The Mean Value Theorem, and its Geometric Meaning

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Abstract Cauchy Mean Value Theorem applies to a vector function $(f(x), g(x))$ on an interval (a,b).

We generalize it to vector functions of degrees 3, 4,..., n.

Then, it is characterized by a normality condition for the slope of the tangent at the intermediate point c.

Keywords: Mean Value Theorem, Vector Function, Roll's Theorem,

1.

Mean Value Theorem for Vector Variable

For a real valued $f(x)$ differentiable on (a,b), continuous at a, and b there is a point c in (a,b), so that

$$f(b) - f(a) = f'(c)(b - a)$$

For a real valued $f(x_1, x_2)$ differentiable on $(a_1, b_1) \times (a_2, b_2)$, continuous at (a_1, a_2), and (b_1, b_2). there is a point (c_1, c_2) in $(a_1, b_1) \times (a_2, b_2)$, so that

$$f(b_1, b_2) - f(a_1, a_2) = (\partial_x f)(c_1, c_2)(b_1 - a_1) + (\partial_y f)(c_1, c_2)(b_2 - a_2)$$

And for a real valued $f(\vec{x}) = f(x_1, ..., x_n)$ differentiable on $(a_1, b_1) \times ... \times (a_n, b_n)$, and continuous at $\vec{a}$, and $\vec{b}$, there is a point $\vec{c} = (c_1, c_2, ..., c_n)$ in $(a_1, b_1) \times ... \times (a_n, b_n)$, so that

$$\begin{aligned} f(\vec{b}) - f(\vec{a}) &= (\partial_1 f)(\vec{c})(b_1 - a_1) + ... + (\partial_n f)(\vec{c})(b_n - a_n) \\ &= \nabla f(\vec{c}) \cdot (\vec{b} - \vec{a}) \end{aligned}$$

Then,

$$\boxed{f(\vec{b}) = f(\vec{a}) \Rightarrow \nabla f(\vec{c}) \perp (\vec{b} - \vec{a})}$$

Such Normality Characterizes the Mean Value Theorem.

2.

Mean Value Theorem for a Vector Function

2.1 Vector Function with two Components

For a vector function,

$$\vec{f}(t) = \left[f_1(t), f_2(t) \right], \text{ differentiable on } (a,b),$$

the real valued function

$$F(t) = \begin{vmatrix} 1 & 1 & 1 \\ f_1(a) & f_1(b) & f_1(t) \\ f_2(a) & f_2(b) & f_2(t) \end{vmatrix}$$

satisfies $F(a) = F(b)$.

Therefore, by Roll's Theorem there is c in (a,b) so that

$$F'(c) = 0$$

$$\begin{vmatrix} 1 & 1 & 0 \\ f_1(a) & f_1(b) & f_1'(c) \\ f_2(a) & f_2(b) & f_2'(c) \end{vmatrix} = 0$$

$$-f_1'(c)\left(f_2(b) - f_2(a) \right) + f_2'(c)\left(f_1(b) - f_1(a) \right) = 0$$

Which is Cauchy Mean Value Theorem.

$$\vec{f}\,'(c) \cdot \begin{bmatrix} -\left(f_2(b) - f_2(a) \right) \\ f_1(b) - f_1(a) \end{bmatrix} = 0.$$

$$\boxed{\vec{f}\,'(c) \perp \begin{bmatrix} -\left(f_2(b) - f_2(a) \right) \\ f_1(b) - f_1(a) \end{bmatrix}}$$

The slope $\vec{f}\,'(c) = \left[f_1\,'(c), f_2\,'(c) \right]$ of the tangent at c is perpendicular to the vector $\begin{bmatrix} -\left(f_2(b) - f_2(a) \right) \\ f_1(b) - f_1(a) \end{bmatrix}$

2.2 Vector Function with three Components

For a vector function,

$$\vec{f}(t) = \left[f_1(t), f_2(t), f_3(t) \right], \text{ differentiable on } (a,b),$$

the real valued function

$$F(t) = \begin{vmatrix} f_1(a) & f_1(b) & f_1(t) \\ f_2(a) & f_2(b) & f_2(t) \\ f_3(a) & f_3(b) & f_3(t) \end{vmatrix}$$

satisfies $F(a) = F(b)$.

Therefore, by Roll's Theorem there is c in (a,b) so that

$$F\,'(c) = 0$$

$$\begin{vmatrix} f_1(a) & f_1(b) & f_1\,'(c) \\ f_2(a) & f_2(b) & f_2\,'(c) \\ f_3(a) & f_3(b) & f_3\,'(c) \end{vmatrix} = 0$$

$$f_1\,'(c) \begin{vmatrix} f_2(a) & f_2(b) \\ f_3(a) & f_3(b) \end{vmatrix} - f_2\,'(c) \begin{vmatrix} f_1(a) & f_1(b) \\ f_3(a) & f_3(b) \end{vmatrix} + f_2\,'(c) \begin{vmatrix} f_1(a) & f_1(b) \\ f_2(a) & f_2(b) \end{vmatrix} = 0,$$

which is a generalized Mean Value Theorem for

$$\vec{f}(t) = \left[f_1(t), f_2(t), f_3(t) \right].$$

$$[f_1'(c), f_2'(c), f_3'(c)] \cdot \begin{bmatrix} \begin{vmatrix} f_2(a) & f_2(b) \\ f_3(a) & f_3(b) \end{vmatrix} \\ -\begin{vmatrix} f_1(a) & f_1(b) \\ f_3(a) & f_3(b) \end{vmatrix} \\ \begin{vmatrix} f_1(a) & f_1(b) \\ f_2(a) & f_2(b) \end{vmatrix} \end{bmatrix} = 0.$$

$$\boxed{\vec{f}'(c) \perp \left[\begin{vmatrix} f_2(a) & f_2(b) \\ f_3(a) & f_3(b) \end{vmatrix}, -\begin{vmatrix} f_1(a) & f_1(b) \\ f_3(a) & f_3(b) \end{vmatrix}, \begin{vmatrix} f_1(a) & f_1(b) \\ f_2(a) & f_2(b) \end{vmatrix} \right]}$$

The slope $\vec{f}'(c) = [f_1'(c), f_2'(c), f_3'(c)]$ of the tangent at c is perpendicular to the vector

$$\left[\begin{vmatrix} f_2(a) & f_2(b) \\ f_3(a) & f_3(b) \end{vmatrix}, -\begin{vmatrix} f_1(a) & f_1(b) \\ f_3(a) & f_3(b) \end{vmatrix}, \begin{vmatrix} f_1(a) & f_1(b) \\ f_2(a) & f_2(b) \end{vmatrix} \right].$$

2.2 Vector Function with four Components

For a vector function,

$$\vec{f}(t) = [f_1(t), f_2(t), f_3(t), f_4(t)], \text{ differentiable on } (a,b),$$

the real valued function

$$F(t) = \begin{vmatrix} f_1(a) & f_1(b) & f_1(t) & 1 \\ f_2(a) & f_2(b) & f_2(t) & 1 \\ f_3(a) & f_3(b) & f_3(t) & 1 \\ f_4(a) & f_4(b) & f_4(t) & 1 \end{vmatrix}$$

satisfies $F(a) = F(b)$.

Therefore, by Roll's Theorem there is c in (a,b) so that

$$F'(c) = 0$$

$$\begin{vmatrix} f_1(a) & f_1(b) & f_1'(c) & 1 \\ f_2(a) & f_2(b) & f_2'(c) & 1 \\ f_3(a) & f_3(b) & f_3'(c) & 1 \\ f_4(a) & f_4(b) & f_4'(c) & 1 \end{vmatrix} = 0$$

$$f_1'(c)\begin{vmatrix} f_2(a) & f_2(b) & 1 \\ f_3(a) & f_3(b) & 1 \\ f_4(a) & f_4(b) & 1 \end{vmatrix} - f_2'(c)\begin{vmatrix} f_1(a) & f_1(b) & 1 \\ f_3(a) & f_3(b) & 1 \\ f_4(a) & f_4(b) & 1 \end{vmatrix}$$

$$+ f_3'(c)\begin{vmatrix} f_1(a) & f_1(b) & 1 \\ f_2(a) & f_2(b) & 1 \\ f_4(a) & f_4(b) & 1 \end{vmatrix} - f_4'(c)\begin{vmatrix} f_1(a) & f_1(b) & 1 \\ f_2(a) & f_2(b) & 1 \\ f_3(a) & f_3(b) & 1 \end{vmatrix} = 0$$

Which is a generalized Mean Value Theorem for

$$\vec{f}(t) = \left[f_1(t), f_2(t), f_3(t), f_4(t) \right].$$

The slope $\vec{f}'(c) = \left[f_1'(c), f_2'(c), f_3'(c), f_4'(c) \right]$ **of the tangent at** c **is perpendicular to the vector**

$$\left[\begin{vmatrix} f_2(a) & f_2(b) & 1 \\ f_3(a) & f_3(b) & 1 \\ f_4(a) & f_4(b) & 1 \end{vmatrix}, -\begin{vmatrix} f_1(a) & f_1(b) & 1 \\ f_3(a) & f_3(b) & 1 \\ f_4(a) & f_4(b) & 1 \end{vmatrix}, \begin{vmatrix} f_1(a) & f_1(b) & 1 \\ f_2(a) & f_2(b) & 1 \\ f_4(a) & f_4(b) & 1 \end{vmatrix}, -\begin{vmatrix} f_1(a) & f_1(b) & 1 \\ f_2(a) & f_2(b) & 1 \\ f_3(a) & f_3(b) & 1 \end{vmatrix} \right]$$

2.2 Vector Function with n Components

For a vector function,

$$\vec{f}(t) = \left[f_1(t), f_2(t), \dots, f_n(t) \right], \text{ differentiable on } (a,b),$$

the real valued function

$$F(t) = \begin{vmatrix} f_1(a) & f_1(b) & f_1(t) & 1 & 2 & .... & n \\ f_2(a) & f_2(b) & f_2(t) & 1 & 1 & ... & 1 \\ & & & 1 & 1 & ... & 1 \\ & & & 1 & 1 & ... & 1 \\ & & & 1 & 1 & ... & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ f_n(a) & f_n(b) & f_n(t) & 1 & 1 & ... & 1 \end{vmatrix}$$

satisfies $F(a) = F(b)$.

Therefore, by Roll's Theorem there is c in (a,b) so that

$$F'(c) = 0$$

$$F(t) = \begin{vmatrix} f_1(a) & f_1(b) & f_1'(t) & 1 & 2 & .... & n \\ f_2(a) & f_2(b) & f_2'(t) & 1 & 1 & ... & 1 \\ & & & 1 & 1 & ... & 1 \\ & & & 1 & 1 & ... & 1 \\ & & & 1 & 1 & ... & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ f_n(a) & f_n(b) & f_n'(t) & 1 & 1 & ... & 1 \end{vmatrix} = 0$$

Expanding the determinant in the column of $\vec{f}'(c)$, we obtain a generalized Mean Value Theorem for

$$\vec{f}(t) = \left[f_1(t), f_2(t), ..., f_n(t) \right].$$

The slope $\vec{f}'(c) = \left[f_1'(c), f_2'(c), ..., f_n'(c) \right]$ **of the tangent at** c **is perpendicular to the vector of cofactors in the Laplace expansion of the determinant.**

References

https://en.wikipedia.org/wiki/Rolle%27s_theorem

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