}$$

$$s_n=\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}.\square$$

As n increases, $e^{i(n+1)\theta}$ is not defined. Therefore,

3.2 $1+e^{i\theta}+e^{2i\theta}+e^{3i\theta}+...$ **is Sumless**

3.3 $\big(1+\cos\theta+..+\cos n\theta\big)+i\big(\sin\theta+..+\sin n\theta\big)=$

$$=\operatorname{Re}\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}+i\operatorname{Im}\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}$$

3.4 $\operatorname{Re}\left\{\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}\right\}=\frac{1}{2}+\frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}$

$$\operatorname{Im}\left\{\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}\right\}=\frac{\sin\left[(n+1)\frac{1}{2}\theta\right]\sin\left[n\frac{1}{2}\theta\right]}{\sin\frac{1}{2}\theta}$$

Proof $\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}=\frac{1}{2(1-\cos\theta)}(1-e^{i(n+1)\theta})(1-e^{-i\theta})$

$$=\frac{1}{2(1-\cos\theta)}\left[\big(1-\cos(n+1)\theta\big)-i\sin(n+1)\theta\right]\left[\big(1-\cos\theta\big)+i\sin\theta\right]$$

$$=\frac{\big(1-\cos(n+1)\theta\big)\big(1-\cos\theta\big)+\sin(n+1)\theta\sin\theta}{2(1-\cos\theta)}+$$

$$+i\frac{\big(1-\cos(n+1)\theta\big)\sin\theta-\sin(n+1)\theta\big(1-\cos\theta\big)]}{2(1-\cos\theta)}.\square$$

$$=\frac{1}{2}+\frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}+i\frac{\sin\left[(n+1)\frac{1}{2}\theta\right]\sin\left[n\frac{1}{2}\theta\right]}{\sin\frac{1}{2}\theta}$$

Therefore,

3.5 $\boxed{1+\cos\theta+..+\cos n\theta=\frac{1}{2}+\frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}}$

$$\boxed{\sin\theta + .. + \sin n\theta = \frac{\sin\left[(n+1)\frac{1}{2}\theta\right]\sin\left[n\frac{1}{2}\theta\right]}{\sin\frac{1}{2}\theta}}$$

4.

$1+\cos\theta+\cos 2\theta+...$ is Sumless

4.1 $\theta = 2\pi \Rightarrow 1+\cos 2\pi+\cos 4\pi+... = 1+1+1+... =$ infinite hyper-real

$\theta = \pi \Rightarrow 1+\cos\pi+\cos 2\pi+... = 1-1+1-...=$ no number

$\theta = \frac{1}{2}\pi \Rightarrow 1+\cos\frac{1}{2}\pi+\cos\pi+\cos\frac{3}{2}\pi+\cos 2\pi+... =$

$= 1-1+1-1+... ...=$ no number

$\theta = \frac{1}{4}\pi \Rightarrow 1+\cos\frac{1}{4}\pi+\cos\frac{1}{2}\pi+\cos\frac{3}{4}\pi+\cos\pi+... =$

$= 1+\frac{1}{2}\sqrt{2}-\frac{1}{2}\sqrt{2}-1-\frac{1}{2}\sqrt{2}+\frac{1}{2}\sqrt{2}+1+... =$ no number

4.2 As n increases,

$$1+\cos\theta+..+\cos n\theta = \frac{1}{2}+\frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta} = \text{has no limit}$$

Therefore,

4.3 $1+\cos\theta+\cos 2\theta+\cos 3\theta+...$ **is Sumless**

For $\theta \to \theta+\pi$,

4.4 $1-\cos\theta+\cos 2\theta-\cos 3\theta+...$ **is Sumless**

$-\sin\theta+\sin 2\theta-\sin 3\theta+...$ **is Sumless**

Therefore,

4.5 Differentiation of $\sin\theta-\frac{1}{2}\sin 2\theta+\frac{1}{3}\sin 3\theta+... = \frac{1}{2}\theta$

gives No Equality.

***Proof*:**

On the right,

$$\frac{d}{d\theta}\left(\frac{1}{2}\theta\right) = \frac{1}{2}$$

But on the left,

$$\frac{d}{d\theta}\left(\sin\theta\right) - \frac{d}{d\theta}\left(\tfrac{1}{2}\sin 2\theta\right) + \frac{d}{d\theta}\left(\tfrac{1}{3}\sin 3\theta\right) + ... =$$

$$= \cos\theta - \cos 2\theta + \cos 3\theta - ....$$

that is Sumless.□

Therefore,

4.6 <u>Repeated</u> Differentiation of the Equality

$$\sin\theta - \tfrac{1}{2}\sin 2\theta + \tfrac{1}{3}\sin 3\theta + ... = \tfrac{1}{2}\theta$$

gives No Equality

5.

$\sin\theta + \sin 2\theta + \sin 3\theta + ...$ is Sumless

5.1 $\theta = \frac{1}{2}\pi \Rightarrow \sin\frac{1}{2}\pi + \sin\pi + \sin\frac{3}{2}\pi + \sin 2\pi + ... =$

$$= 1 - 1 + 1 - 1 + ... \; ... = \text{ is Sumless}$$

$$\theta = \frac{1}{4}\pi \Rightarrow \sin\frac{1}{4}\pi + \sin\frac{1}{2}\pi + \sin\frac{3}{4}\pi + \sin\pi + \sin\frac{5}{4}\pi + \sin\frac{6}{4}\pi + ... =$$

$$= \frac{1}{2}\sqrt{2} + 1 + \frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2} - 1 - \frac{1}{2}\sqrt{2} = \text{ is sumless}$$

5.2 As n increases,

$$\sin\theta + .. + \sin n\theta = \frac{\sin\left[(n+1)\frac{1}{2}\theta\right]\sin\left[n\frac{1}{2}\theta\right]}{\sin\frac{1}{2}\theta} \text{ has no limit}$$

Therefore,

5.3 $\sin\theta + \sin 2\theta + \sin 3\theta + ...$ **is sumless**

For $\theta \to \theta + \pi$,

5.4 $-\sin\theta + \sin 2\theta - \sin 3\theta + ...$ **is sumless**

References

Hardy, "Divergent Series", Oxford, 1949. (Reprinted by Chelsea)

https://en.wikipedia.org/wiki/Divergent_series