Modulus Principles for Analytic Functions

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Abstract: We relax the Minimum Modulus Principle.

If $f(z)$ *is analytic, and* $f(z_0) \neq 0$*,*

then $\left|f(z)\right|$ *has no minimum at* z_0*.*

And present what we name the Saddle Modulus Principle:

If $f(z) = u + iv$ ***is analytic on a domain,***

$f'(z_0) = 0$**,** $f(z_0) \neq 0$**,**

and one of the second partial derivatives of u ***or*** v ***is not vanishing at*** z_0***,***

then $\left|f(z)\right|$ ***has a saddle at*** z_0***.***

The Maximum Modulus Principle is due to Landau [1], [2].
It is as late as the 1920's, since it is not mentioned by writers such as Goursat [11], and Forsyth [12].
The Maximum and Minimum Principles tell what the modulus of an analytic function is *not* doing in its domain, but not what it is doing. Here, we aim to clarify the behavior of the modulus function in its domain.

1.

The Modulus Theorem

applies to relax the statement of the minimum modulus principle, and to obtain a new modulus principle.

The Modulus Theorem

At a critical point z_0 of an analytic function f(z),

$$\begin{vmatrix} \partial_{xx}\left|f(z)\right|^2 & \partial_{xy}\left|f(z)\right|^2 \\ \partial_{yx}\left|f(z)\right|^2 & \partial_{yy}\left|f(z)\right|^2 \end{vmatrix}_{z=z_0} = -4\left(u_{xx}^2 + u_{xy}^2\right)\left|f(z)\right|^2\Big|_{z=z_0}.$$

Proof:

For an analytic function $f(z) = u + iv$,

$$u_x = v_y,\ \ u_y = -v_x,$$

$$u_{xx} = -u_{yy},$$

$$\partial_x\left|f(z)\right|^2 = 2\left(u_x u - v u_y\right),$$

$$\partial_y\left|f(z)\right|^2 = 2\left(u_y u + u_x v\right),$$

$$\partial_{xx}\left|f(z)\right|^2 = 2\left(u_{xx}u - u_{xy}v + u_x^2 + u_y^2\right),$$

$$\partial_{yy}\left|f(z)\right|^2 = 2\left(u_{yy}u + u_{xy}v + u_x^2 + u_y^2\right),$$

$$\partial_{xy}\left|f(z)\right|^2 = \partial_{yx}\left|f(z)\right|^2 = 2\left(u_{xy}u + u_{xx}v\right),$$

and

$$\begin{vmatrix} \partial_{xx}\left|f(z)\right|^2 & \partial_{xy}\left|f(z)\right|^2 \\ \partial_{yx}\left|f(z)\right|^2 & \partial_{yy}\left|f(z)\right|^2 \end{vmatrix} = -4\left(u_{xx}^2 + u_{xy}^2\right)\left|f(z)\right|^2 + \left(u_x^2 + u_y^2\right)^2$$

At a critical point in the domain,

$$u_x = u_y = 0,$$

and the determinant is

$$-4\left(u_{xx}^2 + u_{xy}^2\right)\left|f(z)\right|^2. \square$$

From this we will derive

1) The Maximum modulus principle,

2) A relaxed statement of the Minimum modulus principle, and

3) A modulus principle, which we name the <u>Saddle Modulus Principle.</u>

2.

The Maximum Modulus Principle:

If $f(z)$ is analytic, then $\left|f(z)\right|$ has no maximum at any point in its domain.

<u>Proof</u> : A maximum in a domain point z_0, mandates that z_0 is critical.

If $f(z_0) = 0$, then $\left|f(z_0)\right| = 0$ is not a maximum.

If $f(z_0) \neq 0$, then assuming a maximum, one of the second partial derivatives is non-vanishing, and the determinant is < 0.

But at a maximum, the determinant must be > 0.□

3.

The Minimum Modulus Principle.

If $f(z)$ is analytic, and $f(z_0) \neq 0$,

then $\left|f(z)\right|$ has no minimum at z_0.

<u>**Proof**</u>**:** A minimum in a domain point z_0, mandates that z_0 is critical.

If $f(z_0) = 0$, then $\left|f(z_0)\right| = 0$ is a minimum.

If $f(z_0) \neq 0$, then assuming a minimum, one of the second partial derivatives is non-vanishing, and the determinant is < 0.

But at a minimum, the determinant must be > 0.

4.

The Saddle Modulus Principle

If $f(z) = u + iv$ **is analytic on a domain,**

$f\,'(z_0) = 0$**,** $f(z_0) \neq 0$**,**

and one of the second partial derivatives of u **or** v **is not vanishing at** z_0**,**

then $\left|f(z)\right|$ **has a saddle at** z_0**.**

That is,

The modulus of a non-vanishing analytic function at a critical point has a saddle.

<u>Proof</u>: the determinant is <0, and $|f(z)|^2$ has a saddle.□

References

[1]. Copson, E. T. *An Introduction to the Theory of functions of a Complex Variable*, Oxford, 1935, p. 162.

[2]. Hille, Einar, *Analytic Function Theory*, Chelsea, 1982, p.189.

[3]. Goursat, Edouard, *Functions of a Complex Variable*, Ginn and Company, 1916.

[4]. Forsyth, A. R. *Theory of Functions of a Complex Variable,* Cambridge University Press, 1918.