Khinchin's Constant Fallacy, and the Geometric Mean of the Coefficients of Continued Fraction Expansion

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Abstract Khinchin's Claim about his Constant, remains unproved to this date. It is at best a conjecture that fails to hold conclusively for even one real number.

The infinitely many counter-examples to Khinchin's claim, are considered exceptions to the rule. But indeed they are the rule.

Misreading Numerical Experiments may be proof to Khinchin's Conjecture believers. In fact, the Conjecture does not hold for even one number.

We disprove Lebesgue's Measure argument that underlies any of the Conjecture's false Proofs. This demonstrates the non-credibility of the Lebesgue Measure theory.

Furthermore, the Conjecture distinction between rationals and irrationals is not credible under any consideration.

Numerical Experiments suggesting that π, and γ may satisfy that claim, indicate the converse. Namely, that for all real numbers, Khinchin's claim about his constant is a Fallacy.
The Numerical Experiments uncover the Random values attained by the Geometric Means of the Coefficients of Continued Fraction Expansion.

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Contents

1.

Khinchin's Constant

The Khinchin Constant is the converging infinite product

$$K = (1+\tfrac{1}{2(2+2)})(1+\tfrac{1}{3(3+2)})^{\frac{\log 3}{\log 2}}(1+\tfrac{1}{4(4+2)})^{2}(1+\tfrac{1}{5(5+2)})^{\frac{\log 5}{\log 2}}\times$$

$$\times(1+\tfrac{1}{6(6+2)})^{1+\frac{\log 3}{\log 2}}(1+\tfrac{1}{7(7+2)})^{\frac{\log 7}{\log 2}}(1+\tfrac{1}{8(8+2)})^{3}(1+\tfrac{1}{9(9+2)})^{2\frac{\log 3}{\log 2}}\times$$

$$\times(1+\tfrac{1}{10(10+2)})^{1+\frac{\log 5}{\log 2}}(1+\tfrac{1}{11(11+2)})^{\frac{\log 11}{\log 2}}(1+\tfrac{1}{12(12+2)})^{2+\frac{\log 3}{\log 2}}\times$$

$$\times(1+\tfrac{1}{13(13+2)})^{\frac{\log 13}{\log 2}}\times\ldots\times(1+\tfrac{1}{k(k+2)})^{\frac{\log k}{\log 2}}\times\ldots..$$

$$= 2.6854520010\ldots$$

K may seem more tractable written as

$$\prod_{k=2}^{k=\infty}(1+\tfrac{1}{k(k+2)})^{\frac{\log k}{\log 2}}$$

But even then, Khinchin's claim regarding K is non-credible.

Khinchin claimed (1935) that *for almost any real number*, x *represented by its continued fraction expansion*

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ldots}}}},$$

$$(a_1 a_2 \ldots a_n)^{\frac{1}{n}} \quad \textit{converges to } K.$$

In other words, K is *God's Universal Constant*,

2.

Exceptions to Khinchin's Claim

The exceptions are any number that does not satisfy the Claim. There are infinitely many exceptions to Khinchin's Claim, and only finitely many non-exceptions, none of which conclusive.

The Modern Circle Squarers cannot let go of Khinchin's Claim and made any number that violates it into an exception.

It is safe to say that Khinchin Claim allows for most exceptions of all statements ever made under the disguise of Mathematics.

The first exception are numbers in sets that have measure zero.

These include

- the integers, (such as 2),
- the Cantor set, which cardinality is $2^{Card\mathbb{N}}$.

Also believed to be of measure zero, and thus, "exceptions" are

- the rational numbers, (such as $\frac{1}{2}$).

  But in [Dan1] we proved that this set is non-measurable

And our references believe that of measure zero are also

- the quadratic irrationals, (such as $\sqrt{2}$).

  That set may be non-measurable too. But the failure of Lebesgue Measure Theory [Dan1], and [Dan2], renders the question irrelevant.

Lehmer pointed out that Euler's transcendental e does not satisfy the Khinchin Claim, and e became another "exception".

And we observe that any number constitutes a set of length zero, and is an "exception".

We do not know which set of numbers may be defined by

almost every number satisfies what a few numbers may satisfy…

Could it be the definition to the **almost empty set**?

But with no one number that conclusively satisfies Khinchin's claim, his claim is at best a Conjecture.

To date, proofs given to that conjecture are based on the falsehood of Lebesgue's Measure theory that the set of the irrational numbers in the interval $[0,1]$ has a length (In Lebesgue Theory jargon, a "measure").

3.

Khinchin's Conjecture "Proofs"

The wrong proofs given to Khinchin's Conjecture include the ones by

- Khinchin's [Khinchin, pp. 95-101],
- Kac [Kac, pp.88-92],
- Ryll-Nardzewski [Wikipedia]

All "Proofs" assume that the irrationals in $[0,1]$ satisfy the conjecture, ignoring the fact that the irrational e was proven not to satisfy it.

All "Proofs" require the definition of a measure on the irrationals, and are blind to the non-measurability of the irrational numbers in $[0,1]$

The existence of such measure is taken for granted, but according to Lebesgue's definition, the irrationals are non-measurable, and there is no measure that may be defined on them.

By [Lebesgue, p.105],

"A set E is measurable if and only if

for $\varepsilon > 0$, as small as we wish,

E has a cover by $\alpha(\varepsilon)$ open intervals,

and E^c has a cover by $\beta(\varepsilon)$ open intervals *so that the sum of the lengths of the intervals of intersection of the covers is* $< \varepsilon$*"*

For example, the points

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.$$

are separated by the intervals between them.

The length of the intervals is

$$(1 - \tfrac{1}{2}) + (\tfrac{1}{2} - \tfrac{1}{3}) + (\tfrac{1}{3} - \tfrac{1}{4}) + \ldots = 1.$$

and $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.\right\}$ has measure 0.

But the rational numbers in $[0,1]$ cannot be separated from each other by open intervals of irrational numbers.

The rationals, and irrationals have no open covers, that may be refined so that their common intersection shrinks and is $< \varepsilon$.

Any open cover of irrationals, covers the rationals too.

Therefore, by Lebesgue's definition,

The irrationals in $[0,1]$, and the rationals in $[0,1]$, are non- measurable.

Consequently, any "Proof" that depends on the measurability of the irrationals in $[0,1]$ is false. But the Circle Squarers are expected to come up with more such "Proofs"

4.

Implications to Lebesgue Measure

Khinchin [Khinchin, p. 101] does not mention that he had no one conclusive example to support his conjecture. Instead, he speculates further about other power means [Dan3] that may satisfy such conjectures.

In fact, by Lebesgue's definition, the irrationals in $[0,1]$ are non-measurable, and any "Proof" that requires the irrationals to be measurable in $[0,1]$ is false.

Khinchin's Conjecture fails because of it depends on fallacies of Lebesgue Measure theory.

Thus, the failure of Khinchin's Constant to live up to Khinchin's conjecture about it, is the failure of Lebesgue Theory of Measure.

The failure of Lebesgue Measure theory causes also the failure of Lebesgue's Integration theory [Dan2].

5.

Khinchin's Conjecture Meaning to Number Theory

[Finch,p.60], characterizes Khinchin's Conjecture as

"…a profound statement about the nature of real numbers…"

[Kac, p.92] calls it *"…Remarkable Theorem…"*

and adds, that

> *"…the road from kinetic theory … to continued fractions, is a superb example…that mathematics… owes its beauty to other disciplines…"*

But Khinchin Conjecture has to be rejected because of its non-credible implication to Number Theory.

The properties of numbers cannot be determined by the way we group them.

Had it been true that all the rationals in $[0,1]$ fail to satisfy Khinchin's Conjecture, while all the irrationals satisfy it, we would have had a puzzling criteria for rationality, and irrationality.

Namely, that the Khinchin's Constant can serve to determine the rationality or irrationality of any number.

Puzzling, because

- We do not know if Khinchin's Constant is rational, or irrational, algebraic or transcendental.
- If we expect an explanation to come from a proof, this claim is supported by no valid proof.
- If we wish to have one conclusive example, there is none.

Lehmer pointed out that Euler's irrational e does not satisfy the Khinchin Conjecture. That invalidates any of the existing false proofs that assume that the irrationals satisfy the Conjecture.

Instead, the Circle Squarers call Lehmer's disproof another "exception" to the false Khinchin's Conjecture.

Consequently, the new meaning of the Conjecture is that for almost all real numbers, the Khinchin Constant can determine the rationality.

Therefore, for any particular number, the distinction is inapplicable, and Khinchin's Constant cannot detect rationality, or irrationality.

6.

The Myth of "Almost all Real Numbers"

The fallacy that allowed Khinchin to claim that almost all real numbers satisfy his Conjecture is based on Cantor's assumption that there are more real numbers than rationals. Hence, more irrationals than rationals.

But in fact there is no uncountable number of elements anywhere.

The uncountable Cantor Set is the endpoints of mid intervals tossed away in the process of constructing the Cantor set. All those endpoint are rational numbers, which are countable. Therefore, the number of rationals and irrationals is the same $2^{\mathrm{CardN}} = \mathrm{CardN}$. That is, the rationals are "almost all the real numbers" just like the irrationals.

7.

The Numerical Evidence for the Khinchin Fallacy

The evidence that the Khinchin Conjecture is a Fallacy exists in the numerical Experiments that are cited as compelling, yet inconclusive.

To whoever is trained in asymptotic divergence of random numbers, these experiments, that are rather preliminary, are quite conclusive to the detriment of the Khinchin Conjecture.

The experiments raise the false hopes of the Circle Squarers, and confuse the rest who are not trained in such problems.

The expectations that all numbers will violate the Conjecture in one uniform fashion are unsubstantiated.

Some numbers plainly violate the Conjecture. Some violate it in a confusing way. But they all violate it just the same.

Regarding the plain violators [Weinstein] plots with $a_1, a_2, \ldots a_{500}$,

the convergence of the Geometric Means of numbers

$$e,$$

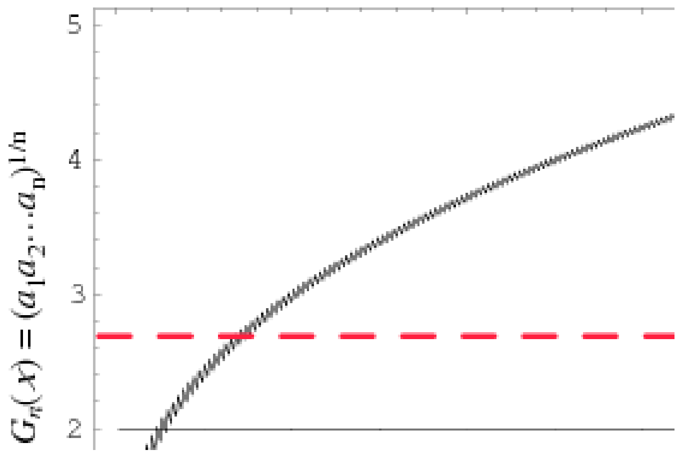
$$\sqrt{2},$$

$$\sqrt{3},$$

and the golden ratio ϕ.

Then, clearly

$$(a_1, a_2, \ldots a_n)^{\frac{1}{n}} \not\to \text{Khinchin's Constant}$$



Regarding the confusing violators [Weinstein] plots with $a_1, a_2, \ldots a_{500}$, the convergence of

π,

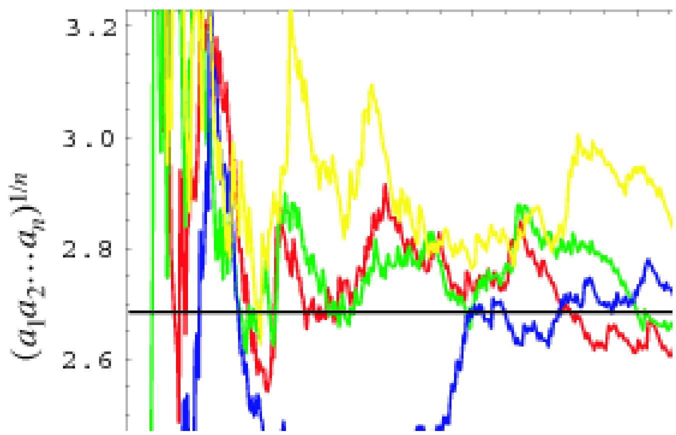
$\sin(1)$,

Euler's other constant γ,

and the Copeland-Erdos constant c.

Then, to the untrained it seems that perhaps

$$(a_1, a_2, \ldots a_n)^{\frac{1}{n}} \xrightarrow{?} \text{Khinchin's Constant???}$$



To the trained, this is the fashion in which the confusing violators diverge away from K, rendering the Khinchin Conjecture a Fallacy.

To interpret correctly the confusing violators graphs, one needs to be familiar with such divergence. We cite [Dan4], and [Dan5].

To make the point clear, we refer to similar convergence that shows up in Riemann's Formula for the Count of the Primes.

In his 1859 Zeta paper, (ref. [Dan5]), Riemann obtained a formula for the count of the primes, that uses all the zeros of the Zeta function on the line $x = \frac{1}{2}$, to solve the problem completely, provided that all the zeros of the Zeta function in $0 < x < 1$, are on the line $x = \frac{1}{2}$.

The Riemann formula has four terms. But only the first and the third of these terms have non-negligible values. The first is a dominant term that can be computed precisely. The third term is smaller and depends on the provision regarding the zeros of the Zeta function.

This provision became known as the Riemann Hypothesis, but it was never hypothesized by Riemann. Not seeing an easy proof for it, Riemann used only the first term of his formula, and obtained an approximation far superior to Gauss for the count of the primes. Thus, the first term in Riemann's Formula is known as the Riemann Approximation term.

We shall refer to the third term that depends on the Hypothesis, and was neglected since Riemann, as the **Riemann-Hypothesis-Series**.

It is obtained provided that all the zeros of the Zeta function in the strip $0 < x < 1$, lie on the line $x = \frac{1}{2}$.

Each term of the Hypothesis Series is evaluated at a zero of the Zeta function on the line $x = \frac{1}{2}$. Since there are infinitely many such zeros, the Series has infinitely many terms.

Riemann wondered about the effect of the Hypothesis series, but left it out of his approximation formula.

Riemann wrote

The finite sum of oscillatory terms

$$-2\frac{t^{-1/2}}{\log t}\sum_{\alpha}\cos(\alpha\log t)$$

cause irregular fluctuations in the density of the primes.

It would be interesting *to trace the fluctuations of the density of the primes* $F'(t)$ *to the particular oscillatory terms in* $f'(t)$ $\diamondsuit$

In [Dan4],we have used the Hypothesis Series, and proved that

Riemann's Formula for the Count of the Primes is valid with Riemann Hypothesis Series, with uncertainty under 10^{-16}.

This allows us to use Riemann's formula for the count of the primes with great certainty.

Actually, our computations indicated that if not for the limitations of the software, Riemann's Formula can be confirmed to any degree of certainty.

In particular we confirmed Riemann's suspicion that

the Hypothesis Series convergence is unpredictable.

To that end we have computed and graphed with the aid of Mathematica the Hypothesis Series for the number of primes up to $10,000,000$ with

the first 50 partial sums, using the first 50 zeta zeros,

the first 1000 partial sums, using the first 1000 zeta zeros,

the first 1000 partial sums, using the first 1000 zeta zeros,

the first 5,000 partial sums, using the first 5,000 zeta zeros,

the first 20,000 partial sums, using the first 20,000 zeta zeros, the first 100,000 partial sums, using the first 100,000 zeta zeros, And performed the Statistical Analysis of Mean and Variance for each case.

At any of these cases, none of the Statistical Moments indicated any Statistical distribution underlying the random convergence of the Hypothesis Series.

Riemann who did compute by hand, before expressing his doubts, was right to suspect that the convergence was random.

Each of the graphs in [Dan4] looks like the preliminary graph posted in [Weinstein]. But the evolution from smaller to larger numbers defies our intuitive perception of convergence. The convergence does not get better with more terms. With less terms, there may be more of the looks of convergence than with more terms.

We note that the preliminary graph posted by [Weinstein] is very preliminary, and may lead to seeing divergence as convergence.

Using 500 terms of $a_1, a_2, ...a_n$ instead of millions, is not a base for any conclusions.

The experienced worker can see the divergence in [Weinstein] preliminary graph, but once the missing work is completed, it will be plain to all that the confusing as-if-convergences are actually divergences, and the Khinchin Conjecture is a fallacy.

Clearly, measure theory is irrelevant to the Khinchin Conjecture. The Conjecture depends on the continued fraction expansion coefficients of the real number x.

8.

Convergence or Divergence of $\left(a_1 a_2 \dots a_n\right)^{\frac{1}{n}}$

Suppose that x has an infinite continued fraction

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\dots}}}},$$

Thus, x must be an irrational number.

Suppose further that

$$(a_1 a_2 \dots a_n)^{\frac{1}{n}} \textit{ converges to } K = 2.6854520010\dots.$$

Then,

$$\frac{\log a_1 + \log a_2 + \dots \log a_n}{n} \to \log K .$$

Denote

$$A_n = \log a_1 + \log a_2 + \dots \log a_n ,$$

$$B_n = n .$$

Since the a_n's are positive integers,

$$A_n \to \infty .$$

Clearly,

$$B_n \uparrow \infty .$$

And

$$\frac{A_n - A_{n-1}}{B_n - B_{n-1}} = \frac{\log a_n}{1} = \log a_n .$$

As $n \to \infty$, we would like to replace

$$\log a_n$$

by

$$\log a_* ,$$

where a_* is a positive integer which existence is guaranteed by the continued fraction expansion of the number x.

<u>Case 1</u> After some n_0, all the a_n's are equal

Then,

$$a_* = a_{n_0} ,$$

and by **Stoltz Rule**, for the indeterminate limit of $\frac{\infty}{\infty}$, we have

$$\frac{A_n}{B_n} \to \log a_{n_0} .$$

On the other hand,

$$\frac{A_n}{B_n} \to \log K .$$

Hence,

$$K = a_{n_0} = \text{positive integer} ,$$

which contradicts $K = 2.6854520010...$

Thus, in Case 1,

$$(a_1 a_2 ... a_n)^{\frac{1}{n}} \to a_{n_0} \neq K .$$

<u>Case 2</u> There is no n_0, after which all the a_n's are equal

Then, we need to consider the indeterminate quotient

$$\frac{A_n}{B_n} = \frac{\log a_1 + \log a_2 + ... \log a_n}{n},$$

and we observe two cases,

<u>Case 2a</u> x is a **periodic** continued fraction.

For example,

$$\sqrt{34} = [5, \overline{1, 4, 1, 10}] = 5 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{10 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{10 + \cfrac{1}{1 + ...}}}}}}}}}$$

Then, for an infinite hyper-real N,

$$\frac{A_{N+1}}{B_{N+1}} = \frac{\log 5 + \frac{N}{4}\log 1 + \frac{N}{4}\log 4 + \frac{N}{4}\log 1 + \frac{N}{4}\log 10}{N+1}$$

$$= \frac{\log 5 + \frac{N}{4}\log 4 + \frac{N}{4}\log 10}{N+1}$$

$$\approx \frac{\frac{N}{4}\log 4 + \frac{N}{4}\log 10}{N+1}$$

$$= \frac{\log(40)^{\frac{1}{4}}}{1+\frac{1}{N}}$$

$$\approx \log(40)^{\frac{1}{4}}$$

Thus, in our example for case 2a,

$$(a_1 a_2 ... a_n)^{\frac{1}{n}} \to (40)^{\frac{1}{4}} = 2.514866859 \neq K$$

Case 2b x is a **non-periodic** continued fraction.

Such is the Khinchin Constant K. According to [Weinstein], the 110,000 first coefficients of K were computed in 1997.

The first 96 coefficients are [Weinstein]

2, 1, 2, 5, 1, 1, 2, 1, 1, 3, 10, 2, 1, 3, 2, 24, 1, 3, 2, 3, 1, 1, 1, 90, 2, 1, 12, 1, 1, 1, 1, 5, 2, 6, 1, 6, 3, 1, 1, 2, 5, 2, 1, 2, 1, 1, 4, 1, 2, 2, 3, 2, 1, 1, 4, 1, 1, 2, 5, 2, 1, 1, 3, 29, 8, 3, 1, 4, 3, 1, 10, 50, 1, 2, 2, 7, 6, 2, 2, 16, 4, 4, 2, 2, 3, 1, 1, 7, 1, 5, 1, 2, 1, 5, 3, 1

The first spiking coefficients are [Weinstein]

2, 5, 10, 24, 90, 770, 941, 11759, 54097, 231973

The coefficients of a non-periodic fraction follow no pattern, and no formula, and are unpredictable like any random numbers.

Consequently, Khinchin Conjecture that these random numbers satisfy

$$(a_1 a_2 ... a_n)^{\frac{1}{n}} \text{ converges to } K = 2.6854520010...$$

is at best baseless.

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